Mathematical
Inequalities
& Applications
Volume 18, Number 3 (2015), 901–913 doi:10.7153/mia-18-67

TWO TRIGONOMETRIC INTEGRAL INEQUALITIES

YI ZHAO

(*Communicated by L. Leindler*)

Abstract. The present paper establishes some important trigonometric integral inequalities related to functions of mean value bounded variation in real sense.

1. Introduction

One important tool in Fourier analysis is the following well-known trigonometric inequality (see, e.g., [8])

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| = O(1), \tag{1}$$

where and in the sequel, $O(1)$ always indicates a bound independent of n.

To generalize this inequality, a number of authors (see, for example, the references in [2], [3], [5] and [4]) discussed various conditions to relax the monotonicity. The following class of sequences of real numbers was defined in [1].

DEFINITION 1. A real sequence $A = \{a_n\}$ is said to satisfy the mean value bounded variation condition (in real sense) if there is a $\lambda \geqslant 2$ and a positive constant M depending upon the sequence A and λ only such for all $n = 1, 2, \cdots$ that

$$\sum_{k=n}^{2n}|\Delta a_k| := \sum_{k=n}^{2n}|a_k - a_{k+1}| \leqslant \frac{M}{n}\sum_{k=n/\lambda}^{\lambda n}|a_k|, \tag{2}$$

where $\sum\limits_{k=n/\lambda}^{\lambda n}$ means $\sum\limits_{n/\lambda\leqslant k\leqslant \lambda n}$. We denote the set of real sequences satisfying (2) as MVBVS (Mean Value Bounded Variation Sequences).

Recently, Feng and Zhou proved the following Theorem in [2].

Mathematics subject classification (2010): 42A05, 42A38.
Keywords and phrases: Sine integral, trigonometric integral inequality, mean value bounded variation.

Paper MIA-18-67

THEOREM 1. *Let a real sequence* $\{a_n\} \in \mathrm{MVBVS}$*. Then, for all* n *and* $x \in [0,\pi]$*,*

$$\sum_{k=1}^{n} a_k \sin kx = O(1)$$

holds if and only if

$$na_n = O(1).$$

In Fourier analysis, positivity and monotonicity are two most important prior factors setting on coefficients of a trigonometric (Fourier) series to guarantee various properties. Theorem 1 is a very general extension of the precedent results, especially of (1). Indeed, the mean value bounded variation concept in real sense is not only a successful ultimate generalization in monotonicity (it was shown in [9] that the MVBV condition generalizes decreasing monotonicity, and cannot be further weakened), but also an efficient replacement of positivity.

Also in [2], Feng and Zhou established the following weighted trigonometric inequality.

THEOREM 2. *Let a real sequence* $\{a_n\} \in \mathrm{MVBVS}$*,* $0 < \gamma < 1$*, then for any natural number* n *and* $x \in [0,\pi]$*,*

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(x^{-\gamma})$$

holds if and only if

$$n^{1-\gamma} a_n = O(1).$$

Motivated by the definition of MVBVS, Zhao, Feng and Zhou [6] recently defined the mean value bounded variation (in real sense) for functions:

DEFINITION 2. A Lebesgue measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be of mean value bounded variation in real sense, in symbols $f \in \mathrm{MVBVF}$, if it is of locally bounded variation and if there exist $\lambda \geqslant 2$, $A > 1$ and a constant $M_0 > 0$ depending only upon f such for all $a > A$ that

$$\int_a^{2a} |\mathrm{d}f| \leqslant \frac{M_0}{a} \int_{a/\lambda}^{\lambda a} |f(x)|\mathrm{d}x \tag{3}$$

holds, where $\mathbb{R}_+ := (0,\infty), \mathbb{R} := (-\infty,\infty)$.

Without loss of generality we may assume $M_0 > 1$ in (3).

