Mathematical
Inequalities
& Applications
Volume 18, Number 3 (2015), 949–958
doi:10.7153/mia-18-71

$M^{\natural}$–CONVEXITY AND ULTRAMODULARITY ON INTEGER LATTICE

SUMBUL AZEEM AND RASHID FAROOQ

(*Communicated by J. Pečarić*)

Abstract. Ultramodular functions defined on a subset of a finite dimensional Euclidean space is a class of functions that generalizes the scalar convexity. On the other hand, $M^{\natural}$-convex functions defined on a subset of integer lattice form a class of integrally convex functions. In this paper, we reveals a relationship between ultramodularity and $M^{\natural}$-convexity on the integer lattice. We show that each $M^{\natural}$-convex set (function) is an ultramodular set (function). The converse, however, may not be true.

1. Introduction

The real-valued convex functions in one variable are important in solving optimization problems. Let $\mathbf{R}$ be the set of reals and $I \subseteq \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

for all $x, y \in I$ and for all $\lambda \in (0,1)$. Equivalently, we say that a function $f : I \to \mathbf{R}$ is convex if and only if

$$f(x+\varepsilon) - f(x) \leqslant f(y+\varepsilon) - f(y) \tag{2}$$

for all $x, y \in I$ and for all $\varepsilon > 0$ with $x < y$ and $y + \varepsilon \in I$. This equivalence, however, may not be true if the function is defined on a subset of $\mathbf{R}^n$, $n > 1$. Marinacci and Montrucchio [5] extended the concept of this one-dimensional convex functions to the functions defined on a subset of $\mathbf{R}^n$ and called them ultramodular functions or the directionally convex functions. Formally, we say that a function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is ultramodular if

$$f(x+h) - f(x) \leqslant f(y+h) - f(y) \tag{3}$$

for all $x, y \in A$ and for all $h \geqslant \mathbf{0}$ with $x \leqslant y$ and $x+h$, $y+h \in A$. From (2) and (3), we see that the concept of convexity and ultramodularity coincides when $n = 1$. However, these two concepts are different from each other in higher dimension.

A non-decreasing function $f : [0,1]^n \to [0,1]$ satisfying $f(0,0\ldots,0) = 0$ and $f(1,1\ldots,1) = 1$ is called an n-ary aggregation function. The reader is referred to

Mathematics subject classification (2010): 26B25, 90C27.
Keywords and phrases: Ultramodularity, $M^{\natural}$-convexity, discrete convexity.

[1, 2] for a detailed study on n-ary aggregation functions. Klement et al [3] introduced the concept of ultramodular aggregation functions which is an extension of one-dimensional convexity. An n-ary aggregation function $f:[0,1]^n \to [0,1]$ is called ultramodular if

$$f(x+y+z)-f(x+y) \geqslant f(x+z)-f(x) \tag{4}$$

for all $x,y,z \in [0,1]^n$ with $x+y+z \in [0,1]^n$. Again, the concepts of one-dimensional convex functions and ultramodular aggregation functions coincide on the real line. However, in higher dimension the two concepts are quite different.

On the other hand, there is a class of integrally convex functions, called the M$^\natural$-convex functions introduced by Murota and Shioura [7]. M$^\natural$-convex functions play a central role in the theory of discrete convex analysis and have close relations to nice properties in mathematical economics, called the gross substitutability and the single improvement property. The reader is referred to Murota [6] for a detailed study on M$^\natural$-convex functions.

Ultramodular functions and M$^\natural$-convex functions apparently seem very different from each other. However, there is a relationship between these two types of functions when defined on a subset of integer lattice. Our main contribution in this paper is to show that each M$^\natural$-convex set (function) is an ultramodular set (function). We give an example to show that an ultramodualr (set) function may not be an M$^\natural$-convex (set) function.

This paper is organized as follows. In Section two, we introduce ultramodular sets and functions. Section three is devoted to M$^\natural$-convex sets and functions. The relationship between ultramodular sets (functions) and M$^\natural$-convex sets (functions) appears in Section four. In Section five, we define a new class of functions, called the componentwise ultramodular functions.

2. Ultramodular sets and functions

In this section, we present ultramodular sets and functions introduced by Marinacci and Montrucchio [5].

A collection (x,y,z,w) of vectors in $\mathbf{R}^n$ is called a *test quadruple* if $x \leqslant y \leqslant w$ and $x+w=y+z$.

