Mathematical
Inequalities
& Applications
Volume 18, Number 3 (2015), 915–922 doi:10.7153/mia-18-68

NUMERICAL RADIUS INEQUALITIES ASSOCIATED WITH THE CARTESIAN DECOMPOSITION

FUAD KITTANEH

(*Communicated by J.-C. Bourin*)

Abstract. We give several sharp numerical radius inequalities associated with the Cartesian decomposition of a Hilbert space operator $A = B + iC$. Among other inequalities, it is shown that

$$\frac{1}{2}\| \, |B|^r + |C|^r \| \leqslant w^r(A) \leqslant \| \, |B|^r + |C|^r \|$$

for $0 < r \leqslant 2$, where $w(\cdot)$ and $\|\cdot\|$ denote the numerical radius and the usual operator norm, respectively. These inequalities generalize and extend earlier numerical radius inequalities.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$. For $A \in \mathbb{B}(\mathbb{H})$, let $w(A)$ and $||A||$ denote the numerical radius norm and the usual operator norm of A, respectively. It is well known that these two norms are equivalent. In fact, for every $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{1}{2}||A|| \leqslant w(A) \leqslant ||A||. \tag{1.1}$$

The inequalities in (1.1) are sharp. If $A^2 = 0$, then the first inequality in (1.1) becomes an equality. If A is normal, then the second inequality in (1.1) becomes an equality.

Several refinements of the inequalities in (1.1) have been recently given (see, e.g., [1]–[4], [7], [8], and references therein).

It has been shown in [7] that for every $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{1}{4}||A^*A + AA^*|| \leqslant w^2(A) \leqslant \frac{1}{2}||A^*A + AA^*||. \tag{1.2}$$

Applications of the inequalities in (1.2) to the theory of power bounded operators have been given in [5].

If $A = B + iC$ is the Cartesian decomposition of A, then B and C are self-adjoint, and $A^*A + AA^* = 2\left(B^2 + C^2\right)$. Thus, the inequalities in (1.2) can be written as

$$\frac{1}{2}\left|\left|B^2 + C^2\right|\right| \leqslant w^2(A) \leqslant \left|\left|B^2 + C^2\right|\right|, \tag{1.3}$$

Mathematics subject classification (2010): 47A12, 47A30, 47A63, 47B15.
Keywords and phrases: Numerical radius, usual operator norm, Cartesian decomposition, convex function, concave function, triangle inequality.

or equivalently, as

$$\frac{1}{4}\left\|(B+C)^2+(B-C)^2\right\| \leqslant w^2(A) \leqslant \frac{1}{2}\left\|(B+C)^2+(B-C)^2\right\|. \qquad (1.4)$$

The second inequality in (1.3) and the inequalities in (1.4) have been generalized in [4] as follows.

THEOREM 1.1. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$. Then*

$$w^r(A) \leqslant \||B|^r + |C|^r\| \qquad (1.5)$$

for $0 < r \leqslant 2$, and

$$w^r(A) \leqslant 2^{r/2-1}\||B|^r + |C|^r\| \qquad (1.6)$$

for $r \geqslant 2$. Here $|X| = (X^*X)^{1/2}$ is the absolute value of X.

THEOREM 1.2. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$. Then*

$$2^{r/2-1}\|\,|B+C|^r + |B-C|^r\| \leqslant w^r(A) \leqslant \frac{1}{2}\|\,|B+C|^r + |B-C|^r\| \qquad (1.7)$$

for $r \geqslant 2$.

Note that Theorem 1.1 is a generalization of the second inequality in (1.3), and Theorem 1.2 is a generalization of the inequalities in (1.4), which deals only with the case $r \geqslant 2$.

The purpose of this paper is to give a generalization of the first inequality in (1.3), and to establish inequalities related to those in (1.7) for the values $0 < r \leqslant 2$. We will formulate our new results in a more general setting involving certain convex and concave functions, from which our desired generalizations follow as special cases.

2. Main results

To achieve our goal, we need the following lemma involving norm inequalities for convex and concave functions of positive operators. A more general form of this lemma can be found in [6] and references therein. Henceforth,we assume that every function is continuous.

LEMMA 2.1. *Let $X, Y \in \mathbb{B}(\mathbb{H})$ be positive. Then*

(a) $\left\|f\left(\frac{X+Y}{2}\right)\right\| \leqslant \frac{1}{2}\|f(X)+f(Y)\|$ for every non-negative convex function f on $[0,\infty)$.

