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A GENERALIZED ČEBYŠEV FUNCTIONAL FOR THE RIEMANN–STIELTJES INTEGRAL

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Abstract. Sharp bounds for a generalised Čebyšev functional for the Riemann-Stieltjes integral are given.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1}$$

In 1935, Grüss [19] showed that

$$|C(f,g)| \leqslant \frac{1}{4}(M-m)(N-n), \tag{2}$$

provided that there exist the real numbers m,M,n,N such that

$$m \leqslant f(t) \leqslant M \quad \text{and} \quad n \leqslant g(t) \leqslant N \quad \text{for a.e. } t\in[a,b]. \tag{3}$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [5], states that

$$|C(f,g)| \leqslant \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{4}$$

provided that f',g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (4) also holds if $f,g:[a,b]\to\mathbb{R}$ are assumed to be absolutely continuous and $f',g'\in L_\infty[a,b]$ while $\|f'\|_\infty = \text{ess}\sup_{t\in[a,b]}|f'(t)|$.

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A mixture between Grüss' result (2) and Čebyšev's one (4) is the following inequality obtained by Ostrowski in 1970, [25]:

$$|C(f,g)| \leqslant \frac{1}{8}(b-a)(M-m)\left\|g'\right\|_{\infty}, \tag{5}$$

provided that f is *Lebesgue integrable* and satisfies (3) while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (5).

The case of *Euclidean norms* of the derivative was considered by A. Lupaş in [22] in which he proved that

$$|C(f,g)| \leqslant \frac{1}{\pi^2}\left\|f'\right\|_2\left\|g'\right\|_2(b-a), \tag{6}$$

provided that f,g are absolutely continuous and $f',g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S. S. Dragomir [3] have proved the following results:

$$|C(f,g)| \leqslant \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \tag{7}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$ and $q=\infty$, and

$$|C(f,g)| \leqslant \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_1 \cdot \frac{1}{b-a}\operatorname*{ess\ sup}_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|, \tag{8}$$

provided that $f\in L_p[a,b]$ and $g\in L_q[a,b]$ $(p>1,\ \frac{1}{p}+\frac{1}{q}=1;\ p=1,\ q=\infty$ or $p=\infty$, $q=1)$.

Notice that for $q=\infty, p=1$ in (7) we obtain

$$\begin{aligned}|C(f,g)| &\leqslant \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\ &\leqslant \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt\end{aligned} \tag{9}$$

and if g satisfies (3), then

$$\begin{aligned}|C(f,g)| &\leqslant \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\ &\leqslant \left\|g-\frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\ &\leqslant \frac{1}{2}(N-n)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\end{aligned} \tag{10}$$

The inequality between the first and the last term in (10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [20], [24] and [26] and the references therein.

Consider the following Čebyšev functional for the *Riemann-Stieltjes integral* [27, p. 219]

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \tag{11}$$

where f,g are continuous on $[a,b]$ and u is of *bounded variation* on $[a,b]$ with $u(b)\neq u(a)$.

The following result that provides sharp bounds for the Čebyšev functional defined above was obtained in [12].

THEOREM 1. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ with $u(a)\neq u(b)$. Assume also that there exist the real constants γ,Γ such that*

$$\gamma\leqslant f(t)\leqslant\Gamma \text{ for each } t\in[a,b]. \tag{12}$$

a) If u is of bounded variation on $[a,b]$, then we have the inequality

$$|T(f,g;u)| \leqslant \frac{1}{2}\cdot\frac{\Gamma-\gamma}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \tag{13}$$

where $\bigvee_a^b(u)$ denotes the total variation of u in $[a,b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller quantity.

b) If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then one has the inequality:

$$|T(f,g;u)| \leqslant \frac{1}{2}\cdot\frac{\Gamma-\gamma}{u(b)-u(a)}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t). \tag{14}$$

The constant $\frac{1}{2}$ is sharp.

c) Assume that $f,g:[a,b]\to\mathbb{R}$ are Riemann integrable functions on $[a,b]$ and f satisfies the condition (12). If $u:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant L, then we have the inequality

$$|T(f,g;u)| \leqslant \frac{1}{2}\cdot\frac{L(\Gamma-\gamma)}{|u(b)-u(a)|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \tag{15}$$

The constant $\frac{1}{2}$ is best possible in (15).