A set $A \subseteq \mathbf{R}^n$ is *ultramodular* if, given any triple $x,y,w \in A$ with $x \leqslant y \leqslant w$, we have the following:

$$x+w=y+z \implies z \in A. \tag{5}$$

This means that A is closed under the formation of test quadruples. The following result gives a good description of ultramodular sets under certain conditions.

PROPOSITION 1. ([5, Proposition 3.1]) *Let $A \subseteq \mathbf{R}^n$. Then A is ultramodular whenever the following condition holds:*

$$x,y \in A \text{ and } y \leqslant x \Longrightarrow [y,x] \subseteq A. \tag{6}$$

Conversely, suppose that A is ultramodular and atleast one of the following properties holds:

(i) *A is open,*

(ii) $\operatorname{int}(A) \neq \emptyset$ *and A has a smallest and largest element.*

Then (6) *holds.*

Next, we present ultramodular functions defined on n-dimensional Euclidean space $\mathbf{R}^n$.

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is said to be *ultramodular* if

$$f(y) + f(z) \leqslant f(x) + f(w) \tag{7}$$

for all test quadruples (x,y,z,w) in A. By setting $h = z - x = w - y$, we can restate the definition of an ultramodular function as follows:

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is said to be *ultramodular* if

$$f(x+h) - f(x) \leqslant f(y+h) - f(y) \tag{8}$$

for all $h \geqslant 0$ and $x \leqslant y$ such that $x+h$, $y+h \in A$. It is noteworthy to mention that no condition on the domain of an ultramodular function is required.

The following lemma gives an equivalent condition for a function to be ultramodular.

LEMMA 1. *A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is ultramodular if and only if*

$$f(x) + f(y) \geqslant f(x+h) + f(y-h) \tag{9}$$

for all $x, y \in A$ and $h \geqslant \mathbf{0}$ with $x+h$, $y-h \in A$ and $x \leqslant y-h \leqslant y$.

Proof. Suppose that f is ultramodular. For any $x, y \in A$ and $h \geqslant \mathbf{0}$ such that $x+h$, $y-h \in A$ and $x \leqslant y-h$, we have

$$f(y) - f(y-h) \geqslant f(x+h) - f(x).$$

The above inequality can be written as

$$f(x) + f(y) \geqslant f(x+h) + f(y-h).$$

Conversely, suppose that (9) holds. We show that f is ultramodular function. Let $x, y \in A$, $x \leqslant y$ and $h \geqslant \mathbf{0}$ with $x+h$, $y+h \in A$. Since $x \leqslant y \leqslant y+h$, the inequality (9) gives

$$f(y) + f(x+h) \leqslant f(y+h) + f(x).$$

The above inequality is equivalent to

$$f(y+h) - f(y) \geqslant f(x+h) - f(x).$$

Thus, f is ultramodular. $\square$

3. M$^\natural$-convex sets and functions

In this section, we present basic definitions of M$^\natural$-convex sets and functions introduced by Murota and Shioura [7].

Let V be a nonempty finite set and let 0 be an element which is not in V. The set of integers is denoted by $\mathbf{Z}$. We denote by $\mathbf{Z}^V$ the set of integral vectors of the form $x = (x(v) \in \mathbf{Z} \mid v \in V)$, where $x(v)$ denotes the v-component of the vector x. Similarly, the set of real vectors of the form $x = (x(v) \in \mathbf{R} \mid v \in V)$ is denoted by $\mathbf{R}^V$. We define the *positive support* $\mathrm{supp}^+(x)$ and the *negative support* $\mathrm{supp}^-(x)$ of a vector $x \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\},$$
$$\mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\},$$

respectively. The *characteristic vector* χ_S of a set $S \subseteq V$ is defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \in V \setminus S, \end{cases}$$

where χ_v is used instead of $\chi_{\{v\}}$ for each $v \in V$. Moreover, we define χ_0 as the zero vector in $\mathbf{Z}^V$.

The *effective domain* $\mathrm{dom} f$ of a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by:

$$\mathrm{dom} f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called *$M^\natural$-convex* if it satisfies the following condition:

(M$^\natural$-EXC) For all $x, y \in \mathrm{dom} f$ and all $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geqslant f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

A nonempty set $B \subseteq \mathbf{Z}^V$ is called $M^\natural$-convex if the following condition is satisfied:

(B$^\natural$-EXC) For all $x, y \in B$ and all $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$x - \chi_u + \chi_v \in B \text{ and } y + \chi_u - \chi_v \in B.$$

PROPOSITION 2. (Murota [6]) *The effective domain of an $M^\natural$-convex function is an $M^\natural$-convex set.*

4. Relationship

In this section, we construct a relationship between M$^\natural$-convexity and Ultramodularity on the integer lattice $\mathbf{Z}^V$. We show that each M$^\natural$-convex set is an ultramodular set. Similarly, we show that each M$^\natural$-convex function is an ultramodular function. We give examples to show that an ultramodular set (function) may not be an M$^\natural$-convex set (function).