(b) $\frac{1}{2}\|f(X)+f(Y)\| \leqslant \left\|f\left(\frac{X+Y}{2}\right)\right\|$ for every non-negative concave function f on $[0,\infty)$.

(c) $\|f(X)+f(Y)\| \leqslant \|f(X+Y)\|$ for every non-negative convex function f on $[0,\infty)$ such that $f(0) = 0$.

(d) $\|f(X+Y)\| \leqslant \|f(X)+f(Y)\|$ for every non-negative concave function f on $[0,\infty)$.

It should be mentioned here that part (c) of Lemma 2.1 will not be used in this paper. It has been given for the sake of completeness.

In view of the facts that $f(t)=t^r$ is convex on $[0,\infty)$ for $r \geqslant 1$ and that $f(t)=t^r$ is concave on $[0,\infty)$ for $0<r\leqslant 1$, Lemma 2.1 admits an important special case. We remark here that if $X \in \mathbb{B}(\mathbb{H})$, and if f is a non-negative increasing function on $[0,\infty)$, then $\|f(|X|)\| = f(\|X\|)$. In particular, $\||X|^r\| = \|X\|^r$ for every $r>0$. Thus, if $X,Y \in \mathbb{B}(\mathbb{H})$ are positive, then

$$||X^r+Y^r|| \leqslant ||X+Y||^r \leqslant 2^{r-1}\,||X^r+Y^r|| \tag{2.1}$$

for $r \geqslant 1$, and

$$2^{r-1}\,||X^r+Y^r|| \leqslant ||X+Y||^r \leqslant ||X^r+Y^r|| \tag{2.2}$$

for $0<r\leqslant 1$.

Our main numerical radius inequalities can be stated as follows.

THEOREM 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A=B+iC$, and let $f(t)$ be a non-negative function on $[0,\infty)$ such that $g(t)=f(\sqrt{t})$ is concave. Then*

$$\frac{1}{2}\,||f(|B|)+f(|C|)|| \leqslant f(w(A)) \leqslant ||\,f(|B|)+f(|C|)|| \tag{2.3}$$

and

$$\frac{1}{2}\left\|f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\| \leqslant f(w(A)) \leqslant \left\|f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\|. \tag{2.4}$$

Proof. Since g is non-negative and concave on $[0,\infty)$, it follows that g is increasing, and so it follows by the second inequality in (1.3) that

$$g(w^2(A)) \leqslant g\left(\left|\left|B^2+C^2\right|\right|\right).$$

Now,

$$\begin{aligned} f(w(A)) &= g(w^2(A)) \\ &\leqslant g\left(\left|\left|B^2+C^2\right|\right|\right) \\ &= g\left(\left|\left|\,|B|^2+|C|^2\right|\right|\right) \\ &= \left|\left|g\left(|B|^2+|C|^2\right)\right|\right| \\ &\leqslant \left|\left|g\left(|B|^2\right)+g\left(|C|^2\right)\right|\right| \quad \text{(by Lemma 2.1(d))} \\ &= ||f(|B|)+g(|C|)||, \end{aligned}$$

which proves the second inequality in (2.3). The first inequality in (2.3) follows from the triangle inequality for both norms $\|\cdot\|$ and $w(\cdot)$, the equality of these norms for self-adjoint operators, and the conclusion that f is increasing. In fact,

$$\begin{aligned} \frac{1}{2}\left\|f(|B|)+f(|C|)\right\| &\leqslant \frac{1}{2}\left(\left\|f(|B|)\right\|+\left\|f(|C|)\right\|\right) \\ &= \frac{1}{2}\left(f(\|B\|)+f(\|C\|)\right) \\ &= \frac{1}{2}\left(f(w(B))+f(w(C))\right) \\ &\leqslant \frac{1}{2}\left(f(w(A))+f(w(A))\right) \\ &= f(w(A)). \end{aligned}$$

This completes the proof of the inequalities in (2.3).

