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ON BACKWARD ALUTHGE ITERATES OF HYPONORMAL OPERATORS

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Abstract. In this paper we study several remarkable properties of the backward Aluthge iterates of a hyponormal operator. In particular, we show that, under suitable conditions, operators in $BAIH(k)$ admit a moment sequence and have nontrivial hyperinvariant subspaces.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the uniquely determined partial isometry satisfying $ker(U) = ker(|T|) = ker(T)$ and $ker(U^*) = ker(T^*)$. Associated with T, an operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the Aluthge transform of T, denoted throughout this paper by $\tilde{T}$. For an arbitrary T in $\mathcal{L}(\mathcal{H})$, the sequence $\{\tilde{T}^{(n)}\}$ of the Aluthge iterates of T is defined by $\tilde{T}^{(0)} = T$ and $\tilde{T}^{(n)} = [\tilde{T}^{(n-1)}]^{\sim}$ for $n \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *p-hyponormal* operator if $(T^*T)^p \geqslant (TT^*)^p$, where $0 < p < \infty$. If $p = 1$, T is called *hyponormal* and if $p = \frac{1}{2}$, T is called *semi-hyponormal*. If an operator $T \in \mathcal{L}(H)$ is invertible and $log(TT^*) \leqslant log(T^*T)$, then T is called a *log-hyponormal* operator (see [27]). Since $log : (0,\infty) \to (-\infty,\infty)$ is operator monotone, every invertible p-hyponormal operator is log-hyponormal. But it is known that there is a log-hyponormal operator which is not p-hyponormal (see Example 1.2 in [27]). Also an operator $T = U|T|$ is called a *w-hyponormal* operator, if $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$.

An operator X in $\mathcal{L}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator S in $\mathcal{L}(\mathcal{H})$ if there is a quasiaffinity X in $\mathcal{L}(\mathcal{H})$ such that $XT = SX$, and this relation of S and T is denoted by $T \prec S$. If both $T \prec S$ and $S \prec T$, then we say that S and T are *quasisimilar*.

DEFINITION. For $k \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is called a *backward Aluthge iterate of a hyponormal operator of order k* if $\tilde{T}^{(k)}$ is a hyponormal operator for some $k \in \mathbb{N}$.

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We denote by $BAIH(k)$ the class of all backward Aluthge iterate of a hyponormal operator of order k. For example, $BAIH(1)$ contains all semi-hyponormal operators and $BAIH(2)$ contains all p-hyponormal $(0 < p < \frac{1}{2})$, log-hyponormal, and w-hyponormal operator, etc (see [2], [3], and [27]). In [21], E. Ko showed that $T \in BAIH(k)$ is a weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers if and only if $(\prod_{j=0}^{k} \alpha_{n+j}^{{}_kC_j})^{1/2^k} \leqslant (\prod_{j=0}^{k} \alpha_{n+j+1}^{{}_kC_j})^{1/2^k}$ holds for $n = 0, 1, 2, \cdots$.

In this paper we study several remarkable properties of the backward Aluthge iterates of a hyponormal operator. In particular, we show that, under suitable conditions, operators in $BAIH(k)$ admit a moment sequence and have nontrivial hyperinvariant subspaces.

2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the *resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T - z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Using local spectra, we define the *local spectral subspace* of T by $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is known from [22] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *upper semi-Fredholm* if T has closed range and dim $ker(T) < \infty$, and T is called *lower semi-Fredholm* if T has closed range and $\dim(\mathscr{H}/ran(T)) < \infty$. When T is either upper semi-Fredholm or lower semi-Fredholm, it is called *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathscr{L}(\mathscr{H})$, denoted index$(T)$, is given by index$(T) = \dim ker(T) - \dim(\mathscr{H}/ran(T))$ and this value is an integer or $\pm\infty$. Also an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Weyl* if it is Fredholm of index zero. For an operator $T \in \mathscr{L}(H)$, if we can choose the smallest positive integer m such that $ker(T^m) = ker(T^{m+1})$, then m is called the *ascent* of T and T is said to have *finite ascent*. Moreover, if there is the smallest positive integer n satisfying $ran(T^n) = ran(T^{n+1})$, then n is called *the descent* of T and T is said to have *finite descent*. We say that $T \in \mathscr{L}(\mathscr{H})$ is *Browder* if it is Fredholm of finite ascent and finite descent. We define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}.$$

It is evident that

$$\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T).$$

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

where $\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \dim ker(T-\lambda) < \infty\}$ and iso $\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. We say that *Browder's theorem holds* for $T \in \mathscr{L}(\mathscr{H})$ if $\sigma_b(T) = \sigma_w(T)$.

3. Main results

In this section, we study some important properties of the backward Aluthge iterates of a hyponormal operator of order k. We first give some elementary properties for such operators.

PROPOSITION 1. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$. Then the following statements hold.*

(i) $\alpha T \in BAIH(k)$ for any $\alpha \in \mathbb{C}$.

(ii) $W^*TW \in BAIH(k)$ where W is unitary.