The following result gives a sufficient condition for a subset of integer lattice to be an ultramodular set. The proof is omitted since it is an easy consequence of Proposition 1 considering the discrete analogue.

PROPOSITION 3. *Let $A \subseteq \mathbf{Z}^V$. Then A is ultramodular whenever the following condition holds:*

$$x, y \in A \text{ and } y \leqslant x \Longrightarrow [y,x]_{\mathbf{Z}} \subseteq A. \tag{10}$$

The Proposition 3 is used to prove the following lemma.

LEMMA 2. *If a set $A \subseteq \mathbf{Z}^V$ is M$^\natural$-convex then it is ultramodular.*

Proof. Assume that $A \subseteq \mathbf{Z}^V$ is an M$^\natural$-convex set. Let $x, y \in A$ with $y \leqslant x$ and $z \in [y,x]_{\mathbf{Z}}$.[1] We show that $z \in A$. If $z = x$ then obviously $z \in A$. Suppose that $z \neq x$. Then $\mathrm{supp}^+(x-z) \neq \emptyset$ and let $u_1 \in \mathrm{supp}^+(x-z)$, that is, $u_1 \in \mathrm{supp}^+(x-y)$. Then (B$^\natural$-EXC) implies $x_1 = x - \chi_{u_1} \in A$. Clearly, $z \leqslant x_1$. If $z \neq x_1$ then take $u_2 \in \mathrm{supp}^+(x_1 - z) \subseteq \mathrm{supp}^+(x-z)$, that is, $u_2 \in \mathrm{supp}^+(x_1 - y)$. Again by virtue of (B$^\natural$-EXC), $x_2 = x_1 - \chi_{u_2} \in A$. In other words, for $u_1, u_2 \in \mathrm{supp}^+(x-z)$ we have $x - \chi_{u_1} - \chi_{u_2} \in A$. Continuing this process, we get $x - \sum_{u \in \mathrm{supp}^+(x-z)} \big(x(u) - z(u)\big)\chi_u \in A$. But

$$z = x - \sum_{u \in \mathrm{supp}^+(x-z)} \big(x(u) - z(u)\big)\chi_u.$$

Thus $z \in A$. This shows that $[y,x]_{\mathbf{Z}} \subseteq A$. By Proposition 3, A is ultramodular. □

REMARK 1. Lemma 3.1 [4] states that if $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is an M$^\natural$-concave function then $[y,x]_{\mathbf{Z}} \subseteq \mathrm{dom} f$ for each $x, y \in \mathrm{dom} f$ with $y \leqslant x$.[2] The proofs of Lemma 3.1 [4] and Lemma 2 are essentially same. Thus, we do not claim the originality on Lemma 2.

The following examples show that the converse of the above lemma does not hold in general.

EXAMPLE 1. Let $a, h \in \mathbf{Z}^V$ such that $h \geqslant \mathbf{0}$. Define a set

$$A = \{a + nh \mid n \in \mathbf{Z}_+\},$$

where $\mathbf{Z}_+$ is the set of non-negative integers. Then A is ultramodular but not an M$^\natural$-convex set.

[1] $[y,x]_{\mathbf{Z}} = \{z \in \mathbf{Z}^V \mid y(u) \leqslant z(u) \leqslant x(u) \forall u \in V\}$.

[2] A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is M$^\natural$-concave if $-f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex.

EXAMPLE 2. Let $|V| = 3$ and define a set $A \subseteq \mathbf{Z}^V$ by:

$$A = \{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\}.$$

Clearly, A is ultramodular. However, it is not an $\mathrm{M}^\natural$-convex set.

COROLLARY 1. *If a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is $M^\natural$-convex then* $\mathrm{dom} f$ *is an ultramodular set.*

Proof. By Proposition 2, $\mathrm{dom} f$ is an $\mathrm{M}^\natural$-convex set. Thus the assertion follows from Lemma 2. □

Now, we give a relationship between $\mathrm{M}^\natural$-convex functions and ultramodular functions on the integer lattice $\mathbf{Z}^V$.