We observe that if $u(t)=t,$ then from (13) we get

$$|C(f,g)| \leqslant \frac{1}{2}(\Gamma-\gamma)\left\|g-\frac{1}{b-a}\int_a^b g(s)\,ds\right\|_\infty$$

while from (14) and (15) we recapture the inequality between the first and last term in (10).

For some recent inequalities for Riemann-Stieltjes integral see [8]–[13] and [21].

Motivated by the above results we consider here a more general Čebyšev functional depending on four functions and defined as

$$\begin{aligned} &T(f,g,h;u) \qquad (16)\\ &:= \frac{1}{u(b)-u(a)}\int_a^b h(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t)\\ &\quad -\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b f(t)h(t)\,du(t), \end{aligned}$$

provided that all the Riemann-Stieltjes integrals incorporated in (16) exist and $u(b)\neq u(a)$. That happens, for instance, when $u:[a,b]\to\mathbb{C}$ is of bounded variation and $f,g,h:[a,b]\to\mathbb{C}$ are continuous on $[a,b]$.

The functional $T(f,g,h;u)$ can be written in a determinant form as

$$\begin{aligned} &T(f,g,h;u) \qquad (17)\\ &=\det\begin{bmatrix} \frac{1}{u(b)-u(a)}\int_a^b h(t)\,du(t) & \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\\ \frac{1}{u(b)-u(a)}\int_a^b f(t)h(t)\,du(t) & \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t)\end{bmatrix}. \end{aligned}$$

We remark that if $h(t)=\mathbf{1}(t)=1$ for all $t\in[a,b]$ then we get $T(f,g,\mathbf{1};u)=T(f,g;u)$. By (17) we then have the determinant form of $T(f,g;u)$ as

$$\begin{aligned} &T(f,g;u) \qquad (18)\\ &=\det\begin{bmatrix} 1 & \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\\ \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t) & \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t)\end{bmatrix}. \end{aligned}$$

We also observe that if e denotes the identity mapping on $[a,b]$, i.e. $e(t)=t$, $t\in[a,b]$ then by choosing $u=e$ in (18) we have $T(f,g;e)=C(f,g)$.

In this paper we establish some sharp bounds for the magnitude of the functional $T(f,g,h;u)$ under various assumptions for the functions involved.

2. The results

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C}\,|\,\mathrm{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geqslant 0 \text{ for each } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \middle| \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leqslant \frac{1}{2} |\Gamma - \gamma| \text{ for each } t \in [a,b] \right\}.$$

The following representation result may be stated.

PROPOSITION 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{19}$$

For some related ideas, see [7].

On making use of the complex numbers field properties we can also state that:

COROLLARY 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f : [a,b] \to \mathbb{C} \mid &(\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ &+ (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geqslant 0 \text{ for each } t \in [a,b] \}. \end{aligned} \tag{20}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geqslant \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geqslant \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geqslant \operatorname{Re} f(t) \geqslant \operatorname{Re}(\gamma) \tag{21}$$

and

$$\operatorname{Im}(\Gamma) \geqslant \operatorname{Im} f(t) \geqslant \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}.$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{22}$$

The following result can be stated.

THEOREM 2. *Assume that* $u : [a,b] \to \mathbb{C}$ *with* $u(b) \neq u(a)$ *and* $f, g, h : [a,b] \to \mathbb{C}$ *are such that the Riemann-Stieltjes integrals in the definition of* $T(f,g,h;u)$ *exist. Assume also that there exist the complex numbers* γ, Γ, $\gamma \neq \Gamma$, *such that*

$$f \in \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{23}$$

i) If u *is of bounded variation on* $[a,b]$, *then we have the inequality*

$$\begin{aligned} |T(f,g,h;u)| \leqslant \frac{1}{2} |\Gamma - \gamma| \bigvee_a^b (u) \frac{1}{|u(b) - u(a)|^2} \\ \times \sup_{t \in [a,b]} \left| \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} \right| \end{aligned} \tag{24}$$

$$\leqslant \frac{1}{2}|\Gamma-\gamma|\left[\bigvee_a^b(u)\right]^2\frac{1}{|u(b)-u(a)|^2}$$
$$\times \sup_{(t,s)^2\in[a,b]^2}\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|.$$