(iii) If T is invertible, then $T^{-1} \in BAIH(k)$ and

$$\| \prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}} (T-\lambda)^{-1} \prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{-\frac{1}{2}} \| = \frac{1}{dist(\lambda, \sigma(T))}$$

for $\lambda \notin \sigma(T)$ where $\prod_{j=0}^{k-1} S_j := S_0 S_1 \cdots S_{k-1}$ and $\prod_{j=k-1}^{0} S_j := S_{k-1} S_{k-2} \cdots S_0$.

(iv) If $T = U|T|$ is the polar decomposition of T, then $Re\ \sigma(T) = \sigma((Re\ U_{k-1})\ |\widetilde{T}^{(k-1)}|)$ where $\widetilde{T}^{(k-1)} = U_{k-1} |\widetilde{T}^{(k-1)}|$ is the polar decomposition of $\widetilde{T}^{(k-1)}$.

Proof. (i) Let $\alpha \in \mathbb{C}$. Since $\widetilde{T}^{(k)}$ is hyponormal and $\widetilde{\alpha T}^{(k)} = \alpha \widetilde{T}^{(k)}$, it holds that $\alpha T \in BAIH(k)$.

(ii) Since $\widetilde{W^*TW} = W^*\tilde{T}W$, it is easy to see that $\widetilde{W^*TW}^{(k)} = W^*\tilde{T}^{(k)}W$ by induction. Since $\widetilde{T}^{(k)}$ is hyponormal, $\widetilde{W^*TW}^{(k)} = W^*\tilde{T}^{(k)}W$ is also hyponormal. Hence $W^*TW \in BAIH(k)$.

(iii) If $T = U|T|$ is the polar decomposition of T and is invertible, then U is a unitary operator. Since $|T| = U^*|T^*|U$, we have

$$T^{-1} = (U|T|)^{-1} = |T|^{-1}U^* = U^*|T^*|^{-1}UU^* = U^*|T^*|^{-1}.$$

Since $|T^{-1}|^2 = (T^{-1})^*(T^{-1}) = (T^*)^{-1}(T^{-1}) = (TT^*)^{-1} = |T^*|^{-2}$, we obtain the identity $|T^{-1}| = |T^*|^{-1}$. Hence $T^{-1} = U^*|T^*|^{-1}$ is the polar decomposition of T^{-1}. So we get

$$\widetilde{T^{-1}} = |T^*|^{-\frac{1}{2}} U^* |T^*|^{-\frac{1}{2}}. \tag{1}$$

Hence (1) implies that

$$\begin{aligned}(\widetilde{T})^{-1} &= (|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^{-1} = (U^*|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}U)^{-1}\\ &= U^*|T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}}U = U^*\widetilde{T^{-1}}U. \end{aligned} \tag{2}$$

Claim. $(\widetilde{T}^{(m)})^{-1} = (\prod_{j=m-1}^{0} U_j^*)\widetilde{T^{-1}}^{(m)}(\prod_{j=0}^{m-1} U_j)$ for all $m \in \mathbb{N}$ where $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ is the polar decomposition of $\widetilde{T}_j$ for each $j \in \mathbb{N}\cup\{0\}$.

If $m = 1$, $(\widetilde{T})^{-1} = U_0^*\widetilde{T^{-1}}U_0$ from (2). If the claim holds when $m = n$, then by induction hypothesis and (2),

$$\begin{aligned}(\widetilde{T}^{(n+1)})^{-1} &= ([\widetilde{T}^{(n)}]^{\sim})^{-1} = U_n^*[(\widetilde{T}^{(n)})^{-1}]^{\sim}U_n = U_n^*[(\prod_{j=n-1}^{0} U_j^*)\widetilde{T^{-1}}^{(n)}(\prod_{j=0}^{n-1} U_j)]^{\sim}U_n\\ &= U_n^*\prod_{j=n-1}^{0} U_j^*[\widetilde{T^{-1}}^{(n)}]^{\sim}\prod_{j=0}^{n-1} U_jU_n = (\prod_{j=n}^{0} U_j^*)\widetilde{T^{-1}}^{(n+1)}(\prod_{j=0}^{n} U_j).\end{aligned}$$

So we complete the proof of our claim.

Since $T \in BAIH(k)$, we get that $(\widetilde{T}^{(k)})^{-1}$ is hyponormal by [28], and so $\widetilde{T^{-1}}^{(k)}$ is also hyponormal by the above claim. Hence $T^{-1} \in BAIH(k)$ by the induction.

Since $\widetilde{T}^{(k)}$ is hyponormal and $\sigma(T) = \sigma(\widetilde{T})$ from [16], [2] implies that

$$\|(\widetilde{T}^{(k)} - \lambda I)^{-1}\| = \frac{1}{dist(\lambda, \sigma(T))}$$

for $\lambda \notin \sigma(T)$. Since $(\widetilde{T}^{(k)} - \lambda I)^{-1} = \prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}(T - \lambda I)^{-1}\prod_{j=0}^{k-1}|\widetilde{T}^{(j)}|^{-\frac{1}{2}}$, we complete our proof.

