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SOME IMPROVEMENTS OF THE POPOVICIU, BELLMAN AND DIAZ—METCALF INEQUALITIES VIA SUPERQUADRATIC FUNCTIONS

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Abstract. We give refinements of the generalized Bellman and Diaz–Metcalf inequalities for isotonic linear functionals by using the concept of superquadratic functions. Further, we give refinements of the generalized Bellman and Diaz–Metcalf inequalities for time scales integrals as applications of the above obtained inequalities for isotonic linear functionals. Moreover we give refinements of the Aczél and Popoviciu inequalities for time scales integrals.

1. Introduction

During past few years numerous generalization, refinements and variants of the classical Aczél, Popoviciu, Bellman, and Diaz–Metcalf inequalities have appeared in the literature. In this paper, we give refinements of the generalized inequalities for isotonic linear functionals and time scales integrals. We first recall the following definition from [8, page 47].

DEFINITION 1.1. (Isotonic linear functional) Let E be a nonempty set and L be a linear class of real-valued functions $f:E\to\mathbb{R}$ having the following properties:

(L_1) If $f,g\in L$ and $a,b\in\mathbb{R}$, then $(af+bg)\in L$.

(L_2) If $f(t)=1$ for all $t\in E$, then $f\in L$.

An *isotonic linear functional* is a functional $I:L\to\mathbb{R}$ having the following properties:

(I_1) If $f,g\in L$ and $a,b\in\mathbb{R}$, then $I(af+bg)=aI(f)+bI(g)$.

(I_2) If $f\in L$ and $f(t)\geqslant 0$ for all $t\in E$, then $I(f)\geqslant 0$.

Popoviciu's inequality for isotonic linear functionals is given in the following theorem.

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THEOREM 1.2. (Popoviciu's inequality [8, Theorem 4.27]) *Let E, L, and I be such that* (L_1), (L_2), (I_1), (I_2) *are satisfied. For* $p \neq 1$, *define* $q = p/(p-1)$. *Assume* $|f|^p, |g|^q, |fg| \in L$. *Suppose* $f_0, g_0 > 0$ *are such that*

$$f_0^p - I(|f|^p) > 0 \quad \text{and} \quad g_0^q - I(|g|^q) > 0.$$

If $p > 1$, *then*

$$\left(f_0^p - I(|f|^p)\right)^{1/p} \left(g_0^q - I(|g|^q)\right)^{1/q} \leqslant f_0 g_0 - I(|fg|).$$

This inequality is reversed if $0 < p < 1$ *and* $I(|g|^q) > 0$, *or if* $p < 0$ *and* $I(|f|^p) > 0$.

If $p = 2$ in the above theorem, we obtain Aczél inequality for isotonic linear functionals (see [8, Theorem 4.26]).

In the following two theorems Bellman and Diaz–Metcalf inequalities are given for isotonic linear functionals.

THEOREM 1.3. (Bellman's inequality [8, Theorem 4.29]) *Let E, L, and I be such that* (L_1), (L_2), (I_1), (I_2) *are satisfied. For* $p \in \mathbb{R}$, *assume* $|f|^p, |g|^p, (|f| + |g|)^p \in L$. *Suppose* $f_0, g_0 > 0$ *are such that*

$$f_0^p - I(|f|^p) > 0 \quad \text{and} \quad g_0^p - I(|g|^p) > 0.$$

If $p > 1$, *then*

$$\left(\left(f_0^p - I(|f|^p)\right)^{1/p} + \left(g_0^p - I(|g|^p)\right)^{1/p}\right)^p \leqslant (f_0 + g_0)^p - I((|f| + |g|)^p).$$

THEOREM 1.4. (Diaz–Metcalf's inequality [8, Theorem 4.14]) *Let E, L, and I be such that* (L_1), (L_2), (I_1), (I_2) *are satisfied. For* $p \neq 1$, *let* $q = p/(p-1)$. *Assume* $|w||f|^p, |w||g|^q, |wfg| \in L$ *and, if* $p \neq 0$,

$$0 < m \leqslant |f(t)||g(t)|^{-q/p} \leqslant M \quad \text{for all} \quad t \in E.$$

If $p > 1$, *or if* $p < 0$ *and* $I(|w||f|^p) + I(|w||g|^q) > 0$, *then*

$$(M - m)I(|w||f|^p) + (mM^p - Mm^p)I(|w||g|^q) \leqslant (M^p - m^p)I(|wfg|).$$

This inequality is reversed if $0 < p < 1$ *and* $I(|w||f|^p) + I(|w||g|^q) > 0$.

