Mathematical
Inequalities
& Applications
Volume 18, Number 3 (2015), 893–899 doi:10.7153/mia-18-66

A GENERALIZED CAUCHY–SCHWARZ INEQUALITY

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(*Communicated by H. Martini*)

Abstract. In the course of realizing certain triangle centers as points that minimize certain quantities, C. Kimberling and P. Moses, in *Math. Mag.* **85** (2012) 221–227, discovered an inequality in three variables that generalizes the Cauchy-Schwarz inequality, and made a conjecture regarding a generalization of that inequality to an arbitrary number of variables. In this paper, we give a proof of a stronger form of that conjecture.

1. Introduction

Given a triangle ABC and a point P in its plane, the *trilinear coordinates* (or simply, the *trilinears*) of P with respect to ABC are the signed distances $x, y,$ and z from P to the sidelines BC, CA, and AB, respectively. Here, the *signed* distance between P and a sideline, say BC, is taken to be positive if A and P lie on the same side of line BC, i.e, if they lie in one of the two half-planes into which the line BC divides the plane; it is taken to be negative otherwise. For example, the trilinears of P are equal if and only if P is the *incenter* of ABC, i.e., the point where the internal angle bisectors of ABC meet. We notice that this is the point at which the quantity $f = x^2 + y^2 + z^2 - (xy + yz + zx)$ attains its minimum. This follows immediately from the representation

$$2f = (x-y)^2 + (y-z)^2 + (z-x)^2.$$

One can also prove that the point at which the quantity $g = x^2 + y^2 + z^2$ attains its minimum is the point known as the *Lemoine* or *symmedian* point of ABC. This is defined to be the point of concurrence of the symmedians of ABC, where the *symmedians* are the reflections of the medians about the respective internal angle bisectors, and where the *medians* are the cevians joining the vertices of ABC to midpoints of the opposite sides. Motivated by these observations, C. Kimberling and P. Moses considered the points that minimize the quantity $(x^2 + y^2 + z^2) - t(xy + yz + zx)$ for arbitrary values of $t \in \mathbb{R}$, and found out that several of the points catalogued in [7] and [9] are realized in this manner.

Mathematics subject classification (2010): 51M16, 51M04.

Keywords and phrases: Cauchy-Schwarz inequality, constrained critial point, extrema, Hessian, Lagrange's multipliers, Lemoine point, positive-definite matrix, quadratic form, symmedian point, triangle centers.

This research is supported by a research grant from Yarmouk University.

Along the way, they discovered an inequality that involves the side lengths a, b, c of a triangle, and then realized that the inequality actually holds for arbitrary numbers a, b, c. The inequality states that if $k, t \in [-1, 2]$ are such that $(k+2)(t+2) = 4$, then

$$\begin{aligned}&[a^2+b^2+c^2+k(bc+ca+ab)][x^2+y^2+z^2+t(yz+zx+xy)]\\ \geqslant&(ax+by+cz)^2(1+k)(1+t)\end{aligned} \tag{1}$$

for all $x, y, z, a, b, c \in \mathbb{R}$, thus generalizing the Cauchy-Schwarz inequality – being nothing but the special case $k = t = 0$. They then made a conjecture regarding a generalization of (1) to an arbitrary number of variables. We state and strengthen this conjecture in the next section, and we give a proof.

We digress here to say that it is fairly natural to feel that a triangle center ought to be the point that *minimizes* (or *maximizes*) a certain quantity of some geometric or/and algebraic significance. For an elaboration on this and other similar feelings, see [2].

2. The conjecture and its proof

We now turn to our main goal. We state, as Conjecture 1, the conjecture posed in [8], and we prove a stronger form in Theorem 1. Lemma 1, which is taken from [1], is used in the proof of Theorem 1. The first equivalence in Lemma 1 is included because it is easier to remember.

CONJECTURE 1. ([8]) *Suppose that $n \geqslant 3$ and that the following conditions hold:*

$$k \text{ and } t \text{ both lie in the interval } \left[\frac{-2}{n-1}, 2\right]; \tag{2}$$

$$\left(k+\frac{n+1}{n-1}\right)\left(t+\frac{n+1}{n-1}\right) = \frac{3n-1}{n-1}; \tag{3}$$

$$a_1, a_2, \cdots, a_n, x_1, x_2, \cdots, x_n \text{ are real numbers}; \tag{4}$$

$$B = \frac{3n-1}{4(n-1)^3}[2+k(n-1)][2+t(n-1)]. \tag{5}$$

Then

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2 + k\sum_{1\leqslant i<j\leqslant n} a_i a_j\right)\left(\sum_{1\leqslant i\leqslant n} x_i^2 + t\sum_{1\leqslant i<j\leqslant n} x_i x_j\right) \geqslant B\left(\sum_{1\leqslant i\leqslant n} a_i x_i\right)^2. \tag{6}$$