The constant $\frac{1}{2}$ is sharp in the first inequality.

ii) If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then

$$|T(f,g,h;u)| \leqslant \frac{1}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2} \tag{25}$$
$$\times\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|du(t)$$
$$\leqslant \frac{1}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2}$$
$$\times\int_a^b\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|du(s)\,du(t)$$
$$\leqslant \frac{\sqrt{2}}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2}$$
$$\times\left(\det\begin{bmatrix} \int_a^b|g(t)|^2du(t) & \int_a^b\left(g(t)\overline{h(t)}\right)du(t) \\ \int_a^b\left(\overline{g(t)}h(t)\right)du(t) & \int_a^b|h(t)|^2du(t)\end{bmatrix}\right)^{1/2}.$$

The multiplicative constant $\frac{1}{2}$ in the first inequality is best possible.

iii) If u is Lipschitzian with the constant $L>0$, then we have the inequality

$$|T(f,g,h;u)| \leqslant \frac{1}{2}|\Gamma-\gamma|\frac{L}{|u(b)-u(a)|^2} \tag{26}$$
$$\times\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(t)\,du(t) & \int_a^b h(t)\,du(t)\end{bmatrix}\right|dt$$
$$\leqslant \frac{1}{2}|\Gamma-\gamma|\frac{L^2}{[u(b)-u(a)]^2}$$
$$\times\int_a^b\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|dsdt$$

$$\leqslant \frac{\sqrt{2}}{2}(\Gamma-\gamma)\frac{L^2}{[u(b)-u(a)]^2}$$
$$\times\left(\det\begin{bmatrix} \int_a^b |g(t)|^2 dt & \int_a^b \left(g(t)\overline{h(t)}\right) dt \\ \int_a^b \left(\overline{g(t)}h(t)\right) dt & \int_a^b |h(t)|^2 dt \end{bmatrix}\right)^{1/2}.$$

The constant $\frac{1}{2}$ in the first inequality is sharp.

REMARK 1. We notice that the above Theorem 2 not only provides a generalization of Theorem 1 but also an extension of that result to the complex valued functions.

Perhaps simpler, however coarser bounds for the magnitude of $T(f,g,h;u)$ can be provided if some connection between the other two functions g and h are known.

COROLLARY 2. *Assume that $u:[a,b]\to\mathbb{C}$ with $u(b)\neq u(a)$ and $f,g,h:[a,b]\to\mathbb{C}$ are such that the Riemann-Stieltjes integrals in the definition of $T(f,g,h;u)$ exist. Assume also that there exist the complex numbers γ,Γ such that (23) holds true.*

a) Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants φ and Φ such that either

$$\operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right]\geqslant 0 \text{ for any } t\in[a,b] \tag{27}$$

or, equivalently,

$$\left|g(t)-\frac{\varphi+\Phi}{2}h(t)\right|\leqslant\frac{1}{2}|\Phi-\varphi||h(t)| \text{ for any } t\in[a,b], \tag{28}$$

holds, then we have

$$|T(f,g,h;u)|\leqslant\frac{\sqrt{2}}{4}|\Gamma-\gamma||\Phi-\varphi|\frac{1}{[u(b)-u(a)]^2}\int_a^b|h(t)|^2du(t). \tag{29}$$

aa) Let $u:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants φ and Φ such that either (27) or (28) hold true, then

$$|T(f,g,h;u)|\leqslant\frac{\sqrt{2}}{4}|\Gamma-\gamma||\Phi-\varphi|\frac{L^2}{[u(b)-u(a)]^2}\int_a^b|h(t)|^2dt. \tag{30}$$

For some similar results see [18].

REMARK 2. The above Corollary 2 can be then used to provide simpler bounds for the functional $T(\cdot,\cdot;\cdot)$ as follows:

b) Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants λ and Λ such that either

$$\operatorname{Re}\left[(\Lambda-g(t))\left(\overline{g(t)}-\overline{\lambda}\right)\right]\geqslant 0 \text{ for any } t\in[a,b] \tag{31}$$

or, equivalently,

$$\left|g(t)-\frac{\lambda+\Lambda}{2}h(t)\right|\leqslant\frac{1}{2}|\Lambda-\lambda| \text{ for any } t\in[a,b], \tag{32}$$

holds, then we have

$$|T(f,g;u)|\leqslant\frac{\sqrt{2}}{4}\frac{|\Gamma-\gamma||\Lambda-\lambda|}{u(b)-u(a)}. \tag{33}$$

bb) Let $u:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants λ and Λ such that either (31) or (32) hold true, then

$$|T(f,g;u)|\leqslant\frac{\sqrt{2}}{4}|\Gamma-\gamma||\Lambda-\lambda|\frac{L^2(b-a)}{[u(b)-u(a)]^2}. \tag{34}$$