(iv) Let $\widetilde{T}^{(k-1)} = U_{k-1}|\widetilde{T}^{(k-1)}|$ be the polar decomposition of $\widetilde{T}^{(k-1)}$. Since $\widetilde{T}^{(k)}$ is hyponormal, it is known from [24] that $\sigma(Re\ \widetilde{T}^{(k)}) = Re\ \sigma(\widetilde{T}^{(k)})$. Since $\sigma(T) = \sigma(\widetilde{T}^{(k)})$ form [16], we get that

$$Re\ \sigma(T) = \sigma(Re\ \widetilde{T}^{(k)}) = \sigma\Big(\frac{\widetilde{T}^{(k)} + \widetilde{T}^{(k)*}}{2}\Big) = \sigma(|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}\Big(\frac{U_{k-1} + U_{k-1}^*}{2}\Big)|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}).$$

From some applications of Proposition 1 in [14], we get that

$$\begin{aligned}\sigma(|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}\Big(\frac{U_{k-1} + U_{k-1}^*}{2}\Big)|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}) &= \sigma\Big(\frac{U_{k-1} + U_{k-1}^*}{2}|\widetilde{T}^{(k-1)}|\Big)\\ &= \sigma((Re\ U_{k-1})|\widetilde{T}^{(k-1)}|).\end{aligned}$$

Hence we complete the proof. □

As some applications of the equation (1) in the proof of Proposition 1, we get the following corollary.

COROLLARY 1. *Let $T=U|T|$ be the polar decomposition of T. If T is invertible, then the following statements hold.*

(i) $\widetilde{T^{-1}}=|T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}}$ *and*

(ii) $(\widetilde{T})^{-1}=\widetilde{T^{-1}}$ *if and only if* $[U,|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}]=0$*, where* $[A,B]=AB-BA$ *for any operators* A *and* B.

Proof. (i) The proof follows from (1).

(ii) Since $\widetilde{T^{-1}}=|T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}}$ by (i), we get that

$$\begin{aligned}\widetilde{T}(\widetilde{T^{-1}})&=(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}})=|T|^{\frac{1}{2}}U(U^*|T^*|^{\frac{1}{2}}U)|T^*|^{-\frac{1}{2}}U^*(U|T|^{-\frac{1}{2}}U^*)\\&=|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{-\frac{1}{2}}|T|^{-\frac{1}{2}}U^*.\end{aligned}$$

Hence $\widetilde{T}(\widetilde{T^{-1}})=I$ if and only if $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U=U|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$. □

For $T\in\mathscr{L}(\mathscr{H})$, the *algebraic core* $Alg(T)$ is defined as the greatest (not necessarily closed) subspace $\mathscr{M}$ of $\mathscr{H}$ satisfying $T\mathscr{M}=\mathscr{M}$. The *analytical core* of T is the set $Anal(T)$ of all $x\in\mathscr{H}$ such that there exists a sequence $\{u_n\}\subset\mathscr{H}$ and a constant $\delta>0$ such that $x=u_0$, $Tu_{n+1}=u_n$, and $\|u_n\|\leqslant\delta^n\|x\|$ for every $n\in\mathbb{N}$.

LEMMA 1. *Let $T=U|T|\in\mathscr{L}(\mathscr{H})$ be the polar decomposition of T. Then*

(i) $Alg(\widetilde{T})=|T|^{\frac{1}{2}}Alg(T)$,

(ii) $Alg(T)=U|T|^{\frac{1}{2}}Alg(\widetilde{T})$,

(iii) $Anal(\widetilde{T})=|T|^{\frac{1}{2}}Anal(T)$ *if T is invertible, and*

(iv) $Anal(T)=U|T|^{\frac{1}{2}}Anal(\widetilde{T})$ *if T is invertible.*

Proof. (i) Since $TAlg(T)=Alg(T)$, we get that

$$\widetilde{T}|T|^{\frac{1}{2}}Alg(T)=|T|^{\frac{1}{2}}U|T|Alg(T)=|T|^{\frac{1}{2}}TAlg(T)=|T|^{\frac{1}{2}}Alg(T).$$

Hence $|T|^{\frac{1}{2}}Alg(T)\subseteq Alg(\widetilde{T})$.

On the other hand, since $\widetilde{T}Alg(\widetilde{T})=Alg(\widetilde{T})$, $TU|T|^{\frac{1}{2}}Alg(\widetilde{T})=U|T|^{\frac{1}{2}}\widetilde{T}Alg(\widetilde{T})=U|T|^{\frac{1}{2}}Alg(\widetilde{T})$. Hence $U|T|^{\frac{1}{2}}Alg(\widetilde{T})\subseteq Alg(T)$. Therefore

$$Alg(\widetilde{T})=\widetilde{T}Alg(\widetilde{T})=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}Alg(\widetilde{T})\subseteq|T|^{\frac{1}{2}}Alg(T).$$

So we have $Alg(\widetilde{T})=|T|^{\frac{1}{2}}Alg(T)$.

(ii) By (i), we can get $U|T|^{\frac{1}{2}}Alg(\widetilde{T})=TAlg(T)=Alg(T)$.