This paper shall establish trigonometric integral inequalities corresponding to Theorem 1 and Theorem 2. Exactly, we prove that

THEOREM 3. *Assume a real function* $f(x) \in \mathrm{MVBVF}$*, if* $\int_a^{a+1} |f(x)|\mathrm{d}x$ *is uniformly bounded for arbitrary* $a > A$*, then for any* $a > A$*,* $t \in [0,\infty)$*,*

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = O(1)$$

holds if and only if

$$x|f(x)| = O(1),$$

where $A > 1$ *is the constant appearing in Definition 2.*

THEOREM 4. *Let* $0 < \gamma < 1$. *Assume a real function* $f(x) \in \mathrm{MVBVF}$. *If* $\int_a^{a+1} x^{\gamma}|f(x)|\mathrm{d}x$ *is uniformly bounded for arbitrary* $a > A$*, then for any* $t \in (0,\infty)$*,*

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = O(t^{-\gamma})$$

holds if and only if

$$x^{1-\gamma}|f(x)| = O(1).$$

This inequality also holds for cosine integrals.

THEOREM 5. *Let* $0 < \gamma < 1$. *Assume a real function* $f(x) \in \mathrm{MVBVF}$. *If* $\int_a^{a+1} x^{\gamma}|f(x)|\mathrm{d}x$ *is uniformly bounded for arbitrary* $a > A$*, then for any* $t \in (0,\infty)$*,*

$$\left|\int_a^{\infty} f(x)\cos tx\mathrm{d}x\right| = O(t^{-\gamma})$$

holds if and only if

$$x^{1-\gamma}|f(x)| = O(1).$$

Throughout the paper, we always use M_0 to stand for the positive constant appearing in (3), and M denotes a positive constant that may not be necessarily the same at each occurrence.

2. Preliminaries

We need several lemmas.

LEMMA 1. *Assume a real function* $f(x) \in \mathrm{MVBVF}$. *If*

$$x^{1-\gamma}f(x) = O(1), \quad x \in (0,\infty), \quad 0 < \gamma < 1,$$

then for any real number t *satisfying* $1/t > A$*, we have*

$$\frac{1}{t}\int_{1/t}^{\infty} |df| = O(t^{-\gamma}).$$

Proof. By condition (3), we have

$$\frac{1}{t}\int_{1/t}^{\infty}|\mathrm{d}f| \leqslant \frac{1}{t}\sum_{j=0}^{\infty}\int_{2^j/t}^{2^{j+1}/t}|\mathrm{d}f| \leqslant \sum_{j=0}^{\infty}\frac{M_0}{2^j}\int_{2^j/(t\lambda)}^{2^j\lambda/t}|f(x)|\mathrm{d}x$$
$$\leqslant \sum_{j=0}^{\infty}\frac{M}{2^j}\int_{2^j/(t\lambda)}^{2^j\lambda/t}x^{\gamma-1}\mathrm{d}x$$
$$\leqslant \frac{M}{t^\gamma}\sum_{j=0}^{\infty}\frac{1}{2^{(1-\gamma)j}} \leqslant Mt^{-\gamma}. \quad \square$$

LEMMA 2. *Assume a real function* $f(x) \in \mathrm{MVBVF}$. *If*

$$x^{1-\gamma}f(x) = O(1), \quad x \in (0,\infty), \quad 0<\gamma<1, \tag{4}$$

then, for any $a > A$ *and* $t \in [0,\infty)$, *we have*

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = O(t^{-\gamma}).$$

Proof. The case $t = 0$ is trivial. Let $t \in (0,\infty)$,
Case 1. $1/t > a$.

$$\int_a^{\infty} f(x)\sin tx\mathrm{d}x = \int_a^{1/t} f(x)\sin tx\mathrm{d}x + \int_{1/t}^{\infty} f(x)\sin tx\mathrm{d}x =: I_1 + I_2.$$

By (4), we have

$$|I_1| \leqslant t\int_a^{1/t}|f(x)|x^{1-\gamma}x^{\gamma}\mathrm{d}x = O(t^{-\gamma}).$$

Integration by parts with (4) gives

$$|I_2| \leqslant O(t^{-\gamma}) + \frac{1}{t}\int_{1/t}^{\infty}|\mathrm{d}f|,$$

by Lemma 1, we have

$$|I_2| \leqslant Mt^{-\gamma}.$$

Case 2. $1/t \leqslant a$.
In this case, integration by parts gives

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = \lim_{b\to\infty}\left|\int_a^{b} f(x)\sin tx\mathrm{d}x\right| \leqslant \lim_{b\to\infty}\frac{1}{t}\left(|f(a)|+|f(b)|+\int_a^b|\mathrm{d}f|\right).$$

