

RELATIVE SCHUR–CONVEXITY ON GLOBAL NPC SPACES

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*To Josip Pečarić,
on the occasion of publishing more than
1000 scientific mathematical papers*

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Abstract. We introduce the concept of relative convexity on spaces with global nonpositive curvature and illustrate its usefulness by a number of inequalities involving the convex functions on such spaces.

1. Introduction

The concept of Schur convexity plays a major role in the study of majorization. A nice account is offered by the recent book of Marshal, Olkin and Arnold [10]. See also [14].

The aim of our paper is to discuss this concept within a class of spaces with curved geometry that verifies the Semiparallelogram Law (a weaker form of Apollonius’ theorem relating the length of a median of a triangle to the lengths of its sides).

DEFINITION 1. A global NPC space is a complete metric space $M = (M, d)$ for which the following inequality holds true: for every pair of points $x_0, x_1 \in M$ there exists a point $y \in M$ such that for all points $z \in M$,

$$d^2(z, y) \leq \frac{1}{2}d^2(z, x_0) + \frac{1}{2}d^2(z, x_1) - \frac{1}{4}d^2(x_0, x_1). \quad (1)$$

Here “NPC” stands for “nonpositive curvature”. Global NPC spaces are also known as CAT(0) spaces or Hadamard spaces. In what follows we briefly review a number of basic facts on which our paper is based. For more details, the interested reader may consult the excellent survey of Sturm [15] (and also the books of Ballman [1], Bridson and Haefliger [4], and Jost [8]).

In a global NPC space, each pair of points $x_0, x_1 \in M$ can be connected by a geodesic (that is, by a rectifiable curve $\gamma: [0, 1] \rightarrow M$ such that the length of $\gamma|_{[s, t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \leq s \leq t \leq 1$). Moreover, this geodesic is unique.

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The point y that appears in Definition 1 is the *midpoint* of x_0 and x_1 and has the property

$$d(x_0, y) = d(y, x_1) = \frac{1}{2}d(x_0, x_1).$$

We will denote it as $\frac{1}{2}x_0 \boxplus \frac{1}{2}x_1$. The general convex combinations of x_0 and x_1 can be introduced by using the argument of the minimum of a suitable objective function:

$$(1 - \lambda)x_0 \boxplus \lambda x_1 = \arg \min_{z \in M} [(1 - \lambda)d^2(x_0, z) + \lambda d^2(x_1, z)]. \tag{2}$$

See Bhatia [2], Proposition 6.2.8, for the case $\lambda = 1/2$. Here, an important role is played by the inequality (1), which assures the uniform convexity of the square distance.

Every Hilbert space is a global NPC space. Its geodesics are the line segments and $\frac{1}{2}x_0 \boxplus \frac{1}{2}x_1 = \frac{x_0+x_1}{2}$.

A more sophisticated example is provided by the upper half-plane

$$\mathbf{H} = \{z \in \mathbb{C} : \text{Im} z > 0\},$$

when endowed with the Poincaré metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

In this case the geodesics are the semicircles in \mathbf{H} perpendicular to the real axis and the straight vertical lines ending on the real axis.

The space $\text{Sym}^{++}(n, \mathbb{R})$, of all positively definite matrices with real coefficients becomes a global NPC space when endowed with the *trace metric*,

$$d_{\text{trace}}(A, B) = \left(\sum_{k=1}^n \log^2 \lambda_k \right)^{1/2},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of AB^{-1} . In this case the curve

$$\gamma_{AB} : [0, 1] \rightarrow \text{Sym}^{++}(n, \mathbb{R}), \quad \gamma_{AB}(t) = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^t A^{1/2}$$

is the unique minimal geodesic (up to parametrization) joining A and B (so that the midpoint of A and B is in this case $\frac{1}{2}A \boxplus \frac{1}{2}B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$).

Notice that $\text{Sym}^{++}(n, \mathbb{R})$ is also a Riemannian manifold. In general, a Riemannian manifold is a global NPC space if and only if it is complete, simply connected and of nonpositive sectional curvature. Besides manifolds, other important examples of global NPC spaces are the Bruhat-Tits buildings (in particular, the trees). See [4].

DEFINITION 2. A set $C \subset M$ is called *convex* if $\gamma([0, 1]) \subset C$ for each geodesic $\gamma : [0, 1] \rightarrow M$ joining the points $\gamma(0), \gamma(1) \in C$.