(iii) Let $x\in Anal(T)$. Then there exist a sequence $\{u_n\}\subset\mathscr{H}$ and a constant $\delta>0$ such that $x=u_0$, $Tu_{n+1}=u_n$, and $\|u_n\|\leqslant\delta^n\|x\|$ for every $n\in\mathbb{N}$. Since $|T|^{\frac{1}{2}}x=|T|^{\frac{1}{2}}u_0$, $\widetilde{T}|T|^{\frac{1}{2}}u_{n+1}=|T|^{\frac{1}{2}}Tu_{n+1}=|T|^{\frac{1}{2}}u_n$ and

$$\begin{aligned}\||T|^{\frac{1}{2}}u_n\|&\leqslant\||T|^{\frac{1}{2}}\|\|u_n\|\leqslant\||T|^{\frac{1}{2}}\|\delta^n\|x\|\leqslant\||T|^{\frac{1}{2}}\|\||T|^{-\frac{1}{2}}\|\delta^n\||T|^{\frac{1}{2}}x\|\\&\leqslant(\||T|^{\frac{1}{2}}\|\||T|^{-\frac{1}{2}}\|)^n\delta^n\||T|^{\frac{1}{2}}x\|=(\||T|^{\frac{1}{2}}\|\||T|^{-\frac{1}{2}}\|\delta)^n\||T|^{\frac{1}{2}}x\|\end{aligned}$$

for all $n \in \mathbb{N}$, it holds that $|T|^{\frac{1}{2}} Anal(T) \subseteq Anal(\widetilde{T})$.

On the other hand, let $y \in Anal(\widetilde{T})$. Then there exist a sequence $\{v_n\} \subset \mathscr{H}$ and a constant $\delta > 0$ such that $y = v_0$, $\widetilde{T} v_{n+1} = v_n$, and $\|v_n\| \leqslant \delta^n \|y\|$ for every $n \in \mathbb{N}$. Since T is inverible, so is $|T|^{\frac{1}{2}}$. Set $z = |T|^{-\frac{1}{2}} y$ and $s_n = |T|^{-\frac{1}{2}} v_n$ for every $n \in \mathbb{N} \cup \{0\}$. Then $z = s_0$. Since $|T|^{\frac{1}{2}} T s_{n+1} = \widetilde{T} |T|^{\frac{1}{2}} s_{n+1} = \widetilde{T} v_{n+1} = v_n = |T|^{\frac{1}{2}} (|T|^{-\frac{1}{2}} v_n) = |T|^{\frac{1}{2}} s_n$ and $|T|^{\frac{1}{2}}$ is invertible, we have $T s_{n+1} = s_n$. Moreover,

$$\begin{aligned}\|s_n\| &\leqslant \||T|^{-\frac{1}{2}}\| \|v_n\| \leqslant \||T|^{-\frac{1}{2}}\| \delta^n \|y\| \\ &= \||T|^{-\frac{1}{2}}\| \delta^n \||T|^{\frac{1}{2}} z\| \leqslant (\||T|^{-\frac{1}{2}}\| \||T|^{\frac{1}{2}}\|) \delta^n \|z\| \\ &\leqslant (\||T|^{\frac{1}{2}}\| \||T|^{-\frac{1}{2}}\|)^n \delta^n \|z\|\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $z \in Anal(T)$, i.e., $y \in |T|^{\frac{1}{2}} Anal(T)$. Therefore $Anal(\widetilde{T}) \subseteq |T|^{\frac{1}{2}} Anal(T)$.

(iv) By the similar method as in (iii), we obtain that $Anal(T) = U|T|^{\frac{1}{2}} Anal(\widetilde{T})$ if $0 \notin \sigma(T)$. □

Note that if $T \in \mathscr{L}(\mathscr{H})$ is invertible with polar decomposition $T = U|T|$, then U is unitary and $|T|$ is invertible. Since $\sigma(T) = \sigma(\widetilde{T}^{(j)})$ for every $j \in \mathbb{N}$ by [16], $|\widetilde{T}^{(j)}|$ is also invertible for every $j \in \mathbb{N}$. Combining such a fact with Lemma 1, we easily get the following proposition by induction. So we omit its proof.

PROPOSITION 2. *For $T \in \mathscr{L}(\mathscr{H})$, let $\widetilde{T}^{(j)} = U_j |\widetilde{T}^{(j)}|$ be the polar decomposition for every $j \in \mathbb{N} \cup \{0\}$. If $k \in \mathbb{N}$, then the following statments hold.*

(i) $Alg(\widetilde{T}^{(k)}) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Alg(T)$.

(ii) $Alg(T) = (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Alg(\widetilde{T}^{(k)})$.

(iii) $Anal(\widetilde{T}^{(k)}) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Anal(T)$ if T is invertible, and

(iv) $Anal(T) = (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Anal(\widetilde{T}^{(k)})$ if T is invertible.

PROPOSITION 3. *Let $T = U|T| \in BAIH(k)$ be the polar decompositon of T for some $k \in \mathbb{N}$. Then $ker(\lambda I - T) \cap Anal(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$.*

Proof. If $T = U|T| \in BAIH(k)$, then T has the single-valued extension property from [21]. Hence the proof follows from [1]. □

Recall that if $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit* of x under T, denoted by $O(x,T)$. If $O(x,T)$ is dense in $\mathscr{H}$, then x is called a *hypercyclic* vector for T. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *hypercyclic* if there is at least one hypercyclic vector for T, and *hypertransitive* if every nonzero vector in $\mathscr{H}$ is hypercyclic for T. Denote the set of all nonhypertransitive operators in $\mathscr{L}(\mathscr{H})$ by (NHT).

