Mathematical
Inequalities
& Applications
Volume 18, Number 3 (2015), 975–980 doi:10.7153/mia-18-73

THE SHARP UPPER BOUND FOR THE RATIO BETWEEN THE ARITHMETIC AND THE GEOMETRIC MEAN

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(*Communicated by I. Perić*)

Abstract. In this note we present the sharp upper bound for the ratio between the arithmetic and the geometric mean for concave sequences. In this way the sharp reverse of the Arithmetic-Geometric Mean Inequality is obtained.

1. Introduction

Let X_n denote the n-tuple $(x_1,\cdots,x_n)$, where $x_i \geqslant 0$ for $i = 1,2,\cdots,n$. We shall make use of the notations $EX_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, and $\Pi X_n = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$. An infinite sequence of real numbers $\{x_i\}_{i=0}^{\infty}$ is said to be concave if $x_i \geqslant \frac{1}{2}(x_{i-1}+x_{i+1})$, for $i = 1,2,\cdots$. A sequence $\{x_i\}_{i=0}^{n}$ (with $n \geqslant 2$) is said to be concave if $x_i \geqslant \frac{1}{2}(x_{i-1}+x_{i+1})$, for $i = 1,2,\cdots,n-1$.

Some results related to the problems of the difference and the ratio between the arithmetic and the geometric mean are given in [1, 3, 9].

In this note we present the sharp upper bound for the ratio between the arithmetic and the geometric mean for concave sequences. In this way the sharp reverse of the Arithmetic-Geometric Mean Inequality is obtained. Our main result is the following theorem.

THEOREM 1. *Suppose* $n \geqslant 2$*. Let* $\{x_i\}_{i=0}^{n}$ *be a concave sequence with* $0 = x_0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$*, and let* $X_n = (x_1,\cdots x_n)$*. Then, we have*

$$\frac{EX_n}{\Pi X_n} \leqslant \frac{n+1}{2(n!)^{\frac{1}{n}}}. \tag{1}$$

Equality holds in (1) if and only if $x_i = c_0 i,\ i = 1,2,\cdots,n$*, for some constant* $c_0 > 0$.

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Arithmetic-geometric mean inequality, concave sequence.
This work was partially supported by the National Natural Science Foundation of China (Grant No. 11471209).

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The preceding theorem is motivated by the results of the papers by Aldaz [1, 3], which state some interesting inequalities for the difference and the ratio between the arithmetic and the geometric mean (see also [2, 4, 7, 9] for additional refinements and references).

Note that each infinite non-negative concave sequence must be monotone increasing (Lemma 3). Then, for an infinite concave sequence, we have the following corollary:

COROLLARY 1. *Let* $\{x_i\}_{i=0}^{\infty}$ *be a concave sequence with* $x_0 = 0$ *and* $x_i > 0$, $\forall i \in \mathbb{N}$. *For any* $n \in \mathbb{N}$, *let* $X_n = (x_1, \cdots x_n)$. *Then, we have*

$$\frac{EX_n}{\Pi X_n} < \frac{e}{2}. \tag{2}$$

The constant $\frac{e}{2}$ *is the best possible.*

Note that a continuous result which is closely related to (2) is contained in the Berwald inequality [8]. It states that for a non-negative concave function f on $[a,b]$ and $0 \leqslant r \leqslant 1$, it holds

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{(1+r)^{\frac{1}{r}}}{2}\left[\frac{1}{b-a}\int_a^b f^r(x)dx\right]^{\frac{1}{r}}.$$

Letting $r \to 0$ the estimation analogous to (1) follows.

Theorem 1 establishes the ratio relationship between arithmetic and geometric mean of finite concave sequences, and it can be considered as a sharp version of the reverse of the Arithmetic-Geometric Mean Inequality.

2. Proofs

To prove Theorem 1, we need the following lemma.