LEMMA 1. (Theorem 1 (ii) of [1].) *Let $n \geqslant 2$, and let $\alpha, \beta \in \mathbb{R}$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be given by the quadratic form*

$$F(x_1, \cdots, x_n) = \alpha\left(\sum_{1\leqslant i\leqslant n} x_i^2\right) + \beta\left(\sum_{1\leqslant i<j\leqslant n} x_i x_j\right).$$

Then

$$F(x_1,\cdots,x_n)\geqslant 0\ \ \forall\, x_i\in\mathbb{R} \Longleftrightarrow F(x_1,\cdots,x_n)\geqslant 0 \text{ for the 3 } n\text{-tuples } (1,0,\cdots,0),(1,1,\cdots,1),(1,-1,0,\cdots,0) \Longleftrightarrow \alpha\geqslant 0,\ \ 2\alpha+\beta(n-1)\geqslant 0,\ \ 2\alpha-\beta\geqslant 0.$$

THEOREM 1. (A stronger form of Conjecture 1.) *Suppose that $n\geqslant 3$ and that the following conditions hold:*

$$h \text{ and } t \text{ are real numbers that lie in the interval } \left[\frac{-2}{n-1},2\right]; \tag{7}$$

$$\left(h+\frac{2}{n-2}\right)\left(t+\frac{2}{n-2}\right)=\left(\frac{2}{n-2}\right)^2; \tag{8}$$

$$a_1,a_2,\cdots,a_n,x_1,x_2,\cdots,x_n \text{ are real numbers}; \tag{9}$$

$$A=\frac{(2-h)(2-t)}{4}. \tag{10}$$

Then

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2+h\sum_{1\leqslant i<j\leqslant n} a_ia_j\right)\left(\sum_{1\leqslant i\leqslant n} x_i^2+t\sum_{1\leqslant i<j\leqslant n} x_ix_j\right)\geqslant A\left(\sum_{1\leqslant i\leqslant n} a_ix_i\right)^2. \tag{11}$$

Moreover, (11) is stronger than (6).

Proof. Let n, h, t, $a_1,\cdots,a_n$, and A be as given in the theorem. Let $F:\mathbb{R}^n\to\mathbb{R}$ be defined by

$$F(x_1,\cdots,x_n)=\left(\sum_{1\leqslant i\leqslant n} x_j^2\right)+t\left(\sum_{1\leqslant i<j\leqslant n} x_ix_j\right). \tag{12}$$

It follows from Lemma 1 that $F(x_1,\cdots,x_n)\geqslant 0$ for all $x_i\in\mathbb{R}$.

To prove (11), it is clearly enough to prove (11) for all $(x_1,\cdots,x_n)\in\mathbb{R}^n$ that lie in the hyperplane defined by

$$H(x_1,\cdots,x_n)=\left(\sum_{1\leqslant i\leqslant n} a_ix_i\right)-c=0, \tag{13}$$

where c is an arbitrary real number.

Note that if $t=2$, then $A=0$, and (11) is vacuous. If $t=-2/(n-1)$, then $h=2$ and $A=0$, and (11) is again vacuous. If $c=0$, then $\sum_{1\leqslant i\leqslant n} a_ix_i=0$ on H, and (11) is again vacuous. So we assume that

$$\frac{-2}{n-1}<t<2,\ \ c\neq 0. \tag{14}$$

It follows that

$$(a_1,\cdots,a_n)\neq(0,\cdots,0),\qquad \frac{-2}{n-1}<h<2. \tag{15}$$

It also follows from (8) that

$$h = \frac{-2t}{(n-2)t+2}, \qquad 2-h = 2\left(\frac{(n-1)t+2}{(n-2)t+2}\right). \tag{16}$$

Under the assumptions in (14) and (15), we would like to minimize the function F given in (12) under the constraint given by (13). It suffices to prove that the minimum μ satisfies the inequality

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2 + h\sum_{1\leqslant i<j\leqslant n} a_i a_j\right)\mu \geqslant Ac^2. \tag{17}$$

Actually we shall show that equality holds in (17).