It is easy to check that

$$\lim_{b\to\infty}\frac{1}{t}\left(|f(a)|+|f(b)|\right) = O(t^{-\gamma})$$

by utilizing condition (4). Now since $f(x) \in \mathrm{MVBVF}$, by (3) and (4), we deduce that

$$\frac{1}{t}\int_a^{\infty}|\mathrm{d}f| \leqslant \frac{1}{t}\sum_{j=0}^{\infty}\int_{2^j a}^{2^{j+1}a}|\mathrm{d}f| \leqslant \frac{1}{t}\sum_{j=0}^{\infty}\frac{M_0}{2^j a}\int_{2^j a/\lambda}^{2^j a\lambda}|f(x)|\mathrm{d}x$$
$$\leqslant \frac{1}{t}\sum_{j=0}^{\infty}\frac{M}{2^j a}\int_{2^j a/\lambda}^{2^j a\lambda}x^{\gamma-1}\mathrm{d}x \leqslant \frac{M}{t}\frac{1}{a^{1-\gamma}}\sum_{j=0}^{\infty}\frac{1}{2^{(1-\gamma)j}}$$
$$\leqslant \frac{M}{t^{\gamma}}\sum_{j=0}^{\infty}\frac{1}{2^{(1-\gamma)j}} \leqslant Mt^{-\gamma}.$$

Lemma 2 is proved. □

LEMMA 3. *Assume a real function* $f(x) \in \mathrm{MVBVF}$*, then for any* $a > A$*,*

$$|f(a)| \leqslant \frac{2M_0}{a}\int_{a/\lambda}^{\lambda a}|f(x)|\mathrm{d}x.$$

Proof. Suppose to the contrary that for some $a > A$ we have

$$|f(a)| > \frac{2M_0}{a}\int_{a/\lambda}^{\lambda a}|f(x)|\mathrm{d}x =: \frac{2M_0}{a}I_a,$$

then for all $y \in [a, 2a]$,

$$|f(a)| - |f(y)| \leqslant \int_a^y |\mathrm{d}f| \leqslant \int_a^{2a}|\mathrm{d}f| \leqslant \frac{M_0}{a}I_a.$$

Therefore

$$|f(y)| \geqslant |f(a)| - \frac{M_0}{a}I_a \geqslant \frac{M_0}{a}I_a. \tag{5}$$

Integrating both sides of (5) on $[a, 2a]$ gives

$$\int_a^{2a}|f(y)|\mathrm{d}y \geqslant M_0 I_a \geqslant M_0\int_a^{2a}|f(y)|\mathrm{d}y,$$

which is impossible since $M_0 > 1$. □

LEMMA 4. *Assume* $f(x)$ *satisfies* (3)*, then for any* $a > A$ *and* $x \in [a/\lambda, \lambda a]$ *we have*

$$|f(x)| \leqslant \frac{M}{a}\int_{a/\lambda^2}^{\lambda^2 a}|f(t)|\mathrm{d}t.$$

Proof. According to Lemma 3, for $x \in [a/\lambda, \lambda a]$,

$$|f(x)| \leqslant \frac{2M_0}{x}\int_{x/\lambda}^{\lambda x}|f(t)|\mathrm{d}t \leqslant \frac{M}{a}\int_{a/\lambda^2}^{\lambda^2 a}|f(t)|\mathrm{d}t,$$

thus the conclusion follows. □

3. Trigonometric integral inequality

Proof of Theorem 3.
Sufficiency. Similar to the proof of Lemma 2, we first assume $1/t > a$, then

$$\begin{aligned}\left|\int_a^\infty f(x)\sin tx\mathrm{d}x\right| &\leqslant t\int_a^{1/t}|xf(x)|\,\mathrm{d}x+\frac{1}{t}\int_{1/t}^\infty|\mathrm{d}f|\\ &\leqslant M+\frac{1}{t}\sum_{j=0}^\infty\int_{2^j/t}^{2^{j+1}/t}|\mathrm{d}f|\\ &\leqslant M+\sum_{j=0}^\infty\frac{M_0\lambda}{2^j}\int_{2^j/(t\lambda)}^{2^j\lambda/t}|f(x)|\mathrm{d}x\\ &\leqslant M+\sum_{j=0}^\infty\frac{M\lambda}{2^j}\int_{2^j/(t\lambda)}^{2^j\lambda/t}\frac{1}{x}\mathrm{d}x\\ &\leqslant M.\end{aligned}$$