LEMMA 1. *Let* k *be a positive integer, then*

$$\frac{(k+1)^k}{k!} \leqslant 2^{2k}.$$

Proof. We prove this lemma by induction on k. For $k = 1$, it holds trivially. Assume the lemma holds for k, namely, $k! \geqslant \frac{(k+1)^k}{2^{2k}}$. Now consider the case of $k+1$. Noticing the known inequality $\left(1+\frac{1}{n}\right)^n < 4$, $\forall n \in \mathbb{N}$, we have

$$\frac{(k+2)^{k+1}}{(k+1)!} \leqslant \frac{(k+2)^{k+1}}{(k+1)\frac{(k+1)^k}{2^{2k}}} = 2^{2k}\left(1+\frac{1}{k+1}\right)^{k+1} < 2^{2k}\cdot 4 = 2^{2k+2},$$

as desired. □

Proof of Theorem 1. Set $y_i = \frac{x_i}{i}$, $i = 1,2,\cdots,n$. Since $\{x_i\}_{i=0}^n$ is concave and $x_0 = 0$, it follows that

$$y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n. \tag{3}$$

It is easily seen that (1) is equivalent to

$$\sum_{i=1}^{n} iy_i \leqslant \frac{n(n+1)}{2}\prod_{i=1}^{n} y_i^{\frac{1}{n}}. \tag{4}$$

Now we use induction on n to prove (4) under the condition (3). Note: although we have assumed $n \geqslant 2$ in Theorem 1, we can still prove (4) for all $n \geqslant 1$ under the condition (3).

If $n = 1$, (4) is trivial. Assume that (4) is true for $n = k$. For the case of $n = k+1$, we need to prove that

$$\sum_{i=1}^{k+1} iy_i \leqslant \frac{(k+1)(k+2)}{2}\prod_{i=1}^{k+1} y_i^{\frac{1}{k+1}}. \tag{5}$$

By the induction hypothesis, we have

$$\sum_{i=1}^{k} iy_i \leqslant \frac{k(k+1)}{2}\prod_{i=1}^{k} y_i^{\frac{1}{k}}.$$

Hence, to prove (5), we only need to prove that

$$(k+1)y_{k+1} + \frac{k(k+1)}{2}\prod_{i=1}^{k} y_i^{\frac{1}{k}} \leqslant \frac{(k+1)(k+2)}{2}\prod_{i=1}^{k+1} y_i^{\frac{1}{k+1}}. \tag{6}$$

Setting $A = \prod_{i=1}^{k} y_i^{\frac{1}{k}}$, we can rewrite (6) as

$$(k+2)A^{\frac{k}{k+1}} y_{k+1}^{\frac{1}{k+1}} - 2y_{k+1} - kA \geqslant 0. \tag{7}$$

Consider the function $f(x) = (k+2)A^{\frac{k}{k+1}} x^{\frac{1}{k+1}} - 2x - kA,\ x \in (0,\infty)$. Then (7) is equivalent to

$$f(y_{k+1}) \geqslant 0. \tag{8}$$

Now we prove (8). Since

$$f''(x) = -\frac{k(k+2)}{(k+1)^2} A^{\frac{k}{k+1}} x^{-\frac{2k+1}{k+1}} < 0,$$

the function $f(x)$ is strictly concave on $(0,\infty)$. Hence, for any $0 < a < b$ and $x \in [a,b]$, we have

$$f(x) \geqslant \min\{f(a), f(b)\}. \tag{9}$$

On the other hand, we claim that

$$mA \leqslant y_{k+1} \leqslant A, \tag{10}$$

where $m = \frac{(k!)^{\frac{1}{k}}}{k+1}$.

In fact, from (3), it follows that $y_{k+1} \leqslant A$. Since $x_{k+1} \geqslant x_k \geqslant \cdots \geqslant x_1 > 0$, we have

$$(k+1)y_{k+1} \geqslant ky_k \geqslant \cdots \geqslant 2y_2 \geqslant y_1 > 0.$$

Therefore,

$$y_{k+1} \geqslant \frac{(k!)^{\frac{1}{k}}}{k+1}A.$$

So (10) is true, and it follows from (9) that

$$f(y_{k+1}) \geqslant \min\{f(A), f(mA)\} = \min\{0, f(mA)\}. \tag{11}$$

Therefore, to prove the desired inequality, it is sufficient to prove

$$f(mA) \geqslant 0. \tag{12}$$

By a direct computation, we infer that (12) is equivalent to

$$\sum_{i=1}^{k} m^{\frac{i}{k+1}} \geqslant \frac{k}{2}. \tag{13}$$

In fact, applying the Arithmetic-Geometric Mean Inequality and Lemma 1, we obtain

$$\sum_{i=1}^{k} m^{\frac{i}{k+1}} \geqslant km^{\frac{1}{2}} > \frac{k}{2}.$$

So (13) is true, and then (1) is established.

If $x_i = c_0 i$ for some constant $c_0 > 0$, then it is easy to see that equality holds in (1).