Now we quote the definition of superquadratic functions, for other interesting properties of superquadratic functions we refer the reader to [1, 2, 4, 5].

DEFINITION 1.5. (Superquadratic function) A function $\Psi : [0, \infty) \to \mathbb{R}$ is called superquadratic if there exists a function $C : [0, \infty) \to \mathbb{R}$ such that

$$\Psi(y) - \Psi(x) - \Psi(|y - x|) \geqslant C(x)(y - x) \quad \text{for all} \quad x, y \geqslant 0. \tag{1.1}$$

We say that Ψ is subquadratic if $-\Psi$ is superquadratic.

For example, the function $\Psi(x) = x^p$ is superquadratic for $p \geqslant 2$ and subquadratic for $p \in (0,2]$. In the following two theorems we recall functional Minkowski inequality and a converse of Jensen's inequality for superquadratic function, which are used in the proofs of our main results.

THEOREM 1.6. (See [6, Theorem 4.1]) *Let L satisfy conditions (L_1), (L_2) and let I satisfy conditions (I_1), (I_2) on a nonempty set E. If $p \geqslant 2$ and $I((f+g)^p) > 0$, then for all nonnegative functions f,g on E such that $(f+g)^p, f^p, g^p \in L$, the inequality*

$$I^{\frac{1}{p}}((f+g)^p) \leqslant \left(I(f^p) - I\left(\left|f-(f+g)\frac{I(f(f+g)^{p-1})}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}} + \left(I(g^p) - I\left(\left|g-(f+g)\frac{I(g(f+g)^{p-1})}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}$$

holds.

THEOREM 1.7. (See [5, Theorem 15]) *Let L satisfy conditions (L_1), (L_2) and I satisfy conditions (I_1) and (I_2) on a nonempty set E. Let $k \in L$ be a nonnegative function. Suppose that $\Psi : [0,\infty) \to \mathbb{R}$ is a superquadratic function. Then for every $h \in L$, $h : E \to [m,M] \subseteq [0,\infty)$ such that $kh, k(\Psi \circ h) \in L$, we have*

$$I(k\Psi(h)) + \Delta_c \leqslant \frac{MI(k) - I(kh)}{M-m}\Psi(m) + \frac{I(kh) - mI(k)}{M-m}\Psi(M),$$

where

$$\Delta_c = \frac{1}{M-m} I((Mk-kh)\Psi(h-m\cdot 1) + (kh-mk)\Psi(M\cdot 1-h)).$$

A time scale $\mathbb{T}$ is an arbitrary closed subset of $\mathbb{R}$. In [3] it is shown that time scales integrals such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals are isotonic linear functionals. Moreover, in [3] the generalizations of Aczél, Popoviciu, Bellman, and Diaz–Metcalf inequalities are given for time scales integrals. In this paper we give results for multiple Lebesgue delta integrals, but analogue results for other time scales integrals also hold.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1,\ldots,n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1,\ldots,t_n) : t_i \in \mathbb{T}_i, 1 \leqslant i \leqslant n\} \tag{1.2}$$

is an n-dimensional time scale. Let $X \subset \Lambda^n$, $\mathscr{M}$ be the family of Δ-measurable subsets of Λ^n and μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n. Let $(X,\mathscr{M},\mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : X \to \mathbb{R}$, the corresponding Δ-integral of f over X is denoted according to [7, (3.18)] by

$$\int_X f(t_1,\ldots,t_n)\Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_X f(t)\Delta t, \quad \int_X f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_X f(t)\mathrm{d}\mu_\Delta(t).$$

There are many important time scales integrals, for example, in case of 1-dimension, when $\Lambda^n = \mathbb{R}$ the Lebesgue Δ-integral becomes Lebesgue integral; when $\Lambda^n = \mathbb{Z}$ the Lebesgue Δ-integral becomes sum; and when $\Lambda^n = q^{\mathbb{N}}$, $q > 1$, the time scales integral is a Jackson integral.

2. Bellman's inequality

In the following theorem we give a refinement of the functional Bellman inequality (Theorem 1.3).

