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PARTIAL SUMS OF THE NORMALIZED LOMMEL FUNCTIONS

MURAT ÇAĞLAR AND ERHAN DENIZ

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Abstract. The aim of the present paper determine the ratio of the normalized Lommel functions $\mathcal{L}_{\mu,\nu}$ of the form (5) to its sequence of partial sums $\left(\mathcal{L}_{\mu,\nu}\right)_m(z) = z + \sum_{n=1}^{m} a_n z^{n+1}$ when the coefficients of $\mathcal{L}_{\mu,\nu}$ satisfy some conditions. Furthermore we investigate the radii of univalency, starlikeness, convexity and close-to-convexity of the partial sums $\left(\mathcal{L}_{\mu,\nu}\right)_m(z)$. Computational and graphical usages of Maple (Version 17) as well as geometrical descriptions of the image domains in several illustrative examples are also presented.

1. Introduction and preliminary results

For $r > 0$, let $\mathcal{U}_r := \{z : |z| < r\}$ be the open disk of radius r centered at $z = 0$ and $\mathcal{U}_1 = \mathcal{U}$ be the open unit disk. Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{n+1} \tag{1}$$

which are analytic in the open unit disk $\mathcal{U}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathcal{U}$. Also let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$. Analytically, these classes are characterized by the equivalence

$$f \in \mathcal{S}^* \Leftrightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > 0,$$

$$f \in \mathcal{C} \Leftrightarrow \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,$$

and

$$f \in \mathcal{K} \Leftrightarrow \Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad (g \in \mathcal{C}).$$

Special functions as the Bessel, Lommel, Struve functions have proved themselves as some of the most frequently used special functions in mathematical analysis and its applications in physis, mechanic and engineering.

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Lommel's function [23, Sec. 10.7] which was considered by Lommel [9] in 1876

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{\mu-\nu+1}{2})\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(\frac{\mu-\nu+3}{2}+n)\Gamma(\frac{\mu+\nu+3}{2}+n)}\left(\frac{z}{2}\right)^{2n} \qquad (z\in\mathbb{C}), \tag{2}$$

is a particular solution of the inhomogeneous Bessel equation

$$z^2w''(z)+zw'(z)+[z^2-\nu^2]w(z)=z^{\mu+1} \tag{3}$$

where μ and ν complex parameters, $\Gamma(z)$ stands for Euler gamma function and occurs in several places in physics and engineering (see [5] for a list of references).

Normalized Lommel function $\mathscr{L}_{\mu,\nu}:\mathscr{U}\to\mathbb{C}$, which is defined as follows:

$$\mathscr{L}_{\mu,\nu}(z) = 4\left(\frac{\mu-\nu+1}{2}\right)\left(\frac{\mu+\nu+1}{2}\right)z^{\frac{-\mu+1}{2}}s_{\mu,\nu}(\sqrt{z}). \tag{4}$$

By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+n-1)$, we obtain the following series representation for the function $\mathscr{L}_{\mu,\nu}(z)$ given by (4):

$$\mathscr{L}_{\mu,\nu}(z) = z+\sum_{n=1}^{\infty}A_n z^{n+1} \tag{5}$$

where $A_n = \frac{(-1)^n}{4^n(\alpha+1)_n(\beta+1)_n}$ and $\alpha := \frac{\mu-\nu+1}{2} \neq -1,-2,...,$ $\beta := \frac{\mu+\nu+1}{2} \neq -1,-2,...\cdot$

We refer to Watson's treatise [23] for comprehensive information about Lommel function and recall some more recent contributions. In [22], J. Steinig examined the sign of $s_{\mu,\nu}(z)$ for positive z and $\mu,\nu\in\mathbb{R}$. He showed, among other things, that for $\mu<1/2$ the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0,\infty)$. In 2012 Koumandos and Lamprecht [7] obtained sharp estimates for the location of the zeros of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ when $\mu\in(0,1)$. The Turăn type inequalities for $s_{\mu-\frac{1}{2},\frac{1}{2}}$ were established by Baricz and Koumandos in [1]. Very recently, Baricz and Szász [2] have studied starlikeness and close-to-convexity of $l_\mu(z)=\mu(\mu+1)z^{-\frac{\mu}{2}+\frac{3}{4}}s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{z})$. Apart from these, we study partial sums of Lommel functions in this paper.