3. Proofs

We observe that the following identity of interest holds:

$$\begin{aligned}&[u(b)-u(a)]^2T(f,g,h;u) \\ &=\int_a^b h(t)du(t)\cdot\int_a^b f(t)g(t)du(t)-\int_a^b g(t)du(t)\cdot\int_a^b f(t)h(t)du(t) \\ &=\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)du(s) & \int_a^b h(s)du(s)\end{bmatrix}du(t)\end{aligned} \tag{35}$$

for each $\eta\in\mathbb{C}$.

We notice that (35) is a generalization of Sonin's identity that holds for $u(t)=t$, $t\in[a,b]$, see for instance [23, p. 246].

i) It is well known that if the Riemann-Stieltjes integral $\int_a^b p(t)dv(t)$ exists, $p:[a,b]\to\mathbb{C}$ is bounded and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then we have the inequality [2]

$$\left|\int_a^b p(t)dv(t)\right|\leqslant\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v). \tag{36}$$

Taking the modulus in (35) and utilizing (36) we get

$$\begin{aligned}&|u(b)-u(a)|^2|T(f,g,h;u)| \\ &=\left|\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)du(s) & \int_a^b h(s)du(s)\end{bmatrix}du(t)\right|\end{aligned} \tag{37}$$

$$\leqslant \sup_{t\in[a,b]}\left|(f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|$$
$$\leqslant \sup_{t\in[a,b]}|f(t)-\eta|\sup_{t\in[a,b]}\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|.$$

Since $f\in\bar{U}_{[a,b]}(\gamma,\Gamma)$, then

$$\left|f(t)-\frac{\Gamma+\gamma}{2}\right|\leqslant\frac{1}{2}|\Gamma-\gamma| \quad \text{for each } t\in[a,b]. \tag{38}$$

Utilizing (37) for $\eta=\frac{\gamma+\Gamma}{2}$ and (38) we deduce the first inequality in (24). The rest is obvious.

We observe that for $h(t)=1$, $t\in[a,b]$ and f a real function bounded below by γ and above by Γ, we recapture from the first part of (24) the inequality (13) obtained in [12] whose multiplicative constant $\frac{1}{2}$ is best possible. This fact implies that the constant $\frac{1}{2}$ in the first inequality is also best possible.

ii) It is well known that if the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$, $\int_a^b |p(t)|\,dv(t)$ exist and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\leqslant\int_a^b |p(t)|\,dv(t). \tag{39}$$

For instance, when p is continuous and v is monotonic nondecreasing then (39) holds true.

Taking the modulus in (35) and utilizing (39) we get

$$\begin{aligned}&(u(b)-u(a))^2|T(f,g,h;u)| \\ &=\left|\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}du(t)\right| \\ &\leqslant\int_a^b\left|(f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|du(t) \\ &=\int_a^b|f(t)-\eta|\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|du(t),\end{aligned} \tag{40}$$

which, by (38), produces the first inequality in (25).

The sharpness of this inequality follows from (14) which is a particular case of (25).

Further on, observe that

$$\begin{aligned}\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| &= \left|\int_a^b \det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix} du(s)\right| \\ &\leqslant \int_a^b \left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right| du(s),\end{aligned}$$

which, by integration on $[a,b]$ over the monotonic nondecreasing integrator u, produces the second part of (25).