In case $1/t \leqslant a$, integration by parts gives

$$\left|\int_a^\infty f(x)\sin tx\mathrm{d}x\right| = \lim_{b\to\infty}\left|\int_a^b f(x)\sin tx\mathrm{d}x\right| \leqslant \lim_{b\to\infty}\frac{1}{t}\left(|f(a)|+|f(b)|+\int_a^b|\mathrm{d}f|\right).$$

Now we have

$$\lim_{b\to\infty}\frac{1}{t}\left(|f(a)|+|f(b)|\right)=O(1)$$

by the condition $x|f(x)|=O(1)$. Since $f(x)\in$ MVBVF, by (3) and $x|f(x)|=O(1)$, we deduce that

$$\begin{aligned}\frac{1}{t}\int_a^\infty|\mathrm{d}f| &\leqslant \frac{1}{t}\sum_{j=0}^\infty\int_{2^ja}^{2^{j+1}a}|\mathrm{d}f| \leqslant \frac{1}{t}\sum_{j=0}^\infty\frac{M_0}{2^ja}\int_{2^ja/\lambda}^{2^ja\lambda}|f(x)|\mathrm{d}x\\ &\leqslant \frac{1}{t}\sum_{j=0}^\infty\frac{M}{2^ja}\int_{2^ja/\lambda}^{2^ja\lambda}x^{-1}\mathrm{d}x \leqslant \frac{M}{t}\frac{1}{a}\sum_{j=0}^\infty\frac{1}{2^j}\leqslant M.\end{aligned}$$

Necessity. With the condition $\int_a^{a+1}|f(x)|\mathrm{d}x=O(1)$ for arbitrary $a>A$, suppose that

$$\int_a^{a+1}|f(x)|\mathrm{d}x\leqslant M$$

and therefore

$$I_a=\int_{a/\lambda}^{\lambda a}|f(x)|\mathrm{d}x\leqslant Ma. \tag{6}$$

Moreover, since $\left|\int_a^\infty f(x)\sin tx\mathrm{d}x\right|=O(1)$, for any $b>a$ and all $t\in[0,\infty)$, we have

$$\left|\int_a^b f(x)\sin tx\mathrm{d}x\right|<M. \tag{7}$$

Now define a set E_a for any $a/\lambda > A$ as follows:

$$E_a = \left\{ x \in \left[\frac{a}{\lambda}, \lambda a\right] : |f(x)| \geqslant \frac{1}{2\lambda a} I_a \right\}.$$

First we need to estimate the measure of E_a, denoted by $m(E_a)$. Applying Lemma 4, we see that

$$\begin{aligned} I_a &\leqslant \int_{E_a} |f(x)|\mathrm{d}x + \int_{[a/\lambda,\lambda a]\setminus E_a} |f(x)|\mathrm{d}x \\ &\leqslant m(E_a)\frac{M}{a}\int_{a/\lambda^2}^{\lambda^2 a} |f(x)|\mathrm{d}x + \left(a\left(\lambda - \frac{1}{\lambda}\right) - m(E_a)\right)\frac{1}{2\lambda a} I_a \\ &\leqslant \frac{m(E_a)}{a} M \int_{a/\lambda^2}^{\lambda^2 a} |f(x)|\mathrm{d}x + \frac{1}{2} I_a. \end{aligned}$$

Write $I_a^* := \int_{a/\lambda^2}^{\lambda^2 a} |f(t)|\mathrm{d}t$, we have

$$m(E_a) \geqslant \frac{1}{2}\frac{a}{M}\frac{I_a}{I_a^*}. \tag{8}$$

Also it is easy to see that the set E_a is not empty since $m(E_a) > 0$.

We select disjoint subintervals (except for endpoints) $S_1, \ldots, S_{\kappa_a}$ of $[a/\lambda, \lambda a]$ as follows. Set $m_1 = \inf E_a$, and select ν_1 according to the following procedure:

(i) Select

$$\nu_1 = \inf\{x > m_1 : f(m_1+0)f(x) \geqslant 0,\ \ |f(x)| < I_a/(4\lambda a),\ \ x \in [a/\lambda, \lambda a]\},$$

where $f(x+0) = \lim_{t \to x+0} f(t)$.