Let $\left(\mathscr{L}_{\mu,\nu}\right)_m(z) = z+\sum_{n=1}^{m}a_n z^{n+1}$ be the sequence of partial sums of normalized Lommel functions $\mathscr{L}_{\mu,\nu}(z)$ given by (5). The aim of the present paper is to determine lower bounds for $\Re\left\{\frac{\mathscr{L}_{\mu,\nu}(z)}{\left(\mathscr{L}_{\mu,\nu}\right)_m(z)}\right\}$, $\Re\left\{\frac{\left(\mathscr{L}_{\mu,\nu}\right)_m(z)}{\mathscr{L}_{\mu,\nu}(z)}\right\}$, $\Re\left\{\frac{\mathscr{L}'_{\mu,\nu}(z)}{\left(\mathscr{L}_{\mu,\nu}\right)'_m(z)}\right\}$ and $\Re\left\{\frac{\left(\mathscr{L}_{\mu,\nu}\right)'_m(z)}{\mathscr{L}'_{\mu,\nu}(z)}\right\}$. Furthermore as an application of our main results, we examine the radii of univalency, starlikeness, convexity and close-to-convexity of the partial sums $\left(\mathscr{L}_{\mu,\nu}\right)_m(z)$. Computational and graphical usages of Maple (Version 17) as well as geometrical descriptions of the image domains in several illustrative examples are also presented.

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Brickman et al. [3], Çağlar and Orhan [4], Lin and Owa [8], Deniz and Orhan [12], [13], Orhan and Yağmur [14], Owa et al. [15], Sheil-Small [17], Silverman [19] and Silvia [21].

For convenience throughout in the sequel, we use the following notations:

$$\mathscr{F} = (\alpha+1)(\beta+1)$$

and

$$\mathscr{G} = (\alpha+2)(\beta+2)$$

where $\alpha := \frac{\mu-\nu+1}{2} \neq -1,-2,...,$ $\beta := \frac{\mu+\nu+1}{2} \neq -1,-2,...$ and $\mu,\nu \in \mathbb{R}$.

LEMMA 1. *Let $\mu,\nu \in \mathbb{R}$. Then the function*

$$\mathscr{L}_{\mu,\nu} : \mathscr{U} \longrightarrow \mathbb{C}$$

given by (5) satisfies the following inequalities:

(i) If $\mathscr{G} > \frac{1}{4}$ then

$$\left|\mathscr{L}_{\mu,\nu}(z)\right| \leqslant 1 + \frac{\mathscr{G}}{\mathscr{F}(4\mathscr{G}-1)} \quad (z \in \mathscr{U}),$$

(ii) If $\mathscr{G} > \frac{1}{2}$ then

$$\left|\mathscr{L}'_{\mu,\nu}(z)\right| \leqslant 1 + \frac{\mathscr{G}}{\mathscr{F}(2\mathscr{G}-1)} \quad (z \in \mathscr{U}).$$

Proof. (i) By using the well-known triangle inequality:

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

and the inequalities

$$(\alpha+1)_n \geqslant (\alpha+1)^n, \ (\beta+1)_n \geqslant (\beta+1)^n \quad (n \in \mathbb{N} = \{1,2,...\}),$$

we have

$$\begin{aligned} \left|\mathscr{L}_{\mu,\nu}(z)\right| &= \left| z + \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n (\alpha+1)_n (\beta+1)_n} z^{n+1} \right| \leqslant 1 + \sum_{n=1}^{\infty} \frac{1}{4^n (\alpha+1)_n (\beta+1)_n} \\ &= 1 + \frac{1}{4\mathscr{F}} \sum_{n=1}^{\infty} \left(\frac{1}{4\mathscr{G}}\right)^{n-1} \\ &= 1 + \frac{\mathscr{G}}{\mathscr{F}[4\mathscr{G}-1]}, \quad \left(\mathscr{G} > \frac{1}{4}\right). \end{aligned}$$