THEOREM 2.1. *Let L satisfy conditions (L_1), (L_2) and let I satisfy conditions (I_1), (I_2) on a nonempty set E. For $p \geqslant 2$, assume f,g are nonnegative functions on E such that $(f+g)^p, f^p, g^p \in L$ and $I((f+g)^p) > 0$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - I(f^p) > 0 \quad \textit{and} \quad g_0^p - I(g^p) > 0. \tag{2.1}$$

Then the following inequality holds.

$$\begin{aligned} &\left(\left(f_0^p - I(f^p)\right)^{1/p} + \left(g_0^p - I(g^p)\right)^{1/p}\right)^p \\ &\leqslant \left[\left(f_0^p - I\left(\left|f - (f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right. \\ &\quad \left. + \left(g_0^p - I\left(\left|g - (f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p - I((f+g)^p). \end{aligned} \tag{2.2}$$

Proof. Let x_1, x_2, y_1, y_2 be nonnegative real numbers. Now from the discrete Minkowski inequality we have,

$$\left((x_1+y_1)^p + (x_2+y_2)^p\right)^{\frac{1}{p}} \leqslant \left(x_1^p + x_2^p\right)^{\frac{1}{p}} + \left(y_1^p + y_2^p\right)^{\frac{1}{p}}. \tag{2.3}$$

By applying the substitution

$$\begin{aligned} &x_1^p \to f_0^p - I(f^p), \quad y_1^p \to g_0^p - I(g^p), \\ &x_2^p \to I(f^p) - I\left(\left|f - (f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right), \\ &y_2^p \to I(g^p) - I\left(\left|g - (f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right) \end{aligned}$$

in (2.3), and by using Theorem 1.6 and (2.1), we have

$$
\begin{aligned}
&\left(\left(f_0^p - I(f^p)\right)^{1/p} + \left(g_0^p - I(g^p)\right)^{1/p}\right)^p \\
&\leqslant \left[\left(f_0^p - I\left(\left|f-(f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(g_0^p - I\left(\left|g-(f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p \\
&\quad - \left[\left(I(f^p) - I\left(\left|f-(f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{1/p}\right. \\
&\quad \left. + \left(I(g^p) - I\left(\left|g-(f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{1/p}\right]^p \\
&\leqslant \left[\left(f_0^p - I\left(\left|f-(f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(g_0^p - I\left(\left|g-(f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p - I((f+g)^p). \quad \square
\end{aligned}
$$

REMARK 2.2. Since

$$
f_0^p > I(f^p) \geqslant I\left(\left|f-(f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right) \geqslant 0,
$$

we have

$$
\left(f_0^p - I\left(\left|f-(f+g)\frac{I\left(f(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}} \leqslant f_0.
$$

Similarly

$$
\left(g_0^p - I\left(\left|g-(f+g)\frac{I\left(g(f+g)^{p-1}\right)}{I((f+g)^p)}\right|^p\right)\right)^{\frac{1}{p}} \leqslant g_0.
$$

It follows that (2.2) is a refinement of the Bellman inequality.

The following corollary is an immediate consequence of Theorem 2.1 by using the fact that Δ-integral is an isotonic linear functional.

COROLLARY 2.3. *Let $(X,\mathscr{M},\mu_\Delta)$ be a time scale measure space. For $p \geqslant 2$, assume f,g are nonnegative functions on X such that $(f+g)^p, f^p, g^p$ are Δ-integrable on X and $\int_X (f(t)+g(t))^p \Delta t > 0$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - \int_X f^p(t)\Delta t > 0 \quad \textit{and} \quad g_0^p - \int_X g^p(t)\Delta t > 0.$$

Then the following inequality holds.

$$\begin{aligned}
&\left(\left(f_0^p - \int_X f^p(t)\Delta t\right)^{1/p} + \left(g_0^p - \int_X g^p(t)\Delta t\right)^{1/p}\right)^p \qquad (2.4)\\
&\leqslant \left[\left(f_0^p - \int_X \left| f(s) - (f(s)+g(s))\frac{\int_X f(t)(f(t)+g(t))^{p-1}\Delta t}{\int_X (f(t)+g(t))^p \Delta t}\right|^p \Delta s\right)^{\frac{1}{p}}\right.\\
&\quad \left. + \left(g_0^p - \int_X \left| g(s) - (f(s)+g(s))\frac{\int_X g(t)(f(t)+g(t))^{p-1}\Delta t}{\int_X (f(t)+g(t))^p \Delta t}\right|^p \Delta s\right)^{\frac{1}{p}}\right]^p\\
&\quad - \int_X (f(t)+g(t))^p \Delta t.
\end{aligned}$$

REMARK 2.4. Corollary 2.3 is a refinement of [3, Theorem 10.2].