(ii) If case (i) is not satisfied, then let

$$\nu_1 = \inf\{x > m_1 : f(m_1+0)f(x) < 0,\ \ x \in [a/\lambda, \lambda a]\}.$$

(iii) If neither (i) nor (ii) happens, then simply let $\nu_1 = \lambda a$. Define

$$S_1 = [m_1, \nu_1].$$

Next, set $m_2 = \inf(E_a \setminus S_1)$ if this latter set is not empty, and using the same procedure we select ν_2 and define

$$S_2 = [m_2, \nu_2].$$

We continue this procedure until we reach an S_{κ_a} for which $E_a \setminus (S_1 \cup \cdots \cup S_{\kappa_a}) = \emptyset$.

First we give an estimate for κ_a, i.e. for the number of these S_j's. Note first of all that for all $1 \leqslant j < \kappa_a$ we have

$$\int_{m_j}^{\nu_j} |\mathrm{d}f| = \bigvee_{m_j}^{\nu_j}(f) \geqslant \frac{I_a}{4\lambda a}$$

by the choice of the ν_j's (but for $j=\kappa_a$ this property may not be true). We check that (3) implies

$$\int_{a/\lambda}^{\lambda a}|\mathrm{d}f| \leqslant \frac{M\lambda^3}{a}\int_{a/\lambda^2}^{\lambda^2 a}|f(x)|\mathrm{d}x = \frac{M\lambda^3}{a}I_a^*,$$

from which

$$\frac{M\lambda^3}{a}I_a^* \geqslant \int_{a/\lambda}^{\lambda a}|\mathrm{d}f| \geqslant \sum_{j=1}^{\kappa_a-1}\int_{m_j}^{\nu_j}|\mathrm{d}f| \geqslant \sum_{j=1}^{\kappa_a-1}\frac{I_a}{4\lambda a} = (\kappa_a-1)\frac{I_a}{4\lambda a},$$

i.e.,

$$\kappa_a \leqslant 4M\lambda^4\frac{I_a^*}{I_a}+1 \leqslant 5M\lambda^4\frac{I_a^*}{I_a}, \tag{9}$$

this also implies that κ_a is finite.

Note now that all $x\in(m_j,\nu_j)$ (S_j excluding m_j and ν_j) are of the same sign, hence upon substituting $t=\pi/(2a\lambda)$ and for $a/\lambda \leqslant x \leqslant \lambda a$ we have

$$\sin\frac{x\pi}{2a\lambda} \geqslant \frac{2}{\pi}\frac{x\pi}{2a\lambda} \geqslant \frac{1}{\lambda^2},$$

thus by (7),

$$\frac{1}{\lambda^2}\int_{m_j}^{\nu_j}|f(x)|\mathrm{d}x \leqslant \left|\int_{m_j}^{\nu_j}f(x)\sin tx\mathrm{d}x\right| < M$$

provided $a/\lambda > A$. On summing up for all $1\leqslant j\leqslant \kappa_a$ and using (9) we get

$$\int_{E_a}|f(x)|\mathrm{d}x \leqslant \sum_{j=1}^{\kappa_a}\int_{m_j}^{\nu_j}|f(x)|\mathrm{d}x < 5M\lambda^6\frac{I_a^*}{I_a}.$$

From here, in view of the definition of E_a and (8), we deduce that

$$\frac{I_a^3}{(I_a^*)^2} \leqslant M. \tag{10}$$

Let $\Lambda_m := I_{\lambda^m}$, then

$$\Lambda_m^3 \leqslant M(\Lambda_{m-1}+\Lambda_{m+1})^2, \tag{11}$$

by taking the $\limsup$ on both sides of (11), we have

$$\limsup_{m\to\infty}\Lambda_m^3 \leqslant M\limsup_{m\to\infty}\Lambda_m^2.$$

Then either $\Lambda_m = O(1)$ or $\limsup\limits_{m\to\infty}\Lambda_m = \infty$.

If $\limsup\limits_{m\to\infty}\Lambda_m=\infty$, we discuss the following two cases.