(ii) Suppose that $\mathscr{G} > \frac{1}{2}$, by using well-known triangle inequality and the following inequality:

$$2^n (\alpha+1)_n (\beta+1)_n \geqslant (n+1)(\alpha+1)^n (\beta+1)^n \quad (n \in \mathbb{N}),$$

we obtain

$$\begin{aligned}\left|\mathcal{L}'_{\mu,\nu}(z)\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(n+1)(-1)^n}{4^n(\alpha+1)_n(\beta+1)_n}z^n\right| \leqslant 1+\sum_{n=1}^{\infty}\frac{(n+1)}{4^n(\alpha+1)_n(\beta+1)_n}\\ &= 1+\frac{1}{2\mathcal{F}}\sum_{n=1}^{\infty}\frac{(n+1)}{2^{n-1}2^n(\alpha+2)_{n-1}(\beta+2)_{n-1}}\\ &\leqslant 1+\frac{1}{2\mathcal{F}}\sum_{n=1}^{\infty}\left(\frac{1}{2\mathcal{G}}\right)^{n-1}\\ &= 1+\frac{\mathcal{G}}{\mathcal{F}[2\mathcal{G}-1]}, \quad \left(\mathcal{G}>\frac{1}{2}\right).\end{aligned}$$

Thus the proof of Lemma 1 is completed. □

2. Main results

THEOREM 1. *If* $\mu,\nu\in\mathbb{R}$ *such that* $\frac{1}{\mathcal{F}}+\frac{1}{\mathcal{G}}<4$ *then*

$$\Re\left\{\frac{\mathcal{L}_{\mu,\nu}(z)}{\left(\mathcal{L}_{\mu,\nu}\right)_m(z)}\right\}\geqslant\frac{\mathcal{F}(4\mathcal{G}-1)-\mathcal{G}}{\mathcal{F}(4\mathcal{G}-1)}\quad(z\in\mathcal{U}) \tag{6}$$

and

$$\Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}\right)_m(z)}{\mathcal{L}_{\mu,\nu}(z)}\right\}\geqslant\frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{F}(4\mathcal{G}-1)+\mathcal{G}}\quad(z\in\mathcal{U}). \tag{7}$$

Proof. We observe from part (i) of Lemma 1 that

$$1+\sum_{n=1}^{\infty}|A_n|\leqslant 1+\frac{\mathcal{G}}{\mathcal{F}(4\mathcal{G}-1)},$$

which is equivalent to

$$\frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{G}}\sum_{n=1}^{\infty}|A_n|\leqslant 1,$$

where $A_n=\frac{(-1)^n}{4^n(\alpha+1)_n(\beta+1)_n}$.

Define the function $w(z)$, we may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{G}}\frac{\mathcal{L}_{\mu,\nu}(z)}{\left(\mathcal{L}_{\mu,\nu}(z)\right)_m(z)}-\left(\frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{G}}-1\right) \\ &= \frac{1+\sum\limits_{n=1}^{m}A_nz^n+\frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{G}}\sum\limits_{n=m+1}^{\infty}A_nz^n}{1+\sum\limits_{n=1}^{m}A_nz^n}.\end{aligned} \tag{8}$$

Then from (8) we can obtain

$$w(z)=\frac{\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}A_nz^n}{2+2\sum\limits_{n=1}^{m}A_nz^n+\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}A_nz^n}$$

and

$$|w(z)|\leqslant\frac{\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}|A_n|}{2-2\sum\limits_{n=1}^{m}|A_n|-\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}|A_n|}.$$

Now $|w(z)|\leqslant 1$ if and only if

$$\frac{2\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum_{n=m+1}^{\infty}|A_n|\leqslant 2-2\sum_{n=1}^{m}|A_n|,$$

which is equivalent to

$$\sum_{n=1}^{m}|A_n|+\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum_{n=m+1}^{\infty}|A_n|\leqslant 1. \tag{9}$$

It suffices to show that the left hand side of (9) is bounded above by $\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=1}^{\infty}|A_n|$, which is equivalent to

$$\frac{\mathscr{F}(4\mathscr{G}-1)-\mathscr{G}}{\mathscr{G}}\sum_{n=1}^{m}|A_n|\geqslant 0.$$