THEOREM 1. *If $k \in \mathbb{N}$, then the following statements hold.*

(i) $BAIH(k)$ is closed in the uniform operator topology.

(ii) The spectrum σ is continuous in the Hausdorff metric at every $T \in BAIH(k)$.

(iii) Every $T \in BAIH(k)$ is nonhypertransitive.

(iv) If $T \in BAIH(k)$ and $\sigma(T)$ is a Lebesgue null set, then $\widetilde{T}^{(k)}$ is normal and $\widetilde{T}^{(k)} = \widetilde{T}^{(k+n)}$ for every $n \in \mathbb{N}$.

Proof. (i) If $\lim_{n\to\infty} \|T_n - T\| = 0$ and $T_n \in BAIH(k)$, then $\lim_{n\to\infty} \|\widetilde{T_n}^{(k)} - \widetilde{T}^{(k)}\| = 0$ by [11]. Since $\widetilde{T_n}^{(k)}$ is hyponormal, $\widetilde{T}^{(k)}$ is also hyponormal. Therefore $T \in BAIH(k)$.

(ii) If $\lim_{n\to\infty} \|T_n - T\| = 0$, then $T \in BAIH(k)$ by (i) and $\lim_{n\to\infty} \|\widetilde{T_n}^{(k)} - \widetilde{T}^{(k)}\| = 0$ by [11]. Since the spectrum σ is continuous at hyponormal operators, $\lim_{n\to\infty} \sigma(\widetilde{T_n}^{(k)}) = \sigma(\widetilde{T}^{(k)})$. Since $\sigma(\widetilde{T_n}^{(k)}) = \sigma(T_n)$ and $\sigma(\widetilde{T}^{(k)}) = \sigma(T)$ by [16], we have $\lim_{n\to\infty} \sigma(T_n) = \sigma(T)$.

(iii) If T is not a quasiaffinity, then $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$. Hence T has a nontrivial invariant subspace, and so $T \in (NHT)$. Otherwise, suppose that T is a quasiaffinity. Since $\widetilde{T}^{(k)}$ is not hypercyclic from [20], there exists a nonzero vector $x \in \mathscr{H}$ such that $O(x, \widetilde{T}^{(k)})$ is not dense in $\mathscr{H}$. Let $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ be the polar decomposition for $j = 0, 1, \ldots, k-1$. Since $U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}}\widetilde{T}^{(j+1)} = \widetilde{T}^{(j)}U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}}$, it follows that

$$\begin{aligned}\widetilde{T}^{(k-1)}(U_{k-1}|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}O(x,\widetilde{T}^{(k)})) &= U_{k-1}|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}(\widetilde{T}^{(k)}O(x,\widetilde{T}^{(k)})) \\ &\subseteq U_{k-1}|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}O(x,\widetilde{T}^{(k)}).\end{aligned}$$

Since T is a quasiaffinity, so is $\widetilde{T}^{(k-1)}$. Hence $|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}$ is a quasiaffinity and U_{k-1} is unitary. Therefore, $U_{k-1}|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}O(x,\widetilde{T}^{(k)})$ is not dense in $\mathscr{H}$. So $\widetilde{T}^{(k-1)} \in (NHT)$ because $|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}\{U_{k-1}|\widetilde{T}^{(k-1)}|^{\frac{1}{2}}O(x,\widetilde{T}^{(k)})\} = \widetilde{T}^{(k)}O(x,\widetilde{T}^{(k)}) \subset O(x,\widetilde{T}^{(k)})$. Repeating the same arguments as above and using [5] or [17], we can show that $T \in (NHT)$.

(iv) Since $\widetilde{T}^{(k)}$ is hyponormal, we obtain from [24] that

$$||(\widetilde{T}^{(k)})^*(\widetilde{T}^{(k)}) - (\widetilde{T}^{(k)})(\widetilde{T}^{(k)})^*|| \leqslant \frac{1}{\pi}\mu(\sigma(\widetilde{T}^{(k)})) = \frac{1}{\pi}\mu(\sigma(T))$$

where μ denotes the Lebegue measure. Thus, if $\sigma(T)$ is a Lebesgue null set, then $\widetilde{T}^{(k)}$ is normal. Since $\widetilde{T}^{(k)}$ is normal, the proof follows from [16]. □

COROLLARY 2. *If $T \in BAIH(k)$ for some $k \in \mathbb{N}$, then the Weyl spectrum σ_w and the Browder spectrum σ_b are continuous at T.*

Proof. If $T \in BAIH(k)$, T is subscalar by [21]. Hence T satisfies Weyl's theorem from [1]. Since σ is continuous at T by Theorem 1, the proof follows from [10]. □

Recall that an operator T in $\mathscr{L}(\mathscr{H})$ *admits a moment sequence* if there exists nonzero vectors x and y in $\mathscr{H}$ and a (finite, regular) Borel measure μ suppoerted on $\sigma(T)$ such that

$$\langle T^n x, y\rangle = \int_{\sigma(T)} \lambda^n d\mu, \ \ n \in \mathbb{N} \cup \{0\}.$$

(We use the term *measure* here in the usual sense of a nonnegative-valued set function.)