(I) $\{\Lambda_j\}$ has the only accumulation point ∞. Then $\lim\limits_{j\to\infty}\Lambda_j=\infty$. We show that in this case, there exists a subsequence $\{j_k\}$ such that

$$\frac{\Lambda_{j_k-1}+\Lambda_{j_k+1}}{\Lambda_{j_k}} \leqslant M.$$

Suppose to the contrary,

$$\lim_{j\to\infty}\frac{\Lambda_{j-1}+\Lambda_{j+1}}{\Lambda_j}=\infty.$$

Hence there exists a $J\in\mathbb{N}$ such for all $j>J$ that

$$\frac{\Lambda_{j-1}+\Lambda_{j+1}}{\Lambda_j}>2\lambda^4.$$

Since $\lim\limits_{j\to\infty}\Lambda_j=\infty$, we may assume that for some $j_0>J$, it holds that $\Lambda_{j_0+1}>\Lambda_{j_0}$. Now denoting the two roots of the equation $x^2-2\lambda^4x+1=0$ as x_1 and x_2, where $0<x_1<\frac{1}{2}<\lambda^4<x_2$, by Vieta's formulas, we have

$$\Lambda_{j+1}-x_1\Lambda_j\geqslant x_2\left(\Lambda_j-x_1\Lambda_{j-1}\right)\geqslant 0,\quad j>J,$$

in particular,

$$\Lambda_{j_0+1}-x_1\Lambda_{j_0}\geqslant x_2\left(\Lambda_{j_0}-x_1\Lambda_{j_0-1}\right)\geqslant 0,$$

therefore,

$$\Lambda_{j+1}-x_1\Lambda_j\geqslant x_2^{j-j_0+1}\left(\Lambda_{j_0}-x_1\Lambda_{j_0-1}\right),\quad j>j_0>J,$$

hence for $j>j_0$,

$$\Lambda_{j+1}>Mx_2^{j-j_0+1}\left(\Lambda_{j_0}-x_1\Lambda_{j_0-1}\right)>M\lambda^{4j-4j_0+4}.$$

However, it makes a contradiction since by (6) that

$$\Lambda_{j+1}\leqslant M\lambda^{j+1}.$$

Altogether, we show that when $\{\Lambda_j\}$ has the only accumulation point ∞, (10) yields $\Lambda_{j_k}\leqslant M$, which makes a contradiction to the assumption

$$\lim_{j\to\infty}\Lambda_j=\infty.$$

(II) $\{\Lambda_j\}$ has at least one finite accumulation point, then, there exists a number L and a subsequence of natural numbers $\{\tilde{j}_k^{(1)}\}$ satisfying

$$\lim_{k\to\infty}\Lambda_{\tilde{j}_k^{(1)}}=L,$$

hence $\{\Lambda_{\tilde{j}_k^{(1)}}\}$ has an upper bound S. Furthermore, select a subsequence $\{\tilde{j}_k^{(2)}\}$ such for all $k\in\mathbb{N}$ that $\Lambda_{\tilde{j}_k^{(2)}}>S$, and

$$\lim_{k\to\infty}\Lambda_{\tilde{j}_k^{(2)}}=\infty.$$

Set $j_1^{(1)}=\tilde{j}_k^{(1)}$, take

$$j_i^{(2)}=\min\{\tilde{j}_k^{(2)}>j_i^{(1)}:k\in\mathbb{N}\},\quad i=1,2,\cdots,$$

$$j_{i+1}^{(1)} = \min\{\tilde{j}_k^{(1)} > j_i^{(2)} : k \in \mathbb{N}\}, \quad i = 1, 2, \cdots,$$

and define $\tilde{j}_{2k-1} = j_k^{(1)}$, while define $\tilde{j}_{2k}$ as the number satisfying

$$\Lambda_{\tilde{j}_{2k}} = \max_{\tilde{j}_{2k-1} < j < \tilde{j}_{2k+1}} \Lambda_j.$$

It is clear that $\Lambda_{\tilde{j}_{2k}} \geqslant \Lambda_{j_k^{(2)}}$. Furthermore, $\lim_{k\to\infty} \Lambda_{\tilde{j}_{2k}} = \infty$ since $\lim_{k\to\infty} \Lambda_{j_k^{(2)}} = \infty$. Setting $j_k = \tilde{j}_{2k} - 1$, we obtain

$$\lim_{k\to\infty} \Lambda_{j_k+1} = \lim_{k\to\infty} \Lambda_{\tilde{j}_{2k}} = \infty. \tag{12}$$