We next prove bounds for (7), we write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)}&=\left(\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}+1\right)\frac{\left(\mathscr{L}_{\mu,\nu}\right)_m(z)}{\mathscr{L}_{\mu,\nu}(z)}-\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\\&=\frac{1+\sum\limits_{n=1}^{m}A_nz^n-\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}A_nz^n}{1+\sum\limits_{n=1}^{\infty}A_nz^n}\end{aligned}$$

where

$$|w(z)|\leqslant\frac{\frac{\mathscr{F}(4\mathscr{G}-1)+\mathscr{G}}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}|A_n|}{2-2\sum\limits_{n=1}^{m}|A_n|-\frac{\mathscr{F}(4\mathscr{G}-1)-\mathscr{G}}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}|A_n|}\leqslant 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}|A_n|+\frac{\mathscr{F}(4\mathscr{G}-1)}{\mathscr{G}}\sum_{n=m+1}^{\infty}|A_n|\leqslant 1. \tag{10}$$

Since the left hand side of (10) is bounded above by $\frac{\mathcal{F}(4\mathcal{G}-1)}{\mathcal{G}}\sum_{n=1}^{\infty}|A_n|$, the proof is completed. □

THEOREM 2. *If $\mu,\nu\in\mathbb{R}$ such that $\frac{1}{\mathcal{F}}+\frac{1}{\mathcal{G}}<2$ then,*

$$\Re\left\{\frac{\mathcal{L}'_{\mu,\nu}(z)}{\left(\mathcal{L}_{\mu,\nu}\right)'_m(z)}\right\}\geqslant\frac{\mathcal{F}(2\mathcal{G}-1)-\mathcal{G}}{\mathcal{F}(2\mathcal{G}-1)}\quad(z\in\mathcal{U}),\tag{11}$$

and

$$\Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}\right)'_m(z)}{\mathcal{L}'_{\mu,\nu}(z)}\right\}\geqslant\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{F}(2\mathcal{G}-1)+\mathcal{G}}\quad(z\in\mathcal{U}).\tag{12}$$

Proof. From part (ii) of Lemma 1 we observe that

$$1+\sum_{n=1}^{\infty}(n+1)|A_n|\leqslant1+\frac{\mathcal{G}}{\mathcal{F}(2\mathcal{G}-1)},$$

which is equivalent to

$$\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=1}^{\infty}(n+1)|A_n|\leqslant1,$$

where $A_n=\frac{(-1)^n}{4^n(\alpha+1)_n(\beta+1)_n}$.

Now, we write

$$\frac{1+w(z)}{1-w(z)}=\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\frac{\mathcal{L}'_{\mu,\nu}(z)}{\left(\mathcal{L}_{\mu,\nu}\right)'_m(z)}-\left(\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}-1\right)$$
$$=\frac{1+\sum_{n=1}^{m}(n+1)A_nz^n+\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum_{n=1}^{m}(n+1)A_nz^n},$$

where

$$|w(z)|\leqslant\frac{\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=m+1}^{\infty}(n+1)|A_n|}{2-2\sum_{n=1}^{m}(n+1)|A_n|-\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=m+1}^{\infty}(n+1)|A_n|}\leqslant1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)|A_n|+\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=m+1}^{\infty}(n+1)|A_n|\leqslant1.\tag{13}$$

It suffices to show that the left hand side of (13) is bounded above by

$$\frac{\mathcal{F}(2\mathcal{G}-1)}{\mathcal{G}}\sum_{n=1}^{\infty}(n+1)|A_n|\tag{14}$$

which is equivalent to

$$\frac{\mathscr{F}(2\mathscr{G}-1)-\mathscr{G}}{\mathscr{G}}\sum_{n=1}^{m}(n+1)|A_n| \geqslant 0.$$