LEMMA 2. *Let $T \in \mathcal{L}(\mathcal{H})$ with invertible $|T|$. If $\widetilde{T}$ admits a moment sequence, then T also admits a moment sequence.*

Proof. Since $\widetilde{T}$ admits a moment sequence, there exists nonzero vectors x and y in $\mathcal{H}$ and a (finite, regular) Borel measure $\mu_{x,y}$ supported on $\sigma(\widetilde{T})$ such that

$$\langle (\widetilde{T})^n x, y\rangle = \int_{\sigma(\widetilde{T})} \lambda^n d\mu_{x,y}, \ \ n \in \mathbb{N} \cup \{0\}.$$

Set $s = |T|^{-\frac{1}{2}}x$ and $t = |T|^{\frac{1}{2}}y$. Then s and t are nonzero vectors in $\mathcal{H}$ satisfying that

$$\begin{aligned} \langle T^n s, t\rangle &= \langle T^n |T|^{-\frac{1}{2}}x, |T|^{\frac{1}{2}}y\rangle = \langle |T|^{\frac{1}{2}} T^n |T|^{-\frac{1}{2}}x, y\rangle \\ &= \langle (\widetilde{T})^n x, y\rangle = \int_{\sigma(\widetilde{T})} \lambda^n d\mu_{x,y}. \end{aligned} \tag{3}$$

Set $d\nu_{s,t} = d\mu_{|T|^{\frac{1}{2}}s, |T|^{-\frac{1}{2}}t}$. Then $\nu_{s,t}$ is a (finte, regular) Borel measure supported on $\sigma(T)(=\sigma(\widetilde{T}))$. Then we get that

$$\langle T^n s, t\rangle = \int_{\sigma(\widetilde{T})} \lambda^n d\mu_{x,y} = \int_{\sigma(T)} \lambda^n d\mu_{|T|^{\frac{1}{2}}s, |T|^{-\frac{1}{2}}t} = \int_{\sigma(T)} \lambda^n d\nu_{s,t},$$

i.e., T admits a moment sequence. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *binormal* if $[|T|, |T^*|] = 0$. An operator T is said to be *centered* if the following sequence

$$\cdots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \cdots$$

is commutative. As some applications of Lemma 2, we get the following theorem.

THEOREM 2. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$. Suppose that one of the following statements hold: (i) T is invertible, (ii) $\sigma(T)$ has nonempty interior, and (iii) T is centered. Then T admits a moment sequence.*

Proof. (i) Since $\widetilde{T}^{(k)}$ is hyponormal and invertible, $\widetilde{T}^{(k)}$ admits a moment sequence by [12]. Since T is invertible for by [16], we have $0 \notin \sigma(|T|^{\frac{1}{2}})$. Hence T admits a moment sequence from Lemma 2.

(ii) Since $\widetilde{T}^{(k)}$ is hyponormal and $\sigma(\widetilde{T}^{(k)})$ has nonempty interior, $\widetilde{T}^{(k)}$ has a nontrivial invariant subspace by [6]. By [16], T has a nontrivial invariant subspace. If $\mathcal{M}$ is a nontrivial invariant subspace of T, take $x \in \mathcal{M} \setminus \{0\}$ and $y \in \mathcal{M}^{\perp} \setminus \{0\}$. If we define $\mu \equiv 0$ on $\sigma(T)$, then $\langle T^n x, y\rangle = 0 = \int_{\sigma(T)} \lambda^n d\mu$. Thus T admits a moment sequence.

(iii) Since T is centered, $\widetilde{T}^{(k)}$ is binormal by [15, Theorem F]. Since $\widetilde{T}^{(k)}$ is binormal and hyponormal, $\widetilde{T}^{(k)}$ has a nontrivial invariant subspace by [7] and so does T by [16]. Hence T admits a moment sequence as in the proof of (ii). □

COROLLARY 3. *Let $T \in BAIH(k)$ be invertible for some $k \in \mathbb{N}$. If $\sigma(T)$ contains at least one isolated point, then T has a nontivial invariant subspace.*

Proof. Since T admits a moment sequence from Theorem 2, the proof follows from [9]. □

As some applications of [26], we obtain the following theorem.

THEOREM 3. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$ and let $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If there exists a nonzero vector $x \in \mathscr{H}$ such that (i) $\sigma_T(x) \subsetneqq \sigma(T)$ or (ii) $\|T^n x\| \leqslant Cr^n$ for every $n \in \mathbb{N}$, some constants $C > 0$, and $0 < r < \|\widetilde{T}^{(k)}\|$, then T has a nontrivial hyperinvariant subspace.*

Proof. (i) If there exists a nonzero vector $x \in \mathscr{H}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, set

$$\mathscr{M} := \mathscr{H}_T(\sigma_T(x)), \text{ i.e., } \mathscr{M} = \{y \in \mathscr{H} : \sigma_T(y) \subseteq \sigma_T(x)\}.$$

Since T has Dunford's property (C) by [21], $\mathscr{M}$ is a hyperinvariant subspace of T from [22]. Since $x \in \mathscr{M}$, we get $\mathscr{M} \neq \{0\}$. To show $\mathscr{M} \neq \mathscr{H}$, suppose that this is false. Since T has the single-valued extension property, it follows from [22] that

$$\sigma(T) = \bigcup \{\sigma_T(y) : y \in \mathscr{H}\} \subseteq \sigma_T(x) \subsetneqq \sigma(T).$$

But this is a contradiction, and hence $\mathscr{M}$ is a nontrivial hyperinvariant subspace of T.