LEMMA 3. *If a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is $M^\natural$-convex then it is ultramodular on* $\mathrm{dom} f$.

Proof. Let $x, y \in \mathrm{dom} f$ such that $x \geqslant y$ and $h \geqslant \mathbf{0}$ with $x+h,\ y+h \in \mathrm{dom} f$. Take $u_1 \in \mathrm{supp}^+(x-y)$. Then by Proposition 2, we have $x_1 = x - \chi_{u_1} \in \mathrm{dom} f$. Since $u_1 \in \mathrm{supp}^+(x+h-x_1)$ and f is $\mathrm{M}^\natural$-convex, we have

$$f(x+h) + f(x_1) \geqslant f(x+h-\chi_{u_1}) + f(x).$$

This is equivalent to

$$f(x+h) - f(x) \geqslant f(x_1+h) - f(x_1). \tag{11}$$

Clearly $y \leqslant x_1$. If $x_1 = y$ then (11) implies that f is ultramodular. If $x_1 \neq y$ then we take $u_2 \in \mathrm{supp}^+(x_1-y) \subseteq \mathrm{supp}^+(x-y)$. Proposition 2 gives $x_2 = x_1 - \chi_{u_2} \in \mathrm{dom} f$ and (11) implies $x_1 + h \in \mathrm{dom} f$. As $u_2 \in \mathrm{supp}^+(x_1+h-x_2)$ and f is $\mathrm{M}^\natural$-convex, we obtain

$$f(x_1+h) + f(x_2) \geqslant f(x_1+h-\chi_{u_2}) + f(x_1).$$

This is equivalent to

$$f(x_1+h) - f(x_1) \geqslant f(x_2+h) - f(x_2). \tag{12}$$

The inequalities (11) and (12) give

$$f(x+h) - f(x) \geqslant f(x+h-\chi_{u_1}-\chi_{u_2}) - f(x-\chi_{u_1}-\chi_{u_2}),$$

where $u_1, u_2 \in \mathrm{supp}^+(x-y)$. By applying this argument repeatedly, we get

$$f(x+h) - f(x) \geqslant f(x+h-\sum_{u\in\mathrm{supp}^+(x-y)} \lambda_u \chi_u) - f(x-\sum_{u\in\mathrm{supp}^+(x-y)} \lambda_u \chi_u), \tag{13}$$

where $\lambda_u = x(u) - y(u)$ for each $u \in \text{supp}^+(x-y)$. But $x - \sum_{u\in\text{supp}^+(x-y)} \lambda_u \chi_u = y$. Therefore (13) implies

$$f(x+h) - f(x) \geqslant f(y+h) - f(y).$$

Thus f is ultramodular. □

The following example shows that the converse of above lemma is not true, that is, an ultramodular function needs not to be an M$^\natural$-convex function.

EXAMPLE 3. Let a_i and b be non-negative real numbers, where $i \in V$. Define $f : A \subseteq \mathbf{R}^V \longrightarrow \mathbf{R}$ by

$$f(x) = \sum_{i\in V} a_i x(i)^2 + b \sum_{i<j} x(i)x(j).$$

Let (x,y,z,w) be a test quadruple in A. Then by using the fact that $x+w = y+z$, we get

$$\begin{aligned} f(x)+f(w)-f(y)-f(z) &= \sum_{i\in V} a_i (x(i)^2 + w(i)^2 - y(i)^2 - z(i)^2) \\ &\quad + b\sum_{i<j}(x(i)x(j) + w(i)w(j) - y(i)y(j) - z(i)z(j)) \\ &= 2\sum_{i\in V} a_i (x(i)-y(i))(y(i)-w(i)) \\ &\quad + b\sum_{i<j}((y(i)-x(i))(z(j)-x(j)) \\ &\quad + b\sum_{i<j}(z(i)-x(i))(y(j)-x(j)). \end{aligned}$$

Since $x \leqslant y \leqslant w$ and $x \leqslant z \leqslant w$, it holds that $f(x)+f(w)-f(y)-f(z) \geqslant 0$. Thus, f is an ultramodular function. Generally, f is not an M$^\natural$-convex function on $\mathbf{Z}^V$. It is M$^\natural$-convex if $0 \leqslant b \leqslant 2\min_{i\in V} a_i$ (see Murota [6]).

5. Component-wise ultramodular functions

In this section, we define a new class of functions, called the component-wise ultramodular functions. This class of functions is contained in the class of ultramodular functions. Moreover, on integer lattice we will show that each component-wise ultramodular function is an M$^\natural$-convex function. We give examples to show that an M$^\natural$-convex function may not be a component-wise ultramodular function.