To prove the result (12), we write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}}+1\right)\frac{\left(\mathscr{L}_{\mu,\nu}\right)'_m(z)}{\mathscr{L}'_{\mu,\nu}(z)}-\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}}\\ &= \frac{1+\sum\limits_{n=1}^{m}(n+1)A_nz^n-\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum\limits_{n=1}^{\infty}(n+1)A_nz^n}\end{aligned}$$

where

$$|w(z)| \leqslant \frac{\frac{\mathscr{F}(2\mathscr{G}-1)+\mathscr{G}}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}(n+1)|A_n|}{2-2\sum\limits_{n=1}^{m}(n+1)|A_n|-\frac{\mathscr{F}(2\mathscr{G}-1)-\mathscr{G}}{\mathscr{G}}\sum\limits_{n=m+1}^{\infty}(n+1)|A_n|} \leqslant 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)|A_n|+\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}}\sum_{n=m+1}^{\infty}(n+1)|A_n| \leqslant 1. \tag{15}$$

Since the left hand side of (15) is bounded above by $\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}}\sum\limits_{n=1}^{\infty}|A_n|$, the proof is completed. □

3. Illustrative examples and image domains

In this section, we present several illustrative examples along with the geometrical descriptions of the image domains of the appropriately chosen disk by the partial sums which we considered in our main theorems in Sections 2. In Theorem 1 and Theorem 2, we obtain the following corollaries for special cases of μ and ν.

COROLLARY 1. *If we take* $\mu=\nu=\frac{1}{2}$, *we have*

$$\mathscr{L}_{\frac{1}{2},\frac{1}{2}}(z)=2-2\cos\sqrt{z}, \qquad \mathscr{L}'_{\frac{1}{2},\frac{1}{2}}(z)=\frac{\sin\sqrt{z}}{\sqrt{z}}$$

and for $m=0$ *we get*

$$\left(\mathscr{L}_{\frac{1}{2},\frac{1}{2}}(z)\right)_0(z)=z, \qquad \left(\mathscr{L}'_{\frac{1}{2},\frac{1}{2}}(z)\right)_0(z)=1,$$

so,

$$\Re\left\{\frac{1-\cos\sqrt{z}}{z}\right\} \geqslant \frac{53}{116} \approx 0,45 \quad (z\in\mathscr{U}), \tag{16}$$

$$\Re\left\{\frac{z}{1-\cos\sqrt{z}}\right\} \geqslant \frac{116}{63} \approx 1,84 \quad (z\in\mathscr{U}), \tag{17}$$

$$\Re\left\{\frac{\sin\sqrt{z}}{\sqrt{z}}\right\} \geqslant \frac{23}{28} \approx 0,82 \quad (z\in\mathscr{U}), \tag{18}$$

$$\Re\left\{\frac{\sqrt{z}}{\sin\sqrt{z}}\right\} \geqslant \frac{28}{33} \approx 0,85 \quad (z\in\mathscr{U}). \tag{19}$$

COROLLARY 2. *For* $\mu = \frac{3}{2}$ *and* $\nu = \frac{1}{2}$, *we obtain*

$$\mathscr{L}_{\frac{3}{2},\frac{1}{2}}(z) = \frac{6\sqrt{z}-6\sin\sqrt{z}}{\sqrt{z}}, \qquad \mathscr{L}'_{\frac{3}{2},\frac{1}{2}}(z) = \frac{3\sin\sqrt{z}-3\sqrt{z}\cos\sqrt{z}}{\sqrt{z}}$$

and for $m=0$ *we have*

$$\left(\mathscr{L}_{\frac{1}{2},\frac{1}{2}}(z)\right)_0(z) = z, \qquad \left(\mathscr{L}'_{\frac{1}{2},\frac{1}{2}}(z)\right)_0(z) = 1,$$

so,

$$\Re\left\{\frac{\sqrt{z}-\sin\sqrt{z}}{z\sqrt{z}}\right\} \geqslant \frac{389}{2460} \approx 0,15 \quad (z\in\mathscr{U}), \tag{20}$$

$$\Re\left\{\frac{z\sqrt{z}}{\sqrt{z}-\sin\sqrt{z}}\right\} \geqslant \frac{2460}{431} \approx 5,7 \quad (z\in\mathscr{U}), \tag{21}$$

$$\Re\left\{\frac{\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}}{z\sqrt{z}}\right\} \geqslant \frac{179}{600} \approx 0,29 \quad (z\in\mathscr{U}), \tag{22}$$

$$\Re\left\{\frac{z\sqrt{z}}{\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}}\right\} \geqslant \frac{600}{221} \approx 2,7 \quad (z\in\mathscr{U}). \tag{23}$$

Now, we give the geometrical descriptions of the image domains of the unit disk by the ratio of normalized Lommel function to its sequence of partial sums or the ratio of its sequence of partial sums to the function which we considered in Corollary 1 and Corollary 2.

