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ADDITIVE FUNCTIONS AND THEIR ACTIONS ON CERTAIN ELEMENTARY FUNCTIONS

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Abstract. The main aim of this note is to provide sufficient conditions for an additive function to be a real derivation. Among others the following implication will be verified: Assume that $\xi\colon \mathbb{R}\to\mathbb{R}$ is a given differentiable function and for the additive function $d\colon \mathbb{R}\to\mathbb{R}$, the mapping

$$x \longmapsto d(\xi(x)) - \xi'(x)d(x)$$

is regular (e. g. measurable, continuous, locally bounded). Then d is a sum of a derivation and a linear function.

1. Introduction

Throughout this paper $\mathbb{N}$ denotes the set of the positive integers, further $\mathbb{Z},\mathbb{Q}$, and $\mathbb{R}$ stand for the set of the integer, the set of the rational and the set of the real numbers, respectively.

The aim of this work is to prove theorems on derivations as well as on linear functions. Therefore, firstly we have to recall some definitions and auxiliary results concerning these notions.

A function $f:\mathbb{R}\to\mathbb{R}$ is called an *additive* function if,

$$f(x+y) = f(x)+f(y)$$

holds for all $x,y\in\mathbb{R}$.

We say that an additive function $f:\mathbb{R}\to\mathbb{R}$ is a *derivation* if

$$f(xy) = xf(y)+yf(x)$$

is fulfilled for all $x,y\in\mathbb{R}$.

Clearly, the identically zero function is a real derivation. It is rather difficult to give another example, since the following statements are valid concerning real derivations. If $f\colon\mathbb{R}\to\mathbb{R}$ is a real derivation, then $f(x)=0$ holds for all $x\in \mathrm{algcl}(\mathbb{Q})$ (the algebraic

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closure of the rationals). Further, if $f\colon \mathbb{R}\to\mathbb{R}$ is a real derivation and f is measurable or bounded (above or below) on a set of positive Lebesgue measure, then f is identically zero. Despite of this very pathological behavior, there exist non identically zero derivations in $\mathbb{R}$, see Kuczma [9, Theorem 14.2.2.].

The additive function $f\colon \mathbb{R}\to\mathbb{R}$ is termed to be a *linear function* if f is of the form

$$f(x) = f(1)\cdot x \qquad (x\in\mathbb{R}).$$

It is easy to see from the above definition that every derivation $f:\mathbb{R}\to\mathbb{R}$ satisfies equation

$$f(x^k) = kx^{k-1}f(x) \quad (x\in\mathbb{R}\setminus\{0\}) \tag{$*$}$$

for arbitrarily fixed $k\in\mathbb{Z}\setminus\{0\}$. Furthermore, the converse is also true, in the following sense: if $k\in\mathbb{Z}\setminus\{0,1\}$ is fixed and an additive function $f:\mathbb{R}\to\mathbb{R}$ satisfies ($*$), then f is a derivation, see e.g., Jurkat [7], Kurepa [10], and Kannappan–Kurepa [8].

Concerning linear functions, Jurkat [7] and, independently, Kurepa [10] proved that every additive function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f\left(\frac{1}{x}\right) = \frac{1}{x^2}f(x) \quad (x\in\mathbb{R}\setminus\{0\})$$

has to be linear.

In [12] A. Nishiyama and S. Horinouchi investigated additive functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the additional equation

$$f(x^n) = cx^k f(x^m) \quad (x\in\mathbb{R}\setminus\{0\}),$$

where $c\in\mathbb{R}$ and $n,m,k\in\mathbb{Z}$ are arbitrarily fixed.

Henceforth we will say that the function in question is *regular* on its domain, if at least one of the following statements are fulfilled.

(i) locally bounded;
(ii) continuous;
(iii) measurable in the sense of Lebesgue.

Concerning rational functions F. Halter-Koch and L. Reich proved similar result for derivations as well as linear functions, see [5], [4]. These results were strengthened in [3] in the following way.

THEOREM 1. *Let* $n\in\mathbb{Z}\setminus\{0\}$ *and* $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathbf{GL}_2(\mathbb{Q})$ *be such that*

– *if* $c=0$, *then* $n\neq 1$;

– *if* $d=0$, *then* $n\neq -1$.