(ii) If $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$, then T has a nontrivial hyperinvariant subspace. Otherwise, T is a quasiaffinity. Assume that there is a nonzero vector $x \in \mathscr{H}$ such that $\|T^n x\| \leqslant Cr^n$ for every $n \in \mathbb{N}$, some constant $C > 0$, and $0 < r < \|\widetilde{T}^{(k)}\|$. Set $f(z) := -\sum_{n=0}^{\infty} z^{-(n+1)} T^n x$, which is analytic for $|z| > r$. If $\omega = z^{-1}$ for $|z| > r$, then $f(\omega) = -\sum_{n=0}^{\infty} \omega^{n+1} T^n x$ for $0 < |\omega| < \frac{1}{r}$. Since $\limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}} \leqslant r$, the radius of convergence for $f(\omega)$ is at least $\frac{1}{r}$. Setting $f(0) := 0$, we have $f(\omega)$ is analytic for $|\omega| < \frac{1}{r}$. Therefore $f(z)$ is analytic for $|z| > r$. Since

$$(T - z) f(z) = -\sum_{n=0}^{\infty} z^{-(n+1)} T^{n+1} x + \sum_{n=0}^{\infty} z^{-n} T^n x = x$$

for all $z \in \mathbb{C}$ with $|z| > r$, we have $\rho_T(x) \supseteq \{z \in \mathbb{C} : |z| > r\}$, i.e.,

$$\sigma_T(x) \subseteq \{z \in \mathbb{C} : |z| \leqslant r\}.$$

Since $\sigma_{\widetilde{T}^{(k)}}(\prod_{j=1}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}} x) \subset \sigma_T(x)$ by [19], we get that

$$\sigma_{\widetilde{T}^{(k)}}(\prod_{j=1}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}} x) \subseteq \{z \in \mathbb{C} : |z| \leqslant r\}.$$

Since $r < \|\widetilde{T}^{(k)}\|$ and $\widetilde{T}^{(k)}$ is normaloid by [13], it holds that

$$\sigma_{\widetilde{T}^{(k)}}(\prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}} x) \subsetneqq \sigma(\widetilde{T}^{(k)}).$$

Since T is a quasiaffinity, so is $|\widetilde{T}^{(j)}|^{\frac{1}{2}}$ for each $j=0,1,2,\cdots,k-1$. Then $\prod_{j=0}^{k-1}|\widetilde{T}^{(j)}|^{\frac{1}{2}}x \neq 0$. By (i), $\widetilde{T}^{(k)}$ has a nontrivial hyperinvariant subspace, which implies that T has a nontrivial hyperinvariant subspace by [16]. □

COROLLARY 4. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$. If T has a nonzero invariant subspace $\mathscr{M}$ such that $\sigma(T|_{\mathscr{M}}) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. For any nonzero $x \in \mathscr{M}$, we have

$$\sigma_T(x) \subseteq \sigma_{T|_{\mathscr{M}}}(x) \subseteq \sigma(T|_{\mathscr{M}}) \subsetneqq \sigma(T).$$

Hence T has a nontrivial hyperinvariant subspace by Theorem 3. □

COROLLARY 5. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$. If there exists a nonzero vector $x \in \mathscr{H}$ such that $\sigma_{\widetilde{T}^{(k)}}(x) \subsetneqq \sigma(\widetilde{T}^{(k)})$, then T has a nontrivial hyperinvariant subspace.*

Proof. If $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$, T has a nontrivial hyperinvariant subspace. Otherwise, T is a quasiaffinity. Since $\sigma_{\widetilde{T}^{(k)}}(x) \subsetneqq \sigma(\widetilde{T}^{(k)})$ for some $x \neq 0$, $\widetilde{T}^{(k)}$ has a nontrivial hyperinvariant subspace by Theorem 3. By [16], T has a nontrivial hyperinvariant subspace. □

COROLLARY 6. *Under the same hypotheses as in Theorem 3, suppose that $R \in \mathscr{L}(\mathscr{H})$ is an operator satisfying the following conditions:*

(i) $T^n = R^n$,

(ii) $T^{n-2}R = R^{n-1}$, $R^{n-2}T = T^{n-1}$, and

(iii) $T^{n-1} + R^{n-1} \neq 0$

for some positive integer $n \geqslant 2$. Then R has a nontrivial invariant subspace.