3. Diaz–Metcalf inequality

In the following theorem we give a refinement of the functional Diaz–Metcalf inequality (Theorem 1.4).

THEOREM 3.1. *Let L satisfy conditions (L_1), (L_2) and let I satisfy conditions (I_1), (I_2) on a nonempty set E. For $p \geqslant 2$, let $q = p/(p-1)$. Assume w, f, g are nonegative functions on E such that $wf^p, wg^q, wfg \in L$ and*

$$0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M \quad \textit{for all} \quad t \in E.$$

Then the following inequality holds.

$$\begin{aligned}
&(M-m)I(wf^p) + (mM^p - Mm^p)I(wg^q) \qquad (3.1)\\
&\leqslant (M^p - m^p)I(wfg)\\
&\quad - I\left[wg^q\left(\left(M - fg^{-\frac{q}{p}}\right)\left(fg^{-\frac{q}{p}} - m\right)^p + \left(fg^{-\frac{q}{p}} - m\right)\left(M - fg^{-\frac{q}{p}}\right)^p\right)\right].
\end{aligned}$$

Proof. The inequality (3.1) follows from Theorem 1.7 by applying the substitution

$$\Psi(x) \to x^p,\ h \to fg^{-q/p}, \quad \text{and} \quad k \to wg^q. \qquad \Box$$

The following corollary is an immediate consequence of Theorem 2.1 by using the fact that Δ-integral is an isotonic linear functional.

COROLLARY 3.2. *Let* $(X, \mathscr{M}, \mu_\Delta)$ *be a time scale measure space. For* $p \geqslant 2$*, let* $q = p/(p-1)$*. Assume* w, f, g *are nonegative functions on* X *such that* wf^p, wg^q, wfg *are* Δ*-integrable on* X *and*

$$0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M \quad \text{for all} \quad t \in X.$$

Then the following inequality holds.

$$\begin{aligned}&(M-m)\int_X w(t)f^p(t)\Delta t + (mM^p - Mm^p)\int_X w(t)g^q(t)\Delta t \\ &\leqslant (M^p - m^p)\int_X w(t)f(t)g(t)\Delta t \\ &\quad - \int_X w(t)g^q(t)\left(\left(M - f(t)g^{-\frac{q}{p}}(t)\right)\left(f(t)g^{-\frac{q}{p}}(t) - m\right)^p \right.\\ &\quad \left. + \left(f(t)g^{-\frac{q}{p}}(t) - m\right)\left(M - f(t)g^{-\frac{q}{p}}(t)\right)^p\right)\Delta t.\end{aligned} \tag{3.2}$$

REMARK 3.3. Corollary 3.2 is a refinement of [3, Theorem 11.2].

4. Aczél and Popoviciu inequalities

We first recall the refinement of Popoviciu's inequality for isotonic linear functionals as given in [5].

THEOREM 4.1. (see [5, Theorem 21]) *Let* L *satisfy conditions* (L_1), (L_2) *and let* I *satisfy conditions* (I_1), (I_2) *on a nonempty set* E*. For* $p \geqslant 2$*, let* $q = p/(p-1)$*. Assume* f, g *are nonegative functions on* E *such that*

$$f^p,\ g^q,\ fg,\ \left|g^{\frac{q}{p}}I(fg) - fI(g^q)\right|^p \in L.$$

Suppose $f_0, g_0 > 0$ *are such that*

$$f_0^p - I(f^p) > 0,\ g_0^q - I(g^q) > 0 \quad \text{and} \quad I(g^q) > 0.$$

Then the following inequality holds,

$$\begin{aligned}f_0g_0 - I(fg) &\geqslant \left[\left(f_0^p - I(f^p)\right)\left(g_0^q - I(g^q)\right)^{\frac{p}{q}} + N_p\right]^{1/p} \\ &\geqslant \left(f_0^p - I(f^p)\right)^{1/p}\left(g_0^q - I(g^q)\right)^{1/q},\end{aligned}$$

where

$$\begin{aligned}N_p = &\frac{\left(g_0^q - I(g^q)\right)^{\frac{p}{q}}}{I^p(g^q)} I\left(\left|g^{\frac{q}{p}}I(fg) - fI(g^q)\right|^p\right) \\ &+ \left|f_0 g_0^{-\frac{q}{p}} I(g^q) - I(fg)\right|^p \left(1 + \frac{\left(g_0^q - I(g^q)\right)^{\frac{p}{q}}}{I^{\frac{p}{q}}(g^q)}\right).\end{aligned}$$

Now we present the refinement of Popoviciu's inequality for Δ-integrals.