At the same time, by noting $\tilde{j}_{2k-1} \leqslant j_k < j_k + 2 \leqslant \tilde{j}_{2k+1}$ with $\Lambda_{\tilde{j}_k^{(2)}} > S \geqslant \Lambda_{\tilde{j}_k^{(1)}}$, we deduce that

$$\int_{\lambda^{j_k-1}}^{\lambda^{j_k+3}} |f(x)|\mathrm{d}x = \Lambda_{j_k} + \Lambda_{j_k+2} \leqslant 2\Lambda_{\tilde{j}_{2k}} = 2\Lambda_{j_k+1} = 2\int_{\lambda^{j_k}}^{\lambda^{j_k+2}} |f(x)|\mathrm{d}x,$$

in other words,

$$I^*_{\lambda^{j_k+1}} \leqslant 2I_{\lambda^{j_k+1}},$$

while (10) shows that

$$\Lambda_{j_k+1} = I_{\lambda^{j_k+1}} \leqslant 4M,$$

which is contradict with (12).

Combining (I) and (II), we already show that $\Lambda_j = O(1)$. Then by Lemma 4 we derive

$$x|f(x)| \leqslant MI_a^* \leqslant M\sum_{k=-1}^{2} \Lambda_{[\log a/\log\lambda]+k} \leqslant M,$$

and the conclusion follows. □

We note that, the sufficiency of Theorem 3 does not require the condition that $\int_a^{a+1} |f(x)|\mathrm{d}x$ is uniformly bounded for arbitrary $a > A$. At the same time, if $f(x)$ keeps sign, and

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = O(1),$$

then by taking $t_0 = \pi/(2a)$, we see that

$$\int_a^{a+1} |f(x)|\mathrm{d}x = \int_a^{a+1} f(x)\mathrm{d}x \leqslant M\left|\int_a^{a+1} f(x)\sin t_0x\mathrm{d}x\right| = O(1).$$

Therefore, we have the following corollary.

COROLLARY 1. *Assume a nonnegative function* $f(x) \in$ MVBVF, *then for any* $a > A$, $t \in [0, \infty)$,

$$\left|\int_a^{\infty} f(x)\sin tx\mathrm{d}x\right| = O(1)$$

holds if and only if

$$x|f(x)| = O(1).$$

4. Weighted trigonometric integral inequality

Proof of Theorem 4. By Lemma 2, we have already proved the sufficiency. For necessity, with the condition $\int_a^{a+1} x^\gamma |f(x)|\mathrm{d}x = O(1)$ for arbitrary $a > A$, we have $\int_a^{a+1} x^\gamma |f(x)|\mathrm{d}x \leqslant M$. Therefore,

$$\int_{a/\lambda}^{a\lambda} |f(x)|\mathrm{d}x \leqslant Ma^{1-\gamma}.$$

For any $b > a$ and all $t \in (0,\infty)$, we have

$$t^\gamma \left| \int_a^b f(x)\sin tx\mathrm{d}x \right| < M. \tag{13}$$

By using the same symbols as in Theorem 3, we have the following estimates:

$$m(E_a) \geqslant \frac{a}{M}\frac{I_a}{I_a^*}, \tag{14}$$

$$\kappa_a \leqslant M\frac{I_a^*}{I_a}. \tag{15}$$

Hence, by (15) and (13) with $t = \pi/(2\lambda a)$, we have

$$\int_{E_a} |f(x)|\mathrm{d}x \leqslant \sum_{j=1}^{\kappa_a} \int_{m_j}^{\nu_j} |f(x)|\mathrm{d}x < M\frac{I_a^*}{I_a}a^\gamma,$$

from which and combining (14) we derive that

$$I_a^3 \leqslant Ma^\gamma (I_a^*)^2. \tag{16}$$

Let $\Lambda_j = I_{\lambda^j}$ and take the $\limsup$ on both sides of (16), we have

$$\limsup_{j\to\infty} \Lambda_j^3 \leqslant M \lim_{j\to\infty} \lambda^{\gamma j} \limsup_{j\to\infty} \Lambda_j^2,$$

then either $\Lambda_j = O(1)$ or $\limsup_{j\to\infty} \Lambda_j = \infty$.