A function $\varphi : C \rightarrow \mathbb{R}$ is called *convex* if C is a convex set and for each geodesic $\gamma : [0, 1] \rightarrow C$ the composition $\varphi \circ \gamma$ is a convex function in the usual sense, that is,

$$\varphi(\gamma(t)) \leq (1 - t)\varphi(\gamma(0)) + t\varphi(\gamma(1))$$

for all $t \in [0, 1]$.

The function φ is called concave if $-\varphi$ is convex.

In a global NPC space $M = (M, d)$, the distance function d is convex on $M \times M$, while the functions $d^\alpha(\cdot, z)$, with $\alpha \geq 1$, are convex on M . See Sturm [15], Corollary 2.5, for details.

The determinant function is log-convex on $\text{Sym}^{++}(n, \mathbb{R})$ due to the identity

$$\det A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^t A^{1/2} = (\det A)^{1-t} (\det B)^t.$$

The same is true for the trace function. See [12], Theorem 2.

It is worth mentioning that Jensen’s inequality works in the context of global NPC spaces (despite the fact that the property of associativity of convex combinations fails). The basic ingredient, the *barycenter* of a discrete probability measures $\lambda = \sum_{i=1}^n \lambda_i \delta_{x_i}$ is defined by the formula

$$\text{bar}(\lambda) = \arg \min_{z \in M} \frac{1}{2} \sum_{i=1}^n \lambda_i d^2(z, x_i).$$

In the case of Hilbert spaces, this coincides with the usual definition of barycenter in flat spaces, that is, $\sum_{i=1}^n \lambda_i x_i$.

THEOREM 1. (The discrete form of Jensen’s Inequality) *For every continuous convex function $f : M \rightarrow \mathbb{R}$ and every discrete probability measure $\lambda = \sum_{i=1}^n \lambda_i \delta_{x_i}$ on M , we have the inequality*

$$f(\text{bar}(\lambda)) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

The result of Theorem 1 is a particular case of the integral form of Jensen’s Inequality, which was first noticed by Jost [7] (and later extended by Eells and Fuglede [5]). A probabilistic approach is made available by the paper of Sturm [15].

2. The Hardy-Littlewood-Pólya theorem of majorization in global NPC spaces

In what follows we shall deal with the relation of weighted majorization \prec , for pairs of discrete probability measures. In the context of Euclidean space \mathbb{R}^N ,

$$\sum_{i=1}^m \lambda_i \delta_{x_i} \prec \sum_{j=1}^n \mu_j \delta_{y_j} \tag{3}$$

means the existence of a $m \times n$ -dimensional matrix $A = (a_{ij})_{i,j}$ such that the following four conditions are fulfilled:

$$a_{ij} \geq 0, \text{ for all } i, j \tag{4}$$

$$\sum_{j=1}^n a_{ij} = 1, \quad i = 1, \dots, m \tag{5}$$

$$\mu_j = \sum_{i=1}^m a_{ij} \lambda_i, \quad j = 1, \dots, n \tag{6}$$

and

$$x_i = \sum_{j=1}^n a_{ij} y_j, \quad i = 1, \dots, m. \tag{7}$$

See Borcea [3] and Marshal, Olkin and Arnold [10]. The matrices verifying the conditions (4) and (5) are called *stochastic on rows*. When $m = n$ and all weights λ_i and μ_j are equal to $1/n$, the condition (6) assures the *stochasticity on columns*, so in that case we deal with doubly stochastic matrices.

REMARK 1. A well known result due to Hardy, Littlewood and Pólya [6] (see also [10], [11] and [14]), asserts that in case $M = \mathbb{R}$, the relation of majorization

$$\frac{1}{n} \sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$$

can be characterized in terms of the supports of the two measures. More precisely, if one denotes by $z_1^\downarrow \geq \dots \geq z_n^\downarrow$ the decreasing rearrangement of the family z_1, \dots, z_n of real numbers, then $\frac{1}{n} \sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$ is equivalent to the fulfillment of the conditions (8) and (9) below:

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow \quad \text{for } k = 1, \dots, n \tag{8}$$

and

$$\sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow. \tag{9}$$

It is worth noticing that the concept of weighted majorization in \mathbb{R}^N is related, via equation (7), to an optimization problem. Indeed,

$$x_i = \operatorname{argmin}_{z \in \mathbb{R}^N} \frac{1}{2} \sum_{j=1}^n a_{ij} \|z - y_j\|^2, \quad \text{for } i = 1, \dots, m.$$