THEOREM 4.2. *Let* $(X,\mathscr{M},\mu_\Delta)$ *be a time scale measure space. For* $p \geqslant 2$*, let* $q = p/(p-1)$*. Assume* f,g *are nonnegative functions on* X *such that*

$$f^p,\ g^q,\ fg,\ \left| g^{\frac{q}{p}} \int_X f(t)g(t)\Delta t - f \int_X g^q(t)\Delta t \right|^p$$

are Δ*-integrable on* X*. Suppose* $f_0, g_0 > 0$ *are such that*

$$f_0^p - \int_X f^p(t)\Delta t > 0,\ g_0^q - \int_X g^q(t)\Delta t > 0 \quad \text{and} \quad \int_X g^q(t)\Delta t > 0.$$

Then the following inequality holds,

$$\begin{aligned} & f_0 g_0 - \int_X f(t)g(t)\Delta t \\ & \geqslant \left[\left(f_0^p - \int_X f^p(t)\Delta t \right) \left(g_0^q - \int_X g^q(t)\Delta t \right)^{\frac{p}{q}} + R_p \right]^{\frac{1}{p}} \\ & \geqslant \left(f_0^p - \int_X f^p(t)\Delta t \right)^{\frac{1}{p}} \left(g_0^q - \int_X g^q(t)\Delta t \right)^{\frac{1}{q}} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} R_p = & \frac{\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{p}{q}}}{\left(\int_X g^q(t)\Delta t\right)^p} \int_X \left| g^{\frac{q}{p}}(s) \int_X f(t)g(t)\Delta t - f(s) \int_X g^q(t)\Delta t \right|^p \Delta s \\ & + \left| f_0 g_0^{-\frac{q}{p}} \int_X g^q(t)\Delta t - \int_X f(t)g(t)\Delta t \right|^p \left(1 + \frac{\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{p}{q}}}{\left(\int_X g^q(t)\Delta t\right)^{\frac{p}{q}}} \right). \end{aligned}$$

Proof. The inequality (4.1) followe from Theorem 4.1 by using the fact that Δ-integral is an isotonic linear functional. □

REMARK 4.3. Theorem 4.2 is a refinement of [3, Theorem 9.2].

For $p = 2$, Theorem 4.2 gives the refinement of Aczél inequality on time scales ([3, Theorem 9.3]).

THEOREM 4.4. *Let* $(X,\mathscr{M},\mu_\Delta)$ *be a time scale measure space. Assume* f,g *are nonnegative functions on* X *such that*

$$f^2,\ g^2,\ fg,\ \left| g \int_X f(t)g(t)\Delta t - f \int_X g^2(t)\Delta t \right|^2$$

are Δ*-integrable on* X*. Suppose* $f_0, g_0 > 0$ *are such that*

$$f_0^2 - \int_X f^2(t)\Delta t > 0,\ g_0^2 - \int_X g^2(t)\Delta t > 0 \quad \text{and} \quad \int_X g^2(t)\Delta t > 0.$$

Then the following inequality holds,

$$\begin{aligned} f_0 g_0 - \int_X f(t)g(t)\Delta t \\ \geqslant \sqrt{\left[\left(f_0^2 - \int_X f^2(t)\Delta t\right)\left(g_0^2 - \int_X g^2(t)\Delta t\right) + R_2\right]} \\ \geqslant \sqrt{\left(f_0^2 - \int_X f^2(t)\Delta t\right)\left(g_0^2 - \int_X g^2(t)\Delta t\right)} \end{aligned} \tag{4.2}$$

where

$$R_2 = \frac{g_0^2 - \int_X g^2(t)\Delta t}{\left(\int_X g^2(t)\Delta t\right)^2} \int_X \left(g(s)\int_X f(t)g(t)\Delta t - f(s)\int_X g^2(t)\Delta t\right)^2 \Delta s$$
$$+ \left(f_0 g_0^{-1} \int_X g^2(t)\Delta t - \int_X f(t)g(t)\Delta t\right)^2 \left(1 + \frac{g_0^2 - \int_X g^2(t)\Delta t}{\int_X g^2(t)\Delta t}\right).$$

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