Let further $f, g\colon \mathbb{R} \to \mathbb{R}$ *be additive functions and define the function* ϕ *by*

$$\phi(x) = f\left(\frac{ax^n + b}{cx^n + d}\right) - \frac{x^{n-1}g(x)}{(cx^n + d)^2} \qquad (x \in \mathbb{R},\ cx^n + d \neq 0).$$

Let us assume ϕ *to be regular. Then, the functions* $F, G\colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(x) = f(x) - f(1)x \quad \text{and} \quad G(x) = g(x) - g(1)x \quad (x \in \mathbb{R})$$

are derivations.

Roughly speaking the above cited papers dealt with a special case of the following problem. Assume that $\xi\colon \mathbb{R} \to \mathbb{R}$ is a given differentiable function and for the additive function $d\colon \mathbb{R} \to \mathbb{R}$, the mapping

$$x \longmapsto d(\xi(x)) - \xi'(x)d(x)$$

is regular on its domain. It is true that in case d admits a representation

$$d(x) = \chi(x) + d(1) \cdot x \quad (x \in \mathbb{R}),$$

where $\chi\colon \mathbb{R} \to \mathbb{R}$ is a real derivation?

In view of the above results, in case $n \in \mathbb{Z} \setminus \{0\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Q})$ and the function ξ is

$$\xi(x) = \frac{ax^n + b}{cx^n + d} \quad (x \in \mathbb{R},\ cx^n + d \neq 0),$$

then the answer is *affirmative*. The main aim of this note is to extend this result to other classes of elementary functions such as the exponential function, the logarithm function, the trigonometric functions and the hyperbolic functions. Concerning such type of investigations, we have to remark the paper of Gy. Maksa (see [11]), where the previous problem was investigated under the supposition that the mapping

$$x \longmapsto d(\xi(x)) - \xi'(x)d(x)$$

is identically zero.

2. The main result

Our main result is contained in the following.

THEOREM 2. *Assume that for the additive function* $d\colon \mathbb{R} \to \mathbb{R}$ *the mapping* φ *defined by*

$$\varphi(x) = d(\xi(x)) - \xi'(x)d(x)$$

is regular. Then the function d *can be represented as*

$$d(x) = \chi(x) + d(1) \cdot x \quad (x \in \mathbb{R}),$$

where $\chi\colon \mathbb{R} \to \mathbb{R}$ *is a derivation, in any of the following cases* *(a)* $\xi(x) = a^x$ *(c)* $\xi(x) = \sin(x)$ *(e)* $\xi(x) = \sinh(x)$.

(b) $\xi(x) = \cos(x)$ *(d)* $\xi(x) = \cosh(x)$

Proof.

Case (a) Let $a \in \mathbb{R} \setminus \{1\}$ be an arbitrary positive real number and suppose that the mapping φ defined by

$$\varphi(x) = d(a^x) - a^x \ln(a) d(x) \quad (x \in \mathbb{R})$$

is regular. A easy calculation shows that

$$\varphi(2x) - 2a^x \varphi(x) = d\left((a^x)^2\right) - 2a^x d(x) \quad (x \in \mathbb{R}),$$

that is

$$\varphi(2\log_a(u)) - 2u\varphi(\log_a(u)) = d(u^2) - 2ud(u) \quad (u \in]0, +\infty[).$$

Due to the regularity of the function φ, the mapping

$$]0, +\infty[\ni u \longmapsto \varphi(2\log_a(u)) - 2u\varphi(\log_a(u))$$

is regular, too. Thus by Theorem 1,

$$d(x) = \chi(x) + d(1) \cdot x \qquad (x \in \mathbb{R}),$$

where the function $\chi \colon \mathbb{R} \to \mathbb{R}$ is a derivation.