To achieve our goal, we use the method of Lagrange to obtain the system of equations

$$M\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \cdots \\ \cdots \\ \lambda a_n \end{bmatrix}, \text{ where } M = \begin{bmatrix} 2 & t & \cdots\cdots & t \\ t & 2 & \cdots\cdots & t \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ t & t & \cdots\cdots & 2 \end{bmatrix}. \tag{18}$$

By Lemma 3.1 of [3], the determinant $\Delta = \det(M)$ of M and the minors $\Delta_{i,j} = \det(M_{i,j})$ are given by

$$\Delta = \det(M) = ((n-1)t+2)(2-t)^{n-1},$$

$$\Delta_{i,j} = \det(M_{i,j}) = \begin{cases} ((n-2)t+2)(2-t)^{n-2} & \text{if } i=j, \\ (-1)^{j-1-i}(2-t)^{n-2}t & \text{if } i\neq j. \end{cases}$$

Since $\Delta \neq 0$, we may use Cramer's rule for solving the system (18). We have

$$\Delta_{i,1} = \begin{cases} ((n-2)t+2)(2-t)^{n-2} & \text{if } i=1, \\ (-1)^{i}(2-t)^{n-2}t & \text{if } i\neq 1. \end{cases}$$

$$\begin{aligned} \Delta_1 := \det &\begin{bmatrix} \lambda a_1 & t & \cdots\cdots & t \\ \lambda a_2 & 2 & \cdots\cdots & t \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \lambda a_n & t & \cdots\cdots & 2 \end{bmatrix} \\ &= \lambda\left(a_1\Delta_{1,1} - a_2\Delta_{2,1} + \cdots + (-1)^{n-1}\Delta_{n-1,1}\right) \\ &= \lambda(2-t)^{n-2}\left[((n-2)t+2)a_1 - (a_2+a_3+\cdots+a_n)t\right] \end{aligned}$$

$$\frac{\Delta_1}{\Delta} = \frac{\lambda\left(((n-2)t+2)a_1 - (a_2+a_3+\cdots+a_n)t\right)}{(2-t)((n-1)t+2)} = \frac{\lambda(a_1u - St)}{(2-t)u},$$

where

$$u = (n-1)t + 2, \qquad S = a_1 + \cdots + a_n. \tag{19}$$

Thus the solution $(\rho_1, \cdots, \rho_n)$ of the system (18) is given by

$$\rho_i = \frac{\lambda(a_i u - St)}{(2-t)u}, \quad 1 \leqslant i \leqslant n. \tag{20}$$

Note also that the leading i-th minor Δ_i of M is given by

$$\Delta_i = \det(M_i) = ((i-1)t + 2)(2-t)^{i-1}.$$

Therefore the leading minors of M are positive for all t in the given interval. By Theorem 7.2.5 of [6], we conclude that M is a positive definite matrix. But M is also the unconstrained Hessian of F (i.e., the Hessian of F on $\mathbb{R}^n$ without being restricted to the constraint H). Therefore F attains a strict local (and actually its absolute) minimum at the critical point $(\rho_1, \cdots, \rho_n)$ found earlier in (20). A comprehensive reference for constrained critical points is Hancock's classic [4]. More accessible accounts can be found in [10], [5], and [11].

Let μ be the value of F at its minimum point $(\rho_1, \cdots, \rho_n)$. Then we can compute μ in two ways. First, we put $x_i = \rho_i$ in our system (18), multiply the resulting equations by ρ_1, $\cdots$, ρ_n, and add. We obtain

$$2\mu = \sum_{1 \leqslant i \leqslant n} \rho_i^2 + t \sum_{1 \leqslant i < j \leqslant n} \rho_i \rho_j = \lambda(a_1\rho_1 + \cdots + a_n\rho_n) = \lambda c. \tag{21}$$

We can also use (20) to obtain

$$\mu = \sum_{1 \leqslant i \leqslant n} \rho_i^2 + t \sum_{1 \leqslant i < j \leqslant n} \rho_i \rho_j. \tag{22}$$

Note that $\mu = 0 \Longleftrightarrow \lambda = 0$, by (21). Also, if $\lambda = 0$, then $\rho_1 = \rho_2 = \cdots = \rho_n = 0$, by (20), and hence $c = a_1\rho_1 + \cdots + a_n\rho_n = 0$, a contradiction. Therefore, neither μ nor λ is 0, and we shall divide by them freely.

Using (22) and (20), we obtain

$$\begin{aligned}\frac{(2-t)^2 u^2 \mu}{\lambda^2} &= \sum_{1 \leqslant i \leqslant n} (a_i u - St)^2 + t \sum_{1 \leqslant i < j \leqslant n} (a_i u - St)^2 (a_j u - St)^2 \\ &= u^2 P - 2S^2 tu + nS^2 t^2 + t\left[\frac{n(n-1)}{2} S^2 t^2 - S^2 tu(n-1) + u^2 R\right],\end{aligned}$$

where

$$P = \sum_{1 \leqslant i \leqslant n} a_i^2, \qquad R = \sum_{1 \leqslant i < j \leqslant n} a_i a_j. \tag{23}$$