When $x_1, \dots, x_m, y_1, \dots, y_n$ are points in a global NPC space M , and $\lambda_1, \dots, \lambda_m$ in $[0, 1]$ are weights that sum to 1, we will define the relation of majorization

$$\sum_{i=1}^m \lambda_i \delta_{x_i} \prec \sum_{j=1}^n \mu_j \delta_{y_j} \tag{10}$$

by asking the existence of an $m \times n$ -dimensional matrix $A = (a_{ij})_{i,j}$ that is stochastic on rows and verifies in addition the following two conditions:

$$\mu_j = \sum_{i=1}^m a_{ij} \lambda_i, \quad j = 1, \dots, n \tag{11}$$

and

$$x_i = \arg \min_{z \in M} \frac{1}{2} \sum_{j=1}^n a_{ij} d^2(z, y_j), \quad i = 1, \dots, m. \tag{12}$$

The existence and uniqueness of the optimization problems (12) is assured by the fact that the objective functions are uniformly convex and positive. See Jost [8], Section 3.1, or Sturm [15], Proposition 1.7, p. 3.

According to our definition,

$$\delta_{\text{bar}(\lambda)} \prec \lambda$$

for every discrete Borel probability measure λ .

The following theorem proved by us in [12] offers a partial extension of the Hardy-Littlewood-Pólya Theorem to the context of global NPC spaces.

THEOREM 2. *If*

$$\sum_{i=1}^m \lambda_i \delta_{x_i} \prec \sum_{j=1}^n \mu_j \delta_{y_j}$$

in the global NPC space M , then for every real-valued continuous convex function f defined on a convex subset $U \subset M$ that contains all points x_i and y_j we have

$$\sum_{i=1}^m \lambda_i f(x_i) \leq \sum_{j=1}^n \mu_j f(y_j).$$

Theorem 2 implies the following result that relates the property of convexity to that of Schur convexity.

COROLLARY 1. (Lim [9] and Niculescu and Roventă [12]) *If $\frac{1}{n} \sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$ in the global NPC space M and $f : M^n \rightarrow \mathbb{R}$ is a continuous convex function invariant under the permutation of coordinates, then*

$$f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n).$$

A function $f : M^n \rightarrow \mathbb{R}$ is called *Schur convex* if $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ whenever $\frac{1}{n} \sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$. According to Corollary 1, every continuous convex function invariant under permutations is Schur convex. The book of Marshall, Olkin and Arnold [10] contains a wealth of information on Schur convex functions and their applications in the case of flat spaces.

In the next section we shall discuss the existence of points of convexity (respectively of Schur convexity) for functions not necessarily convex and will extend to that context both Theorem 2 and Corollary 1.

3. Relative Schur-convexity on global NPC spaces

In a recent paper [13] devoted to the availability of Jensen’s inequality in a certain nonconvex context, we outlined the usefulness of the concept of point of convexity. Here we will consider the case of spaces with a curved geometry.

In what follows M is a global NPC space and $f : M \rightarrow \mathbb{R}$ and $F : M^n \rightarrow \mathbb{R}$ are continuous functions.

DEFINITION 3. A point $a \in M$ is a *point of convexity* of the function f if

$$f(a) \leq \sum_{i=1}^n \lambda_i f(x_i), \tag{J}$$

for every family of points x_1, \dots, x_n in M and every family of positive weights $\lambda_1, \dots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 1$ and $\text{bar}(\sum_{i=1}^n \lambda_i \delta_{x_i}) = a$.

The point a is a *point of concavity* if it is a point of convexity for $-f$ (equivalently, if the inequality (J) works in the reversed way).

DEFINITION 4. A point $(a_1, \dots, a_n) \in M^n$ is a *point of Schur-convexity* of the function $F : M^n \rightarrow \mathbb{R}$ if

$$F(a_1, \dots, a_n) \leq F(x_1, \dots, x_n) \tag{13}$$

for every $(x_1, \dots, x_n) \in M^n$ such that $\frac{1}{n} \sum_{i=1}^n \delta_{a_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$.

The point x is a *point of Schur-concavity* if it is a point of Schur-convexity for $-f$ (equivalently, if the above inequality works in the reversed way).