Now suppose that equality holds in (4), that is,

$$\sum_{i=1}^{n} iy_i = \frac{n(n+1)}{2} \prod_{i=1}^{n} y_i^{\frac{1}{n}}. \tag{14}$$

From the proof above it follows that

$$\sum_{i=1}^{k} iy_i \leqslant \frac{k(k+1)}{2} \prod_{i=1}^{k} y_i^{\frac{1}{k}}, \tag{15}$$

for $k = 1,...,n-1$. Using (14) and (15) for $k = n-1$ we conclude that

$$\frac{n(n-1)}{2} \prod_{i=1}^{n-1} y_i^{\frac{1}{n-1}} + ny_n \geqslant \frac{n(n+1)}{2} \prod_{i=1}^{n} y_i^{\frac{1}{n}}. \tag{16}$$

For $k = n-1$, recall that

$$f(x) = (n+1)A^{\frac{n-1}{n}} x^{\frac{1}{n}} - 2x - (n-1)A,$$

where $A = \prod_{i=1}^{n-1} y_i^{1/(n-1)}$. Inequality (16) is equivalent to $f(y_n) \leqslant 0$. On the other hand, from our proof above it follows that $mA \leqslant y_n \leqslant A$, $m = ((n-1)!)^{1/(n-1)}/n$ and $f(y_n) \geqslant 0$, so $f(y_n) = 0$. Since f is strictly concave, $f(A) = 0$, $f(mA) > 0$, we obtain

$$y_n = A = \prod_{i=1}^{n-1} y_i^{1/(n-1)}$$

and equality in (16), which implies equality in (15) for $k = n-1$ i.e.

$$\sum_{i=1}^{n-1} iy_i = \frac{n(n-1)}{2} \prod_{i=1}^{n-1} y_i^{\frac{1}{n-1}}. \tag{17}$$

Proceeding in this way it follows that

$$\sum_{i=1}^{k} iy_i = \frac{k(k+1)}{2} \prod_{i=1}^{k} y_i^{\frac{1}{k}}$$

for every $k = 1, \dots, n$, which easily gives

$$y_1 = y_2 = \cdots = y_n. \quad \square$$

To prove Corollary 1, we need the following two lemmas.

LEMMA 2. *For any $n \in \mathbb{N}$, we have*

$$\frac{n+1}{(n!)^{\frac{1}{n}}} \leqslant e, \qquad \lim_{n \longrightarrow \infty} \frac{n+1}{(n!)^{\frac{1}{n}}} = e.$$

The proof of this lemma is known, so we omit it here. For a more sophisticated results, see [5].

LEMMA 3. *Let $\{x_i\}_{i=0}^{\infty}$ be a non-negative concave sequence, then $\{x_i\}_{i=0}^{\infty}$ is increasing.*

Proof. To prove this lemma, we suppose the contrary, i.e., there exists a $k \in \mathbb{N}$ such that $x_{k+1} - x_k < 0$. Set $x_{k+1} - x_k = c < 0$. Since $\{x_i\}_{i=0}^{\infty}$ is concave, we know that $x_{j+1} - x_j \leqslant x_{k+1} - x_k = c$ for all $j \geqslant k$, $j \in \mathbb{N}$. Hence, for sufficiently large $m \in \mathbb{N}$, we have

$$x_m = x_k + \sum_{j=k}^{m-1} (x_{j+1} - x_j) < x_k + (m-k)c < 0,$$

a contradiction! $\square$

Proof of Corollary 1. By Lemma 3, we know that $\{x_i\}_{i=0}^{\infty}$ is increasing. Thus, for each $n \in \mathbb{N}$, we have

$$\frac{EX_n}{\Pi X_n} \leqslant \frac{n+1}{2(n!)^{\frac{1}{n}}}.$$

Then, the inequality (2) follows immediately from Lemma 2.

Finally, we show that the constant $\frac{e}{2}$ is best possible. Suppose that there is a universal constant c such that

$$EX_k \leqslant c\Pi X_k,$$

for each $k \in \mathbb{N}$. Taking $x_i = i$ in the above inequality, we get $c \geqslant \frac{k+1}{2(k!)^{\frac{1}{k}}}$, for each $k \in \mathbb{N}$. By Lemma 2, we obtain $c \geqslant \lim_{k\to\infty} \frac{k+1}{2(k!)^{\frac{1}{k}}} = \frac{e}{2}$. □

Acknowledgements. The authors are grateful to the referees for their very valuable comments and advises for improvements.

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(Received September 3, 2013)

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