Before defining component-wise ultramodular functions, we define component-wise ultramodular sets and their relationship with ultramodular sets and M$^\natural$-convex sets.

A collection (x,y,z,w) of vectors in $\mathbf{R}^n$ is said to be a *component-wise test quadruple* (denoted by CW-test quadruple) if it satisfies the following:

(i) $x+w=y+z$,

(ii) for each $i\in\{1,\dots,n\}$, either $x(i)\leqslant y(i)\leqslant w(i)$ or $x(i)\geqslant y(i)\geqslant w(i)$.

A set $A\subseteq\mathbf{R}^n$ is *CW-ultramodular* if given any elements $x,y,w\in A$ with $x(i)\leqslant y(i)\leqslant w(i)$ or $x(i)\geqslant y(i)\geqslant w(i)$ for each $i\in\{1,\dots,n\}$, the following holds:

$$x+w=y+z \text{ implies } z\in A. \tag{14}$$

EXAMPLE 4. The hyperplane $\{x\in\mathbf{R}^n \mid \sum_{i=1}^{n} x(i)=r\}$, where $r\in\mathbf{R}$, is a CW-ultramodular set.

A function $f:A\subseteq\mathbf{R}^n\longrightarrow\mathbf{R}$ is said to be *component-wise ultramodular* (denoted by CW-ultramodular) function if

$$f(x)+f(w)\geqslant f(y)+f(z)$$

holds for all CW-test quadruples (x,y,z,w).

Next, we give few examples to illustrate the CW-ultramodular functions.

EXAMPLE 5. Let a_i be a non-negative real number, $i\in\{1,\dots,n\}$. Define $f:A\subseteq\mathbf{R}^n\longrightarrow\mathbf{R}$ by

$$f(x)=\sum_{i=1}^{n} a_i x(i)^2.$$

Let (x,y,z,w) be a CW-test quadruple in A. Then

$$\begin{aligned} f(x)+f(w)-f(y)-f(z) &= \sum_{i=1}^{n} a_i(x(i)^2+w(i)^2-y(i)^2-z(i)^2) \\ &= 2\sum_{i=1}^{n} a_i(x(i)-y(i))(y(i)-w(i)), \end{aligned}$$

where the last equality is due to the fact that $x+w=y+z$. For each $i\in\{1,\dots,n\}$, either $x(i)\leqslant y(i)\leqslant w(i)$ or $x(i)\geqslant y(i)\geqslant w(i)$. In either case, we have $(x(i)-y(i))(y(i)-w(i))\geqslant 0$. Thus $f(x)+f(w)-f(y)-f(z)\geqslant 0$, that is, f is CW-ultramodular.

REMARK 2. If $A=\{0,1\}^n$ in Example 5 then one can easily show that $f(x)+f(w)-f(y)-f(z)=0$ for each CW-test quadruple (x,y,z,w) in A.

EXAMPLE 6. For each $i\in\{1,\dots,n\}$, let $f_i:\mathbf{R}\longrightarrow\mathbf{R}$ be a CW-ultramodular function. Define $f:A\subseteq\mathbf{R}^n\longrightarrow\mathbf{R}$ by

$$f(x)=\sum_{i=1}^{n} f_i(x(i)),$$

where $x=(x(i)\mid i\in\{i,\dots,n\})\in A$. Then f is a CW-ultramodular function.

LEMMA 4. *If a function* $f:\mathbf{Z}^V\to\mathbf{R}$ *is CW-ultramodular, then it is M*$^\natural$*-convex.*

Proof. Let $x,y\in\mathbf{Z}^V$ and $u\in\operatorname{supp}^+(x-y)$. For each $w\in\operatorname{supp}^-(x-y)$, define $\lambda_w=y(w)-x(w)$. Take

$$h_1=-\sum_{w\in\operatorname{supp}^-(x-y)}\lambda_w\chi_w$$

and set $x_1=x-h_1$. Then $(x_1+h_1,y+h_1,x_1,y)$ is a CW-test quadruple and since f is CW-ultramodular, we have

$$f(y+h_1)-f(y)\leqslant f(x_1+h_1)-f(x_1).$$

The above inequality is equivalent to

$$f(y+h_1)+f(x_1)\leqslant f(x)+f(y). \tag{15}$$

Next, define

$$y_2=y+h_1,\quad x_2=x-\chi_u,\quad h_2=-(h_1-\chi_u).$$

Then $(x_2+h_2,y_2+h_2,x_2,y_2)$ is a CW-test quadruple. By CW-ultramodularity, we get