To prove the inequalities in (2.4), observe that the second inequality in (1.4), together with the fact that g is increasing, implies that

$$g\left(w^2(A)\right) \leqslant g\left(\frac{1}{2}\left\|(B+C)^2+(B-C)^2\right\|\right).$$

Now,

$$\begin{aligned} f(w(A)) &= g\left(w^2(A)\right) \\ &\leqslant g\left(\frac{1}{2}\left\||B+C|^2+|B-C|^2\right\|\right) \\ &= \left\|g\left(\frac{|B+C|^2}{2}+\frac{|B-C|^2}{2}\right)\right\| \\ &\leqslant \left\|g\left(\frac{|B+C|^2}{2}\right)+g\left(\frac{|B-C|^2}{2}\right)\right\| \quad \text{(by Lemma 2.1(d))} \\ &= \left\|f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\|, \end{aligned}$$

which proves the second inequality in (2.4). To prove the first inequality in (2.4), first observe that by the first inequality in (1.4), we have

$$g\left(w^2(A)\right) \geqslant g\left(\frac{1}{4}\left\|(B+C)^2+(B-C)^2\right\|\right).$$

Now,

$$\begin{aligned}
f(w(A)) &= g\left(w^2(A)\right) \\
&\geqslant g\left(\frac{1}{4}\left\|\left|B+C\right|^2+\left|B-C\right|^2\right\|\right) \\
&= \left\|g\left(\frac{|B+C|^2}{4}+\frac{|B-C|^2}{4}\right)\right\| \\
&= \left\|g\left(\frac{\frac{|B+C|^2}{2}+\frac{|B-C|^2}{2}}{2}\right)\right\| \\
&\geqslant \frac{1}{2}\left\|g\left(\frac{|B+C|^2}{2}\right)+g\left(\frac{|B-C|^2}{2}\right)\right\| \quad \text{(by Lemma 2.1(b))} \\
&= \frac{1}{2}\left\|f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\|,
\end{aligned}$$

which proves the first inequality in (2.4). This completes the proof of the theorem. □

THEOREM 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$, and let $f(t)$ be a non-negative function on $[0,\infty)$ such that $g(t) = f(\sqrt{t})$ is convex. Then*

$$\frac{1}{2}\left\|f(|B|)+f(|C|)\right\| \leqslant f(w(A)) \leqslant \frac{1}{2}\left\| f\left(\sqrt{2}\,|B|\right)+f\left(\sqrt{2}\,|C|\right)\right\| \tag{2.5}$$

and

$$\frac{1}{2}\left\|f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\| \leqslant f(w(A)) \leqslant \frac{1}{2}\left\|f(|B+C|)+f(|B-C|)\right\|. \tag{2.6}$$

Proof. Since g is non-negative and convex on $[0,\infty)$, it follows that g is increasing. Now,

$$\begin{aligned}
f(w(A)) &= g\left(w^2(A)\right) \\
&\leqslant g\left(\left\||B|^2+|C|^2\right\|\right) \quad \text{(by the second inequality in (1.3))} \\
&= \left\|g\left(|B|^2+|C|^2\right)\right\| \\
&= \left\|g\left(\frac{2\,|B|^2+2\,|C|^2}{2}\right)\right\| \\
&\leqslant \frac{1}{2}\left\|g\left(2\,|B|^2\right)+g\left(2\,|C|^2\right)\right\| \quad \text{(by Lemma 2.1(a))} \\
&= \frac{1}{2}\left\|f\left(\sqrt{2}\,|B|\right)+f\left(\sqrt{2}\,|C|\right)\right\|,
\end{aligned}$$

which proves the second inequality in (2.5).

The proof of the first inequality in (2.5) is the same as that of the first inequality in (2.3). It does not depend on the convexity or concavity of g, but it depends on the fact that f is increasing in both theorems.

To prove the inequalities in (2.6), observe that

$$\begin{aligned} f(w(A)) &= g\left(w^2(A)\right) \\ &\leqslant g\left(\frac{1}{2}\left\| |B+C|^2+|B-C|^2 \right\|\right) \quad \text{(by the second inequality in (1.4))} \\ &= \left\| g\left(\frac{|B+C|^2+|B-C|^2}{2}\right)\right\| \\ &\leqslant \frac{1}{2}\left\| g\left(|B+C|^2\right)+g\left(|B-C|^2\right)\right\| \quad \text{(by Lemma 2.1(a))} \\ &= \frac{1}{2}\left\| f\left(|B+C|\right)+f\left(|B-C|\right)\right\|, \end{aligned}$$