Now, on utilizing the Cauchy-Bunyakovsky-Schwarz's double integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators, namely

$$\begin{aligned}&\left|\int_a^b\int_a^b h(t,s)\,m(t,s)\,du(t)\,du(s)\right|^2 \\ &\leqslant \int_a^b\int_a^b |h(t,s)|^2\,du(t)\,du(s)\int_a^b\int_a^b |m(t,s)|^2\,du(t)\,du(s)\end{aligned} \tag{41}$$

where $h,m:[a,b]^2\to\mathbb{C}$ are continuous, then we have

$$\begin{aligned}&\frac{1}{[u(b)-u(a)]^2}\int_a^b\int_a^b \left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right| du(s)\,du(t) \\ &\leqslant \left[\frac{1}{[u(b)-u(a)]^2}\int_a^b\int_a^b \left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|^2 du(s)\,du(t)\right]^{1/2}.\end{aligned} \tag{42}$$

Since

$$\begin{aligned}&\int_a^b\int_a^b \left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|^2 du(s)\,du(t) \\ &= \int_a^b\int_a^b \Big\{|g(t)|^2|h(s)|^2 + |g(s)|^2|h(t)|^2 \\ &-2\operatorname{Re}\left[(g(t)\,h(s))\,\overline{(g(s)\,h(t))}\right]\Big\}\,du(s)\,du(t)\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_a^b |g(t)|^2|h(s)|^2\,du(s)\,du(t) &= \int_a^b\int_a^b |g(s)|^2|h(t)|^2\,du(s)\,du(t) \\ &= \int_a^b |g(t)|^2\,du(t)\int_a^b |h(t)|^2\,du(t)\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \int_a^b \operatorname{Re}\left[(g(t)h(s))\overline{(g(s)h(t))}\right] du(s)\,du(t) \\
&= \int_a^b \int_a^b \operatorname{Re}\left[\left(g(t)\overline{h(t)}\right)\overline{\left(g(s)\overline{h(s)}\right)}\right] du(s)\,du(t) \\
&= \operatorname{Re}\left[\int_a^b \left(g(t)\overline{h(t)}\right) du(t) \int_a^b \overline{\left(g(s)\overline{h(s)}\right)} du(s)\right] \\
&= \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2
\end{aligned}$$

then

$$\begin{aligned}
&\int_a^b \int_a^b \left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|^2 du(s)\,du(t) \qquad (43)\\
&= 2\left[\int_a^b |g(t)|^2 du(t) \int_a^b |h(t)|^2 du(t) - \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2\right] \\
&= 2\det\begin{bmatrix} \int_a^b |g(t)|^2 du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)}h(t)\right) du(t) & \int_a^b |h(t)|^2 du(t)\end{bmatrix}.
\end{aligned}$$

Making use of (42) and (43) we obtain the last part of (25).

We notice that (43) is a complex Riemann-Stieltjes integral version of Andréief's identity for $n=2,$ see for instance [27, p. 201].

iii) It is well known that if $p:[a,b]\to\mathbb{C}$ is Riemann integrable and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0,$ then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \leqslant L\int_a^b |p(t)|\,dt. \qquad (44)$$

Taking the modulus in (35) and utilizing (44) we get

$$\begin{aligned}
&|u(b)-u(a)|^2\,|T(f,g,h;u)| \qquad (45)\\
&= \left|\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix} du(t)\right| \\
&\leqslant L\int_a^b \left|(f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| dt \\
&= L\int_a^b |f(t)-\eta| \left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| dt.
\end{aligned}$$

Utilizing (38) and (45) we deduce the first inequality (26).

The second part follows from a similar argument to the one in the second part of the statement ii) by choosing $u(t) = t$ and the details are omitted.

We observe that for $h(t) = 1,\ t \in [a,b]$ we recapture from the first inequality in (26) the inequality (15) which is sharp.

Now, in order to prove the statements a) and aa) we need the following result that is of interest in its turn.

LEMMA 1. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation and $g,h : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants φ and Φ such that*

$$\int_a^b |h(t)|^2 du(t) \int_a^b \operatorname{Re}\left[(\Phi h(t) - g(t))\left(\overline{g(t)} - \overline{\varphi h(t)}\right)\right] du(t) \geqslant 0, \tag{46}$$

then we have

$$\det \begin{bmatrix} \int_a^b |g(t)|^2 du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)} h(t)\right) du(t) & \int_a^b |h(t)|^2 du(t) \end{bmatrix} \tag{47}$$
$$\leqslant \frac{1}{4}|\Phi - \varphi|^2 \left[\int_a^b |h(t)|^2 du(t)\right]^2.$$

The constant $\frac{1}{4}$ in (47) is best possible.