Case (b) Assume now that for the additive function $d \colon \mathbb{R} \to \mathbb{R}$, the mapping φ defined on $\mathbb{R}$ by

$$\varphi(x) = d(\cos(x)) + \sin(x) d(x) \qquad (x \in \mathbb{R})$$

is regular. If so, then

$$\frac{\varphi(2x) - 4\cos(x)\varphi(x) + d(1)}{2} = d\left(\cos^2(x)\right) - 2\cos(x) f(x)$$

holds for all $x \in \mathbb{R}$. Let now $u \in]-1, 1[$ and write $\arccos(u)$ in place of x to get

$$\frac{\varphi(2\arccos(u)) - 4u\varphi(\arccos(u)) + d(1)}{2} = d(u^2) - 2ud(u).$$

Again, due to the regularity of the function φ, the mapping

$$]-1, 1[\ni u \longmapsto \frac{\varphi(2\arccos(u)) - 4u\varphi(\arccos(u)) + d(1)}{2}$$

is regular, as well. Therefore, Theorem 1 again implies that

$$d(x) = \chi(x) + d(1) \cdot x \qquad (x \in \mathbb{R}),$$

is fulfilled with a certain real derivation $\chi\colon \mathbb{R}\to\mathbb{R}$.

Case (c) Suppose that for the additive function d, the mapping

$$\varphi(x) = d(\sin(x)) - \cos(x)d(x) \qquad (x\in\mathbb{R})$$

is regular. In this case

$$\begin{aligned}\varphi\left(x-\frac{\pi}{2}\right) &= d\left(\sin\left(x-\frac{\pi}{2}\right)\right) - \cos\left(x-\frac{\pi}{2}\right) d\left(x-\frac{\pi}{2}\right)\\ &\quad -d(\cos(x)) - \sin(x)d(x) + \sin(x)d\left(\frac{\pi}{2}\right),\end{aligned}$$

that is,

$$-\varphi\left(x-\frac{\pi}{2}\right) + \sin(x)d\left(\frac{\pi}{2}\right) = d(\cos(x)) + \sin(x)d(x) \qquad (x\in\mathbb{R}).$$

In view of Case (b) this yields that the function d has the desired representation as stated.

Case (d) Assume the $d\colon \mathbb{R}\to\mathbb{R}$ is an additive function and the mapping

$$\varphi(x) = d(\cosh(x)) - \sinh(x)d(x) \qquad (x\in\mathbb{R})$$

is regular. The additivity of d and some addition formula of the $\cosh$ function furnish

$$\frac{\varphi(2x) - 4\cosh(x)\varphi(x) + d(1)}{2} = d\left(\cosh^2(x)\right) - 2\cosh(x)d(\cosh(x)) \quad (x\in\mathbb{R}).$$

Let now $u\in]1,+\infty[$ arbitrary and put $x=\operatorname{arcosh}(u)$ into the previous identity to get

$$\frac{\varphi(2\operatorname{arcosh}(u)) - 4u\varphi(\operatorname{arcosh}(u)) + d(1)}{2} = d(u^2) - 2ud(u).$$

Since the function φ is regular, the mapping

$$]1,+\infty[\ni u \longmapsto \frac{\varphi(2\operatorname{arcosh}(u)) - 4u\varphi(\operatorname{arcosh}(u)) + d(1)}{2}$$

will also be regular. Therefore, Theorem 1 implies again the desired decomposition of the function d.

Case (e) Finally, assume the $d\colon \mathbb{R}\to\mathbb{R}$ is an additive function so that

$$\varphi(x) = d(\sinh(x)) - \cosh(x)d(x) \qquad (x\in\mathbb{R})$$

is regular. Let $x,y\in\mathbb{R}$ be arbitrary, then

$$\begin{aligned}\varphi(x+y) &= d(\sinh(x+y)) - \cosh(x+y)d(x+y)\\ &= d(\sinh(x)\cosh(y)) + d(\sinh(y)\cosh(x))\\ &\quad - [\sinh(x)\sinh(y) + \cosh(x)\cosh(y)]\, d(x+y)\\ &= d(\sinh(x)\cosh(y)) + d(\sinh(y)\cosh(x)) - \sinh(x)\sinh(y)d(x+y)\\ &\quad - \cosh(x)d(x)\cosh(y) - \cosh(x)\cosh(y)d(y)\end{aligned}$$