$$f(y_2+h_2)-f(y_2)\leqslant f(x_2+h_2)-f(x_2).$$

The above inequality can further be written as:

$$f(y+\chi_u)+f(x-\chi_u)\leqslant f(y+h_1)+f(x_1). \tag{16}$$

Combining (15) and (16), we get

$$f(x)+f(y)\geqslant f(x-\chi_u)+f(y+\chi_u)$$

which is (M$^\natural$-EXC) for $v=0$. Thus, f is M$^\natural$-convex function. □

The following examples show that the converse of the above lemma is not true.

EXAMPLE 7. Let a_i be a positive real number, $i\in V$, and $b\in\mathbf{R}$ satisfying $0<b\leqslant 2\min_{i\in V}a_i$. Define $f:\mathbf{Z}^V\longrightarrow\mathbf{R}$ by

$$f(x)=\sum_{i\in V}a_ix(i)^2+b\sum_{i<j}x(i)x(j).$$

Take $x,y,z,y\in\mathbf{Z}^V$ and $i,j\in V$ such that $i\neq j$ and $x(j)=y(i)=y(j)=w(i)=1$ and all other coordinates of x,y,z and w are zero. Then (x,y,z,w) is a CW-test quadruple. We see that $f(x)+f(w)-f(y)-f(z)<0$, that is, f is not a CW-ultramodular function. However, f is an M$^\natural$-convex function (see Murota [6]).

EXAMPLE 8. Let $|V|=3$ and define $f:\{0,1\}^V\subseteq \mathbf{Z}^V\to\mathbf{R}$ as follows:

$$f(x)=\begin{cases}2 \text{ if } x=(1,1,1)\\ 1 \text{ otherwise.}\end{cases}$$

Consider the CW-test quadruple (x,y,z,w), where

$$x=(0,1,1),\quad y=(1,1,1),\quad z=(0,0,0),\quad w=(1,0,0).$$

Then $f(x)+f(w)-f(y)-f(z)<0$, that is, f is not a CW-ultramodular function. However, one can easily see that f is an $\mathrm{M}^{\natural}$-convex function.

Conclusion

We have seen that, on the integer lattice, the class of $\mathrm{M}^{\natural}$-convex functions is contained in the class of ultramodular functions. We observe that if $f:A\subseteq\mathbf{R}^n\to\mathbf{R}$ is a function and any two elements of A are incomparable, that is, $x\not\leqslant y$ and $y\not\leqslant x$ for each $x,y\in A$ then f is vacuously ultramodular. This characteristic of ultramodular functions creates a gap between the class of ultramodular functions and the class of $\mathrm{M}^{\natural}$-convex functions.

Acknowledgement. The authors would like to thank the anonymous referee for his valuable comments and suggestions to improve the quality of the paper. The second author is grateful to Luigi Montrucchio for very helpful discussion via email way back in 2007 which paved the way for this research.

REFERENCES

[1] G. BELIAKOV, A. PRADERA, T. CALVO, *Aggregation functions: A guide for practioners*, Springer, Heidelberg, 2007.
[2] M. GRABISCH, J.-L. MARICHAL, R. MESIAR, E. PAP, *Aggregation functions*, Cambridge University Press, Cambridge, 2009.
[3] E. P. KLEMENT, M. MANZI, R. MESIAR, *Ultramodular agregation functions*, Inf. Sci., **181** (2011) 4101–4111.
[4] R. FAROOQ, A. MAHMOOD, *A two-sided discrete-concave market with possibly bounded salaries*, memo 2013.
[5] M. MARINACCI, L. MONTRUCCHIO, *Ultramodular functions*, Math. Oper. Res., **30** (2005) 311–332.
[6] K. MUROTA, *Discrete convex analysis*, SIAM, Philadelphia, 2003.
[7] K. MUROTA, A. SHIOURA, *M-convex function on generalized polymatroid*, Math. Oper. Res., **24** (1999) 95–105.

(Received August 7, 2013)

Sumbul Azeem
F. G. Post Graduate College for Women
Kashmir Road Rawalpindi, Pakistan
e-mail: sumbulazeem@gmail.com

Rashid Farooq
School of Natural Sciences
National University of Sciences and Technology
H-12 Islamabad, Pakistan
e-mail: farook.ra@gmail.com