Therefore

$$\begin{aligned}
\frac{(2-t)^2u^2\mu}{\lambda^2} &= u^2P + S^2\left[nt^2 - 2ut + \frac{n(n-1)t^3}{2} - t^2u(n-1)\right] + tu^2R \\
&= u^2P + S^2\left[\left(nt^2 + \frac{n(n-1)t^3}{2}\right) - tu\left(2+t(n-1)\right)\right] + tu^2R \\
&= u^2P + S^2\left[u\frac{nt^2}{2} - u^2t\right] + tu^2R, \\
\frac{2(2-t)^2u\mu}{\lambda^2} &= 2uP + (P+2R)\left[nt^2 - 2ut\right] + 2tuR \text{ (because } S^2 = P+2R) \\
&= (2u + nt^2 - 2ut)P + 2(nt^2 - ut)R \\
&= ((2-t)u + t(nt-u))P + 2t(nt-u) \\
&= ((2-t)u - t(2-t))P - 2t(2-t)R \ \text{ (by (19))} \\
&= (2-t)\left(((n-2)t+2)P - 2tR\right) \\
&= (2-t)((n-2)t+2)(P+hR) \ \text{ (by (16))}, \\
\frac{c^2(2-t)u}{2((n-2)t+2)\mu} &= (P+hR) \quad \text{(by (21))}, \\
(P+hR)\mu &= \frac{c^2(2-t)u}{2((n-2)t+2)} = \frac{k^2(2-t)((n-1)t+2)}{2((n-2)t+2)} \ \text{ by (19)} \\
&= \frac{c^2(2-t)(2-h)}{4} \ \text{ (by (16))}, \\
&= c^2A.
\end{aligned}$$

This is nothing but the equality version of (17), and the proof of (11) is complete.

It remains to prove the last statement. Letting P, R, and F be defined as in (23) and (12), and letting $L = \sum_{i=1}^{n} a_i x_i$, we see that (6) states that

$$(P+kR)F \geqslant BL^2, \tag{24}$$

while (11) states that

$$(P+hR)F \geqslant AL^2. \tag{25}$$

Thus, to show that (25) implies (24), one needs to show that

$$(P+kR)A - (P+hR)B, \quad \text{i.e.,} \quad (A-B)P + (kA-hB)R$$

is non-negative. In view of Lemma 1, this is equivalent to showing that the three quantities

$$A-B, \quad 2(A-B)+(n-1)(kA-hB), \quad 2(A-B)-(kA-hB) \tag{26}$$

are non-negative. Using (3) and (8), we obtain

$$k = \frac{3n-1}{(n-1)t+n+1} - \frac{n+1}{n-1}, \qquad h = \frac{4}{(n-2)((n-2)t+2)} - \frac{2}{n-2}. \tag{27}$$

Substituting in the quantities in (26) the values of k and h obtained in (27) and the values of A and B given in (10) and (5), and using Maple to simplify the calculations, we obtain

$$kA-hB=2(A-B)=\frac{(-2+t)^2(n-3)n((n-1)t+2)}{2(n-1)((n-1)t+n+1)((n-2)t+2)}.$$

If $t \geqslant 0$, then the right hand side is trivially non-negative. If $t<0$, then the right hand side is again non-negative because

$$(n-2)t+2>(n-1)t+2, \qquad (n-1)t+n+1>(n-1)t+2,$$

and because of the assumption $(n-1)t+2 \geqslant 0$.

This proves that (11) implies (6). $\square$

REFERENCES

[1] M. HAJJA, *Radical and rational means of degree two*, Math. Inequal. Appl. **6** (2003), 581–593.
[2] M. HAJJA, *Review of* [8], Zentralblatt Math. Zbl 1260.51005, 2014.
[3] M. HAJJA, M. HAYAJNEH, *Impurity of the corner angles in certain special families of simplices*, J. Geom. **105** (2014), 539–560.
[4] H. HANCOCK, *Theory of Maxima and Minima*, Dover, N. Y., 1993.
[5] C. HASSELL AND E. REES, *The index of a constrained critical point*, Amer. Math. Monthly **100** (1993), 772–778.
[6] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1994.
[7] C. KIMBERLING, *Triangle centers and central triangles*, Congr. Numer. **129**, (1998), 1–285.
[8] C. KIMBERLING, P. MOSES, *Inequalities involving six numbers, with applications to triangle geometry*, Math. Mag. **85**, (2012), 221–227.
[9] C. KIMBERLING, *Encyclopedia of Triangle Centers*, `http://www.ETC`.
[10] J. A. SCOTT, *On the Lagrange method for constrained critical points*, Math. Gaz. **91** (2009), 81–85.
[11] P. SHUTLER, *Constrained critical points*, Amer. Math. Monthly **102** (1995), 49–52.

(Received June 27, 2014)

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