If we use the definition of the function φ, after some rearrangement, we arrive at

$$\begin{aligned} &\varphi(x+y)-\varphi(x)\cosh(y)-\varphi(y)\cosh(x) \\ =\ & d\left(\sinh(x)\cosh(y)\right)+d\left(\sinh(y)\cosh(x)\right)-\sinh(x)\sinh(y)d(x+y) \\ &-\cosh(y)d\left(\sinh(x)\right)-\cosh(x)d\left(\sinh(y)\right) \end{aligned}$$

for all $x,y\in\mathbb{R}$. If we replace here y by $-y$,

$$\begin{aligned} &\varphi(x-y)-\varphi(x)\cosh(y)-\varphi(-y)\cosh(x) \\ =\ & d\left(\sinh(x)\cosh(y)\right)-d\left(\sinh(y)\cosh(x)\right)+\sinh(x)\sinh(y)d(x-y) \\ &-\cosh(y)d\left(\sinh(x)\right)+\cosh(x)d\left(\sinh(y)\right) \end{aligned}$$

can be concluded, where we have also used that the function cosh is even and the function sinh is odd. Adding this two identities side by side,

$$\begin{aligned} \Phi(x,y) =\ & 2d\left(\sinh(x)\cosh(y)\right)+\sinh(x)\sinh(x)\left[d(x-y)-d(x+y)\right] \\ &-2\cosh(y)d\left(\sinh(x)\right) \end{aligned}$$

for any $x,y\in\mathbb{R}$, where

$$\begin{aligned} \Phi(x,y) =\ & \varphi(x+y)-\varphi(x)\cosh(y)-\varphi(y)\cosh(x) \\ &+\varphi(x-y)-\varphi(x)\cosh(y)-\varphi(-y)\cosh(x) \quad (x,y\in\mathbb{R}). \end{aligned}$$

If we put $x=\operatorname{arsinh}(1)$, we get that

$$\frac{\Phi\left(\operatorname{arsinh}(1),y\right)+2\cosh(y)d(1)}{2}=d\left(\cosh(y)\right)-\sinh(y)d(y) \quad (y\in\mathbb{R}).$$

Due to the regularity of the function φ, the mapping

$$\mathbb{R}\ni y\longmapsto\frac{\Phi\left(\operatorname{arsinh}(1),y\right)+2\cosh(y)d(1)}{2}$$

is regular, too. Hence, Case (d) yields the desired form of the function d. □

In what follows, we would like to extend the list of the functions appearing in the previous statement. Therefore we prove the following.

LEMMA 1. *Let $d\colon\mathbb{R}\to\mathbb{R}$ be an additive function, $I\subset\mathbb{R}$ be a nonvoid open interval and $\xi: I\to\mathbb{R}$ be a continuously differentiable function so that the derivative of the function $\xi^{-1}\colon \xi(I)\to\mathbb{R}$ is nowhere zero. The mapping*

$$I\ni x\longmapsto d(\xi(x))-\xi'(x)d(x)$$

is regular if and only if the mapping

$$\xi(I)\ni u\longmapsto d(\eta(u))-\eta'(u)d(u)$$

is regular, where $\eta=\xi^{-1}$.

Proof. Assume that for the additive function d, we have that the mapping

$$\varphi(x) = d(\xi(x)) - \xi'(x)d(x) \qquad (x \in I)$$

is regular. Let now $u \in \xi(I)$ and put $\xi^{-1}(u)$ in place of x to get

$$-\left(\xi^{-1}\right)'(u)\varphi(\xi^{-1}(u)) = d\left(\xi^{-1}(u)\right) - \left(\xi^{-1}\right)'(u)d(u).$$

Due to the regularity of φ, the mapping appearing in the left hand side is also regular, as stated. □

In view of Theorem 2 and Lemma 1, we immediately obtain the following theorem.

COROLLARY 1. *Assume that for the additive function $d\colon \mathbb{R} \to \mathbb{R}$ the mapping φ defined by*

$$\varphi(x) = d(\xi(x)) - \xi'(x)d(x)$$

is regular. Then the function d can be represented as

$$d(x) = \chi(x) + d(1)\cdot x \quad (x \in \mathbb{R}),$$

where $\chi\colon \mathbb{R} \to \mathbb{R}$ is a derivation, in any of the following cases

(a) $\xi(x) = \ln(x)$

(b) $\xi(x) = \arccos(x)$

(c) $\xi(x) = \arcsin(x)$

(d) $\xi(x) = \operatorname{arcosh}(x)$

(e) $\xi(x) = \operatorname{arsinh}(x)$.