EXAMPLE 1. The image domains of $f_1(z) = \frac{1-\cos\sqrt{z}}{z}$, $f_2(z) = \frac{z}{1-\cos\sqrt{z}}$, $f_3(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$ and $f_4(z) = \frac{\sqrt{z}}{\sin\sqrt{z}}$ are shown in Figure 1.

EXAMPLE 2. We have the image domains of $f_5(z) = \frac{\sqrt{z}-\sin\sqrt{z}}{z\sqrt{z}}$, $f_6(z) = \frac{z\sqrt{z}}{\sqrt{z}-\sin\sqrt{z}}$, $f_7(z) = \frac{\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}}{z\sqrt{z}}$ and $f_8(z) = \frac{z\sqrt{z}}{\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}}$ in Figure 2 and Figure 3.

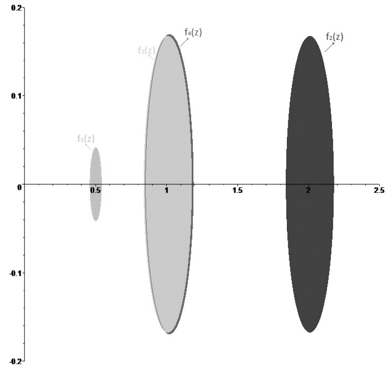

Figure 1

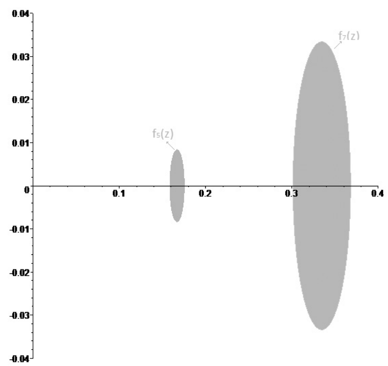

Figure 2

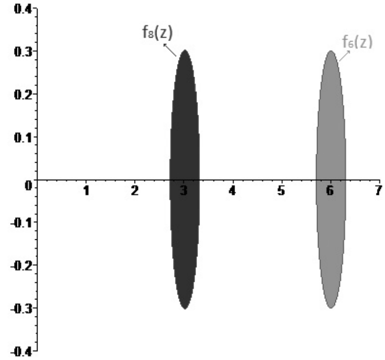

Figure 3

It is therefore of interest to determine the largest disk $\mathscr{U}_\rho$ in which the partial sums $f_n = z + \sum_{k=1}^{n} a_k z^{k+1}$ of the functions $f \in \mathscr{A}$ are univalent, starlike, convex and close-to-convex. Recently, Ravichandran also wrote a survey [16] on geometric properties of partial sums of univalent functions. By the Noshiro-Warschowski Theorem (see e.g [6]) for $m = 0$ in the inequality (11) of Theorem 2 the function $\mathscr{L}_{\mu,\nu}$ is univalent and also close-to-convex. Noshiro [11] showed that the radius of starlikeness of f_n partial sums of the functions $f \in \mathscr{A}$ is $1/M$ if satisfies the inequality $|f'(z)| \leqslant M$. Therefore if we consider the part *ii* of Lemma 1, we conclude that the radius of starlikeness of $\left(\mathscr{L}_{\mu,\nu}\right)_m$ is $\frac{\mathscr{F}(2\mathscr{G}-1)}{\mathscr{G}+\mathscr{F}(2\mathscr{G}-1)}$. For functions whose derivatives has positive real part $(\Re(f'(z)) > 0)$, Silverman [20] and Singh [18] proved that f_n is univalent in $|z| < r_n$, where r_n is the smallest positive root of the equation $1 - r - 2r^n = 0$ and convex in $|z| < 1/4$, respectively. In light of these results, for $m = 0$ in the inequality (11) of Theorem 2, $\left(\mathscr{L}_{\mu,\nu}\right)_m$ is univalent in $|z| < r_n$ and convex in $|z| < 1/4$. According to the result of Miki [10], from (11), $\left(\mathscr{L}_{\mu,\nu}\right)_m$ is close-to-convex in $|z| < 1/4$. The results are all sharp.