Proof. Suppose that R has no nontrivial invariant subspace. Then $Lat\ T \cap Lat\ R$ is trivial. Define $S = T^{n-1} + R^{n-1}$ for some positive integer $n \geqslant 2$. Then we have $ST = (T^{n-1} + R^{n-1})T = T^n + R^{n-1}T$ and $RS = R(T^{n-1} + R^{n-1}) = RT^{n-1} + R^n$. Since $R^{n-1}T = RR^{n-2}T = RT^{n-1}$, we get that $ST = RS$. Similarly, $SR = TS$ holds. Hence S doubly intertwines (T,R). By [23, Lemma], S is 0 or a quasiaffinity. Since $S = T^{n-1} + R^{n-1} \neq 0$, S is a quasiaffinity. Since S is a quasiaffinity and doubly intertwines (T,R), T and R are quasisimilar. Since T has nontrivial a hyperinvariant subspace by Theorem 3, [25, Theorem 6.19], implies that R has a nontrivial hyperinvariant subspace, a contradiction. □

For an operator $T \in \mathscr{L}(\mathscr{H})$, we write $\{T\}' = \{S \in \mathscr{L}(\mathscr{H}) : TS = ST\}$ for the *commutant* of T. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *property* (PS) if there exists a sequence $\{S_n\} \subset \{T\}'$ and $\{K_n\}$ such that $\|S_n - K_n\| \to 0$ where $\{K_n\}$ is a nontrivial sequence of compact operators in $\mathscr{L}(\mathscr{H})$, which means that $\{K_n\}$ converges to a nonzero operator in the weak operator topology. An operator $T \in \mathscr{L}(\mathscr{H})$ has the *property* (A) if there exists a nontrivial sequence $\{K_n\}$ of compact operators in $\mathscr{L}(\mathscr{H})$ such that $\|TK_n - K_nT\| \to 0$.

THEOREM 4. *Let $T \in BAIH(k)$ for some $k \in \mathbb{N}$ and let $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If $\widetilde{T}^{(k)}$ has the property (PS), then T has a nontrivial hyperinvariant subspace. Moreover, if T is a quasiaffinity, then it has the property (PS).*

Proof. Let $T = U|T|$ be the polar decomposition. If T is not a quasiaffinity, then $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$. Hence T has a nontrivial hyperinvariant subspace. If T is a quasiaffinity, then $|T|$ is a quasiaffinity and U is unitary, i.e., $\widetilde{T}$ is a quasiaffinity. By induction, $\widetilde{T}^{(j)}$ is a quasiaffinity for $j = 0, 1, \cdots, k-1$. If $\widetilde{T}^{(k)}$ has the property (PS), then there exists a sequence $\{Q_n\} \subset \{\widetilde{T}^{(k)}\}'$ and $\{H_n\}$ such that $\|Q_n - H_n\| \to 0$ and $\{H_n\}$ is a nontrivial sequence of compact operators. Set

$$S_n := (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Q_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})$$

and

$$K_n := (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) H_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})$$

where $\widetilde{T}^{(j)} = U_j |\widetilde{T}^{(j)}|$ is the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0, 1, \cdots, k-1$. Then $\{K_n\}$ is a nontrivial sequence of compact operators since $\widetilde{T}^{(j)}$ is a quasiaffinity. Moreover, we get that

$$\begin{aligned}
S_n T &= (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Q_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) U |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} \\
&= (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Q_n \widetilde{T}^{(k)} (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) \\
&= (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) \widetilde{T}^{(k)} Q_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) \\
&= T (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Q_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) = T S_n
\end{aligned}$$

and

$$\begin{aligned}
\|S_n - K_n\| &= \|(\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Q_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) - (\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}) H_n (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})\| \\
&\leqslant \|\prod_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}\| \|Q_n - H_n\| \|\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}\| \\
&\to 0.
\end{aligned}$$

Therefore T has the property (PS). From [18] and [8], T has a nontrivial hyperinvariant subspace. $\square$

COROLLARY 7. *Let T be a quasiaffinity in $BAIH(k)$ for some $k \in \mathbb{N}$. If $\widetilde{T}^{(k)}$ has the property (PS) or (A), then T has the property (A).*

Proof. Since every scalar multiple of the identity operator satisfies the property (A), we may assume that $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If $\widetilde{T}^{(k)}$ has the property (PS), then T has the property (PS) from Theorem 4. Hence the proof follows from [18]. If $\widetilde{T}^{(k)}$ has the property (A), then there exist a sequence $\{E_n\}$ of compact operators such that $\|\widetilde{T}^{(k)}E_n - E_n\widetilde{T}^{(k)}\| \to 0$ and $\{E_n\}$ converges to a nonzero operator in the weak operator topology. Set $R_n := (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})E_n(\prod_{j=k-1}^{0}|\widetilde{T}^{(j)}|^{\frac{1}{2}})$. Then $\{R_n\}$ is a seqence of compact operators which converges to a nonzero operator in the weak operator topology since T is a quasiaffinity. Furthermore, we obtain that

$$
\begin{aligned}
&\|TR_n - R_nT\| \\
&= \|(\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})\widetilde{T}^{(k)}E_n(\prod_{j=k-1}^{0}|\widetilde{T}^{(j)}|^{\frac{1}{2}}) - (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})E_n\widetilde{T}^{(k)}(\prod_{j=k-1}^{0}|\widetilde{T}^{(j)}|^{\frac{1}{2}})\| \\
&\leqslant \|\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}}\|\|\widetilde{T}^{(k)}E_n - E_n\widetilde{T}^{(k)}\|\|\prod_{j=k-1}^{0}|\widetilde{T}^{(j)}|^{\frac{1}{2}}\| \\
&\to 0
\end{aligned}
$$

which implies that T has the property (A). $\square$

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