which proves the second inequality in (2.6). To prove the first inequality in (2.6), we recall that $w^2(A) \geqslant \frac{1}{2}\left\| |B \pm C|^2 \right\|$ (see Theorem 1 in [7]). Since g is increasing, it follows that $g\left(w^2(A)\right) \geqslant g\left(\frac{1}{2}\left\| |B \pm C|^2 \right\|\right)$, and so by the triangle inequality for $\|\cdot\|$, we have

$$\begin{aligned} 2g\left(w^2(A)\right) &\geqslant g\left(\frac{1}{2}\left\| |B+C|^2 \right\|\right)+g\left(\frac{1}{2}\left\| |B-C|^2 \right\|\right) \\ &= \left\| g\left(\frac{|B+C|^2}{2}\right)\right\| + \left\| g\left(\frac{|B-C|^2}{2}\right)\right\| \\ &\geqslant \left\| g\left(\frac{|B+C|^2}{2}\right)+g\left(\frac{|B-C|^2}{2}\right)\right\| \\ &= \left\| f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\|. \end{aligned}$$

Hence,

$$\begin{aligned} f(w(A)) &= g\left(w^2(A)\right) \\ &\geqslant \frac{1}{2}\left\| f\left(\frac{|B+C|}{\sqrt{2}}\right)+f\left(\frac{|B-C|}{\sqrt{2}}\right)\right\|. \end{aligned}$$

This completes the proof of the theorem. □

Specializing Theorem 2.2 to the function $f(t)=t^r$ for $0<r\leqslant 2$, and specializing Theorem 2.3 to the function $f(t)=t^r$ for $r\geqslant 2$, we obtain our promised new numerical radius inequalities associated with the Cartesian decomposition of an operator. These inequalities include complements of the inequalities given in Theorem 1.1, and they also include extensions of Theorem 1.2.

COROLLARY 2.4. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$. Then*

$$\frac{1}{2} \| \, |B|^r + |C|^r \| \leqslant w^r(A) \leqslant \| \, |B|^r + |C|^r \| \tag{2.7}$$

for $0 < r \leqslant 2$, and

$$\frac{1}{2} \| \, |B|^r + |C|^r \| \leqslant w^r(A) \leqslant 2^{r/2-1} \| \, |B|^r + |C|^r \| \tag{2.8}$$

for $r \geqslant 2$.

COROLLARY 2.5. *Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$. Then*

$$2^{-r/2-1} \| \, |B+C|^r + |B-C|^r \| \leqslant w^r(A) \leqslant 2^{-r/2} \| \, |B+C|^r + |B-C|^r \| \tag{2.9}$$

for $0 < r \leqslant 2$, and

$$2^{-r/2-1} \| \, |B+C|^r + |B-C|^r \| \leqslant w^r(A) \leqslant \frac{1}{2} \| \, |B+C|^r + |B-C|^r \| \tag{2.10}$$

for $r \geqslant 2$.

Finally, we remark that the inequalities in (2.7)–(2.10) are sharp as demonstrated by two-dimensional examples. In fact, the sharpness of the first inequalities in (2.7)–(2.10) can be verified by considering $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. The sharpness of the second inequalities in (2.7) and (2.10) can be verified by considering $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the sharpness of the second inequalities in (2.8) and (2.9) can be verified by considering $A = \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix}$.

Acknowledgement. The author is greatful to J.-C. Bourin for his suggestions.

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *A numerical radius inequality involving the generalized Aluthge transform*, Studia Math. **216** (2013), 69–75.

[2] A. ABU-OMAR AND F. KITTANEH, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mountain J. Math., in press.

[3] S. S. DRAGOMIR, *Inequalities for the norm and numerical radius of linear operators in Hilbert space*, Demonstratio Math. **40** (2007), 411–417.

[4] M. EL-HADDAD AND F. KITTANEH, *Numerical radius inequalities for Hilbert space operators II*, Studia Math. **182** (2007), 133–140.

[5] A. GOMILKO, I. WROBEL, AND J. ZEMANEK, *Numerical ranges in a strip, in operator theory 20*, Theta, Bucharest, 2006, pp. 111–121.
[6] O. HIRZALLAH AND F. KITTANEH, *Non-commutative Clarkson inequalities for n-tuples of operators*, Integral Equations Operator Theory **60** (2008), 369–379.
[7] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), 73–80.
[8] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007), 83–89.

(Received September 24, 2014)

Fuad Kittaneh
Department of Mathematics
The University of Jordan
Amman, Jordan
e-mail: fkitt@ju.edu.jo