We proceed an analogue to Theorem 3 to achieve that $\Lambda_j = O(1)$. Therefore, since by Lemma 4,

$$|f(x)| \leqslant \frac{Ma^\gamma}{x} I_a^* \leqslant \frac{M}{x^{1-\gamma}} I_a^*, \quad x > a > A, \quad 0 < \gamma < 1,$$

we have

$$x^{1-\gamma}|f(x)| \leqslant MI_a^* \leqslant M \sum_{k=-1}^{2} \Lambda_{[\log a/\log \lambda]+k} \leqslant M,$$

which concludes the proof. □

COROLLARY 2. *Let $0<\gamma<1$. Assume a nonnegative function $f(x)\in$ MVBVF, then for any $t\in(0,\infty)$,*

$$\left|\int_a^\infty f(x)\sin tx\mathrm{d}x\right| = O(t^{-\gamma})$$

holds if and only if

$$x^{1-\gamma}|f(x)| = O(1).$$

5. Remark

One could note in the proof of the inequalities that, if let $a=0$, then the case $1/t<a$ in Lemma 2 and Theorem 3 could not happen. Therefore, by revising Definition 2 and copying the other parts of the proof, we have the another version of the inequalities.

DEFINITION 3. A Lebesgue measurable function $f:\mathbb{R}_+\to\mathbb{R}$ is said to be of mean value bounded variation, in symbols $f\in$ MVBVF*, if it is of locally bounded variation and if there exist $\lambda\geqslant 2$ and a constant $M_0>0$ depending only upon f such for all $a>0$ that

$$\int_a^{2a}|\mathrm{d}f| \leqslant \frac{M_0}{a}\int_{a/\lambda}^{\lambda a}|f(x)|\mathrm{d}x \tag{17}$$

holds.

THEOREM 6. *Assume a real function $f(x)\in$ MVBVF*, if $\int_a^{a+1}|f(x)|\mathrm{d}x$ is uniformly bounded for arbitrary $a>0$, then for $t\in[0,\infty)$,*

$$\left|\int_0^\infty f(x)\sin tx\mathrm{d}x\right| = O(1)$$

holds if and only if

$$x|f(x)| = O(1).$$

This includes the typical classical result that

$$\int_0^\infty \frac{\sin tx}{x}\mathrm{d}x = O(1)$$

for $t\in[0,\infty)$.

THEOREM 7. *Let $0<\gamma<1$. Assume a real function $f(x)\in$ MVBVF*. If $\int_a^{a+1}x^\gamma|f(x)|\mathrm{d}x$ is uniformly bounded for arbitrary $a>0$, then for any $t\in(0,\infty)$,*

$$\left|\int_0^\infty f(x)\sin tx\mathrm{d}x\right| = O(t^{-\gamma})$$

holds if and only if

$$x^{1-\gamma}|f(x)| = O(1).$$

This also holds for cosine integrals.

REFERENCES

[1] L. FENG, V. TOTIK AND S. P. ZHOU, *Trigonometric series with a generalized monotonicity condition*, Acta. Math. Sinica, English Ser., **30**, 8 (2014), 1289–1296.

[2] L. FENG AND S. P. ZHOU, *Trigonometric inequalities in the MVBV condition*, Math. Ineq. Appl., accepted.

[3] L. LEINDLER, *On the uniform convergence and boundedness of a certain class of sine series*, Anal. Math., **27**, (2001), 279–285.

[4] S. A. TELYAKOVSKII, *On partial sums of Fourier series of functions of bounded variation*, Proc. Steklov. Inst. Math., **219**, (1997), 372–381.

[5] M. Z. WANG AND Y. ZHAO, *Generalization of some classical results under MVBV condition*, Math. Ineq. Appl., **12**, (2009), 433–440.

[6] Y. ZHAO, L. FENG AND S. P. ZHOU, *The uniform convergence of sine integrals*, to appear.

[7] S. P. ZHOU, *An important application of mean value bounded variation concept in weighted integrability*, to appear.

[8] S. P. ZHOU, *Monotonicity Condition of Trigonometric Series: Development and Application*, Science Press, Beijing, 2012, in Chinese.

[9] S. P. ZHOU, P. ZHOU AND D. S. YU, *Ultimate generalization to monotonicity for uniform convergence of trigonometric series*, Science China Math., **53**, (2010), 1853–1862, available: arXiv: math.CA/0611805 v127 Nov 2006.

(Received July 11, 2014)

Yi Zhao
School of Science
Hangzhou Dianzi University
Hangzhou 310018 China
e-mail: zhaoyi@hdu.edu.cn