A simple example illustrating the Definitions 3 and 4 is offered by the function $f(x) = xe^x$. This function is concave on $(-\infty, -2]$ and convex on $[-2, \infty)$ (attaining a global minimum at $x = -1$). Every point $a \geq -1$ is a point of convexity because the tangent line $y = L(x)$ at the point (a, ae^a) is a supporting line for the graph. Indeed,

$$f(a) = L(a) = L\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i L(x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

whenever $x_1, \dots, x_n \in \mathbb{R}$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$ verify the conditions $\sum_{i=1}^n \lambda_i = 1$ and $a = \sum_{i=1}^n \lambda_i x_i \geq -1$.

The next result shows that every n -tuple $(a_1, \dots, a_n) \in [-1, \infty)^n$ is a point of Schur-convexity for the function

$$F(x_1, \dots, x_n) = \sum_{i=1}^n x_i e^{x_i}, \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

THEOREM 3. Let $a_1, \dots, a_n \in M$ be points of convexity of the function f . Then

$$\sum_{i=1}^m \lambda_i \delta_{a_i} \prec \sum_{j=1}^n \mu_j \delta_{x_j} \text{ implies } \sum_{i=1}^m \lambda_i f(a_i) \leq \sum_{j=1}^n \mu_j f(x_j).$$

In particular, $(a_{\pi(1)}, \dots, a_{\pi(n)})$ is a point of Schur-convexity of the function

$$F(x_1, \dots, x_n) = \sum_{k=1}^n f(x_k),$$

whenever π is a permutation of $\{1, \dots, n\}$.

Proof. If $\sum_{i=1}^m \lambda_i \delta_{a_i} \prec \sum_{j=1}^n \mu_j \delta_{x_j}$, then there exists an $m \times n$ -dimensional matrix $A = (a_{ij})_{i,j}$ which is stochastic on rows and verifies the conditions (11) and (12). The last condition shows that each point a_i is the barycenter of the probability measure $\sum_{j=1}^n a_{ij} \delta_{x_j}$, so by Jensen's inequality we infer that

$$f(a_i) \leq \sum_{j=1}^n a_{ij} f(x_j).$$

Multiplying each side by λ_i and then summing up over i from 1 to m , we conclude that

$$\sum_{i=1}^m \lambda_i f(a_i) \leq \sum_{i=1}^m \left(\lambda_i \sum_{j=1}^n a_{ij} f(x_j) \right) = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \lambda_i \right) f(x_j) = \sum_{j=1}^n \mu_j f(x_j)$$

and the proof is done. \square

The Schur companion of Theorem 3 is as follows.

THEOREM 4. *Suppose that $F : M^n \rightarrow \mathbb{R}$ is a continuous function invariant under the permutation of coordinates. Then every point of convexity of the function F is also a point of Schur-convexity.*

Proof. For the sake of simplicity we will restrict here to the case where $n = 3$. Suppose that (a_1, a_2, a_3) is a point of convexity of the function F and

$$\frac{1}{3} \sum_{i=1}^3 \delta_{a_i} \prec \frac{1}{3} \sum_{i=1}^3 \delta_{x_i}.$$

According to the definition of majorization, there exists a doubly stochastic matrix $A = (a_{ij})_{i,j=1}^3$ such that

$$a_i = \text{bar} \left(\sum_{j=1}^3 a_{ij} \delta_{x_j} \right) \quad \text{for } i = 1, 2, 3.$$

As A can be uniquely represented under the form

$$A = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_3 + \lambda_5 & \lambda_4 + \lambda_6 \\ \lambda_3 + \lambda_4 & \lambda_1 + \lambda_6 & \lambda_2 + \lambda_5 \\ \lambda_5 + \lambda_6 & \lambda_2 + \lambda_4 & \lambda_1 + \lambda_3 \end{pmatrix},$$

where all λ_k are nonnegative and $\sum_{k=1}^6 \lambda_k = 1$ (a simple matter of linear algebra) we can represent the elements a_j as

$$\begin{aligned} a_1 &= \text{bar}((\lambda_1 + \lambda_2) \delta_{x_1} + (\lambda_3 + \lambda_4) \delta_{x_2} + (\lambda_5 + \lambda_6) \delta_{x_3}), \\ a_2 &= \text{bar}((\lambda_3 + \lambda_5) \delta_{x_1} + (\lambda_1 + \lambda_6) \delta_{x_2} + (\lambda_2 + \lambda_4) \delta_{x_3}), \\ a_3 &= \text{bar}((\lambda_4 + \lambda_6) \delta_{x_1} + (\lambda_2 + \lambda_5) \delta_{x_2} + (\lambda_1 + \lambda_3) \delta_{x_3}). \end{aligned}$$