3. Stability of derivations

As a starting point of the proof of the main result of this section the theorem of Hyers will be used. Originally this statement was formulated in terms of functions that are acting between Banach spaces, see Hyers [6]. However, we will use this theorem only in the particular case when the domain and the range are the set of reals. In this setting we have the following.

THEOREM 3. *Let $\varepsilon \geqslant 0$ and suppose that the function $f:\mathbb{R} \to \mathbb{R}$ fulfills the inequality*

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon$$

for all $x, y \in \mathbb{R}$. Then there exists an additive function $a:\mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - a(x)| \leqslant \varepsilon$$

holds for arbitrary $x \in \mathbb{R}$.

In other words, Hyers' theorem states that if a function $f\colon \mathbb{R}\to\mathbb{R}$ fulfills the inequality appearing above, then it can be represented as

$$f(x)=a(x)+b(x) \qquad (x\in\mathbb{R}),$$

where $a\colon \mathbb{R}\to\mathbb{R}$ is an additive and $b\colon \mathbb{R}\to\mathbb{R}$ is a bounded function. Moreover, for all $x\in\mathbb{R}$, we also have $|b(x)|\leqslant\varepsilon$.

With the aid of Hyers' theorem and the results of the previous section, the following stability type result can be proved. Concerning stability properties of derivations the interested reader may consult Badora [1] and Boros–Gselmann [2].

THEOREM 4. *Let $\varepsilon>0$ be arbitrarily fixed, $f\colon \mathbb{R}\to\mathbb{R}$ be a function and suppose that*

(A) for all $x,y\in\mathbb{R}$ we have

$$|f(x+y)-f(x)-f(y)|\leqslant\varepsilon.$$

(B) the mapping

$$x\longmapsto f(\xi(x))-\xi'(x)f(x)$$

is locally bounded on its domain, where the function ξ is one of the functions

(a) a^x
(b) $\cos(x)$
(c) $\sin(x)$
(d) $\cosh(x)$
(e) $\sinh(x)$
(f) $\ln(x)$
(g) $\arccos(x)$
(h) $\arcsin(x)$
(i) $\operatorname{arcosh}(x)$
(j) $\operatorname{arsinh}(x)$.

Then there exist $\lambda\in\mathbb{R}$ and a real derivation $\chi\colon \mathbb{R}\to\mathbb{R}$ such that

$$|f(x)-[\chi(x)+\lambda\cdot x]|\leqslant\varepsilon$$

holds for all $x\in\mathbb{R}$.

Proof. Due to assumption (A), we immediately have that

$$f(x)=a(x)+b(x) \qquad (x\in\mathbb{R}),$$

where $a\colon \mathbb{R}\to\mathbb{R}$ is an additive and $b\colon \mathbb{R}\to\mathbb{R}$ is a bounded function. If we use supposition (B), from this we get that the mapping

$$x\longmapsto \left[a(\xi(x))-\xi'(x)a(x)\right]+\left[b(\xi(x))-\xi'(x)b(x)\right]$$

is locally bounded. From this however the local boundedness of the function

$$x\longmapsto a(\xi(x))-\xi'(x)a(x)$$

can be deduced. In view of the previous statements (see Theorem 2 and Corollary 1),

$$a(x) = \chi(x) + a(1) \cdot x \qquad (x \in \mathbb{R})$$

is fulfilled for any $x \in \mathbb{R}$, where $\chi : \mathbb{R} \to \mathbb{R}$ is a certain real derivation. For the function f this means that there exists $\lambda \in \mathbb{R}$ and a real derivation $\chi : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \chi(x) + \lambda \cdot x + b(x) \qquad (x \in \mathbb{R}),$$

or equivalently

$$|f(x) - [\chi(x) + \lambda \cdot x]| \leqslant \varepsilon$$

is satisfied for any $x \in \mathbb{R}$. □

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