Proof. Consider the quantities

$$K_1 := \operatorname{Re}\left\{\left[\Phi \int_a^b |h(t)|^2 du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right] \times \left[\overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} - \overline{\varphi}\int_a^b |h(t)|^2 du(t)\right]\right\}$$

and

$$K_2 := \int_a^b |h(t)|^2 du(t) \int_a^b \operatorname{Re}\left[(\Phi h(t) - g(t))\left(\overline{g(t)} - \overline{\varphi h(t)}\right)\right] du(t).$$

We have by simple calculation that

$$K_1 = \int_a^b |h(t)|^2 du(t) \operatorname{Re}\left[\Phi \overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} + \overline{\varphi}\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right] - \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2 - \left[\int_a^b |h(t)|^2 du(t)\right]^2 \operatorname{Re}\left(\Phi\overline{\varphi}\right)$$

and

$$K_2 = \int_a^b |h(t)|^2 \operatorname{Re}\left[\Phi \overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} + \overline{\varphi} \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right]$$
$$- \int_a^b |g(t)|^2 du(t) \int_a^b |h(t)|^2 du(t) - \left[\int_a^b |h(t)|^2 du(t)\right]^2 \operatorname{Re}(\Phi\overline{\varphi}),$$

which produces the equality of interest

$$K_1 - K_2 = \int_a^b |g(t)|^2 du(t) \int_a^b |h(t)|^2 du(t) - \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2 \tag{48}$$
$$= \det \begin{bmatrix} \int_a^b |g(t)|^2 du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)} h(t)\right) du(t) & \int_a^b |h(t)|^2 du(t) \end{bmatrix}.$$

Since, by (46), we have $K_2 \geqslant 0$, then it follows from (48) that

$$\det \begin{bmatrix} \int_a^b |g(t)|^2 du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)} h(t)\right) du(t) & \int_a^b |h(t)|^2 du(t) \end{bmatrix} \tag{49}$$
$$\leqslant \operatorname{Re}\left\{\left[\Phi \int_a^b |h(t)|^2 du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right] \times \left[\overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} - \overline{\varphi} \int_a^b |h(t)|^2 du(t)\right]\right\}$$
$$= \operatorname{Re}\left\{\left(\Phi \int_a^b |h(t)|^2 du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right) \times \overline{\left(\int_a^b \left(g(t)\overline{h(t)}\right) du(t) - \varphi \int_a^b |h(t)|^2 du(t)\right)}\right\}.$$

On utilizing the elementary inequality for complex numbers

$$\operatorname{Re}(u\overline{v}) \leqslant \frac{1}{4}|u+v|^2, u, v \in \mathbb{C}$$

we have

$$\operatorname{Re}\left\{\left(\Phi \int_a^b |h(t)|^2 du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right) \times \overline{\left(\int_a^b \left(g(t)\overline{h(t)}\right) du(t) - \varphi \int_a^b |h(t)|^2 du(t)\right)}\right\}$$
$$\leqslant \frac{1}{4}|\Phi - \varphi|^2 \left[\int_a^b |h(t)|^2 du(t)\right]^2$$

which together with (49) produces the desired result (47). □

The following particular case is of interest since it provides a reverse inequality for the Cauchy-Bunyakovsky-Schwarz's integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators:

COROLLARY 3. *Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exist the complex constants φ and Φ such that either*

$$\operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right]\geqslant 0 \quad \textit{for any} \quad t\in[a,b] \tag{50}$$

or, equivalently,

$$\left|g(t)-\frac{\varphi+\Phi}{2}h(t)\right|\leqslant\frac{1}{2}|\Phi-\varphi||h(t)| \quad \textit{for any} \quad t\in[a,b], \tag{51}$$

holds, then we have

$$\begin{aligned}0&\leqslant\det\begin{bmatrix}\int_a^b|g(t)|^2du(t) & \int_a^b\left(g(t)\overline{h(t)}\right)du(t)\\ \int_a^b\left(\overline{g(t)}h(t)\right)du(t) & \int_a^b|h(t)|^2du(t)\end{bmatrix} \\ &\leqslant\frac{1}{4}|\Phi-\varphi|^2\left[\int_a^b|h(t)|^2du(t)\right]^2.\end{aligned} \tag{52}$$

The constant $\frac{1}{4}$ in (52) is best possible.

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