It is easy to see that (a_1, a_2, a_3) is the barycenter of the measure

$$\begin{aligned} \mu &= \lambda_1 \delta_{(x_1, x_2, x_3)} + \lambda_2 \delta_{(x_1, x_3, x_2)} + \lambda_3 \delta_{(x_2, x_1, x_3)} \\ &\quad + \lambda_4 \delta_{(x_2, x_3, x_1)} + \lambda_5 \delta_{(x_3, x_1, x_2)} + \lambda_6 \delta_{(x_3, x_2, x_1)}. \end{aligned}$$

Since (a_1, a_2, a_3) is a point of convexity of F and F is invariant under permutations we conclude that

$$\begin{aligned} F(a_1, a_2, a_3) = F(b_\mu) &\leq \lambda_1 F(x_1, x_2, x_3) + \lambda_2 F(x_1, x_3, x_2) + \lambda_3 F(x_2, x_1, x_3) \\ &\quad + \lambda_4 F(x_2, x_3, x_1) + \lambda_5 F(x_3, x_1, x_2) + \lambda_6 F(x_3, x_2, x_1) \\ &= (\lambda_1 + \dots + \lambda_6) F(x_1, x_2, x_3) = F(x_1, x_2, x_3). \end{aligned}$$

The proof is done. \square

Theorem 3 and Theorem 4 yield numerous new inequalities through the following consequence of them:

COROLLARY 2. *If $f : M \rightarrow \mathbb{R}$ is a continuous convex function and φ is a continuous nondecreasing convex function defined on an interval including the range of f , then*

$$\frac{1}{n} \sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{y_i} \text{ in } M \text{ implies } \sum_{i=1}^n \varphi(f(x_i)) \leq \sum_{i=1}^n \varphi(f(y_i)).$$

The last inequality works in the reverse way when φ is concave and nonincreasing.

The function $h(t) = \sqrt{t} \log t$ is convex on $(0, 1]$ and concave on $[1, \infty)$. Moreover, it is increasing on the interval $[e^{-2}, \infty)$. The tangent line at $x = e^{-1}$ to the graph of h intersects the graph again at $(c^*, h(c^*))$, where c^* is the solution of the equation $2\sqrt{ex} \log x - ex + 3 = 0$ lying in the interval $(9, 10)$. All points $t \in [e^{-2}, e^{-1}]$ are points of convexity for the restriction of h to $[e^{-2}, c^*]$. As a consequence, all points of $\{A \in \text{Sym}^{++}(n, \mathbb{R}) : \det A \in [e^{-2}, e^{-1}]\}$ are points of convexity for the restriction of $h \circ \det$ to the set $\{A \in \text{Sym}^{++}(n, \mathbb{R}) : \det A \in [e^{-2}, c^*]\}$.

According to Corollary 2, if $A_1, \dots, A_n, B_1, \dots, B_n$ are matrices in $\text{Sym}^{++}(n, \mathbb{R})$ such that

- $\frac{1}{n} \sum_{i=1}^n \delta_{A_i} \prec \frac{1}{n} \sum_{i=1}^n \delta_{B_i}$ in $\text{Sym}^{++}(n, \mathbb{R})$ and
- $\det A_i \in [e^{-2}, e^{-1}]$ and $\det B_i \in [e^{-2}, c^*]$ for all i ,

then

$$\sum_{i=1}^n \sqrt{\det A_i} \log \det A_i \leq \sum_{i=1}^n \sqrt{\det B_i} \log \det B_i$$

that is,

$$\prod_{i=1}^n (\det A_i)^{\sqrt{\det A_i}} \leq \prod_{i=1}^n (\det B_i)^{\sqrt{\det B_i}}.$$

Under the same hypotheses,

$$\prod_{i=1}^n (\operatorname{trace} A_i)^{\sqrt{\operatorname{trace} A_i}} \leq \prod_{i=1}^n (\operatorname{trace} B_i)^{\sqrt{\operatorname{trace} B_i}}.$$

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