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REFERENCES

[1] A. BARICZ, S. KOUMANDOS, *Turán type inequalities for some Lommel functions of the first kind*, arXiv:1308.6477.
[2] A. BARICZ, R. SZÁSZ, *Close-to-convexity of some special functions and their derivatives*, Bull. Malays. Math. Sci. Soc. (in press).
[3] L. BRICKMAN, D. J. HALLENBECK, T. H. MACGREGOR AND D. WILKEN, *Convex hulls and extreme points of families of starlike and convex mappings*, Trans. Amer. Math. Soc. **185** (1973), 413–428.
[4] M. ÇAĞLAR, H. ORHAN, *On neighborhood and partial sums problem for generalized Sakaguchi type functions*, The Scientifc Annals of Al. I. Cuza University of Iasi, (2013), (in press).
[5] M. L. GLASSER, *Integral representations for the exceptional univariate Lommel functions*, J. Phys. A. **43**, 15 (2010), 155207, 4 pp. MR2608275 (2011e:33011).
[6] A. W. GOODMAN, *Univalent Functions*, Vol I. Mariner Publ. Comp., Tampa, Florida, 1984.
[7] S. KOUMANDOS, M. LAMPRECHT, *The zeros of certain Lommel functions*, Proc. Amer. Math. Soc. **140**, 9 (2012), 3091–3100.
[8] L. J. LIN AND S. OWA, *On partial sums of the Libera integral operator*, J. Math. Anal. Appl. **213**, 2 (1997), 444–454.
[9] E. LOMMEL, *Ueber eine mit den Besseischen Functionen verwandte Function*, Math. Ann. **9** (1876), 425–444.
[10] Y. MIKI, *A note on close-to-convex functions*, J. Math. Soc. Japan, **8** (1956), 256–268.
[11] K. NOSHIRO, *On the starshaped mapping by an analytic function*, Proc. Imp. Acad. **8**, 7 (1932), 275–277.
[12] E. DENIZ, H. ORHAN, *Some properties of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator*, Czech. Math. J. **60**, 3 (2010), 699–713.
[13] E. DENIZ, H. ORHAN, *Certain subclasses of multivalent functions defined by new multiplier transformations*, Arab. J. Sci. Eng. **36**, 6 (2011), 1091–1112.
[14] H. ORHAN AND N. YAĞMUR, *Partial Sums of generalized Bessel functions*, J. Math. Inequal. **8**, 4 (2014), 863–877.

[15] S. OWA, H. M. SRIVASTAVA AND N. SAITO, *Partial sums of certain classes of analytic functions*, Int. J. Comput. Math. **81**, 10 (2004), 1239–1256.
[16] V. RAVICHANDRAN, *Geometric properties of partial sums of univalent functions*, arXiv: 1207.4302v1.
[17] T. SHEIL-SMALL, *A note on partial sums of convex schlicht functions*, Bull. London Math. Soc. **2** (1970), 165–168.
[18] R. SINGH, *Radius of convexity of partial sums of a certain power series*, J. Austral. Math. Soc. **11** (1970), 407–410.
[19] H. SILVERMAN, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. **209** (1997), 221–227.
[20] H. SILVERMAN, *Partial sums of a class of univalent functions*, Tamkang J. Math. **29**, 3 (1998), 171–174.
[21] E. M. SILVIA, *On partial sums of convex functions of order* α, Houston J. Math. **11** (1985), 397–404.
[22] J. STEINIG, *The sign of Lommel's function*, Trans. Amer. Math. Soc. **163** (1972), 123–129.
[23] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.

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Murat Çağlar
Department of Mathematics
Faculty of Science and Letters
Kafkas University
36100, Kars, Turkey
e-mail: mcaglar25@gmail.com

Erhan Deniz
Department of Mathematics
Faculty of Science and Letters
Kafkas University
36100, Kars, Turkey
e-mail: edeniz36@gmail.com