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LINEAR OPERATORS INEQUALITY FOR n–CONVEX FUNCTIONS AT A POINT

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Abstract. We study necessary and sufficient conditions on linear operators A and B for inequality $Af \leqslant Bf$ to hold for every function f that is n-convex at a point.

1. Introduction

Levinson's inequality states that if $f:[a,b]\to\mathbb{R}$ is a 3-convex function and p_i, x_i, y_i, $i=1,2,\ldots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a\leqslant x_i, y_i\leqslant b$, $\max_i x_i\leqslant \min_i y_i$ and

$$x_1+y_1=x_2+y_2=\ldots=x_n+y_n=2c \tag{1}$$

for some $c\in[a,b]$, then the following holds

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\leqslant \sum_{i=1}^n p_i f(y_i)-f(\overline{y}), \tag{2}$$

where $\overline{x}=\sum_{i=1}^n p_i x_i$ and $\overline{y}=\sum_{i=1}^n p_i y_i$ denote the weighted arithmetic means. Mercer [4] showed that inequality (2) still holds if the assumption (1) of symmetric distribution with respect to the point c is replaced with the weaker one

$$\sum_{i=1}^n p_i (x_i-\overline{x})^2=\sum_{i=1}^n p_i (y_i-\overline{y})^2, \tag{3}$$

i.e. that the variances of the two sequences are equal. Witkowski [9] extended these results to more a general probabilistic setting.

On the other hand, Baloch, Pečarić and Praljak [1] showed that, for fixed $c\in(a,b)$, inequality (2) under the equal-variances assumption (3) holds for a larger class of functions they introduced and called 3-convex functions at a point. Here, we give a more general definition for this class and, in doing so, we introduce new classes of $(n+1)$-convex functions at a point that we will use in this paper.

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DEFINITION 1.1. Let I be an interval in $\mathbb{R}$, c a point in the interior of I and $n \in \mathbb{N}_0$. A function $f : I \to \mathbb{R}$ is said to be $(n+1)$-convex at point c if there exists a constant K_f such that the function

$$F(x) = f(x) - \frac{K_f}{n!} x^n \tag{4}$$

is n-concave on $I \cap (-\infty, c]$ and n-convex on $I \cap [c, \infty)$. We denote the family of $(n+1)$-convex functions at point c by $\mathscr{K}_{n+1}^c(I)$. A function f is said to be $(n+1)$-concave at point c if the function $-f$ is $(n+1)$-convex at point c.

Probabilistic version of Levinson's inequality under the equal-variances assumption for the class of 3-convex functions at a point was proven by Pečarić, Praljak and Witkowski [5] and is given in the following theorem.

THEOREM 1.2. *Let* $X : \Omega \to I \cap (-\infty, c]$ *and* $Y : \Omega \to I \cap [c, \infty)$ *be two random variables such that* $\mathrm{Var}(X) = \mathrm{Var}(Y) < \infty$*. Then, for every* $f \in \mathscr{K}_3^c(I)$ *such that* $\mathbb{E}(f(X))$ *and* $\mathbb{E}(f(Y))$ *are finite, the following holds*

$$\mathbb{E}(f(X)) - f(\mathbb{E}(X)) \leqslant \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)). \tag{5}$$

Notice that inequality (5) can be restated as

$$Af \leqslant Bf, \tag{6}$$

where A and B are the linear operators

$$Af = \mathbb{E}(f(X)) - f(\mathbb{E}(X)), \qquad Bf = \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)).$$

Actually, if one looks at the proof of Theorem 1.2 in [5], it was proven that the following inequalities hold

$$Af \leqslant \frac{K_f}{n!} h \leqslant Bf, \tag{7}$$

where $n = 2$ and $h = \mathrm{Var}(X) = \mathrm{Var}(Y)$.

Next, we will cite some known results regarding necessary and sufficient conditions on a linear operator to be non-negative on a cone of n-convex functions. Before stating the relevant results we introduce some notation. With $C([u,v])$ we denote continuous functions on $[u,v]$ and $\|\cdot\|$ is the norm $\|f\| = \max_{u \leqslant x \leqslant v} |f(x)|$. A sequence of functions $f_n \in C([u,v])$ converges uniformly to a function $f \in C([u,v])$ if $\lim_n \|f_n - f\| = 0$. Let us assume that $D \subset \mathbb{R}$, and let $S(D)$ be one of the normed subspaces of the space of all real functions defined on D, where the norm of a function $f \in S(D)$ is denoted by $\|f\|_D$ (for example, for a one-element set D the space $S(D)$ is equivalent to $\mathbb{R}$). We consider operators A of the following form $A : C([u,v]) \to S(D)$, and say that A is continuous if $\lim_n \|f_n - f\| = 0$ implies $\lim_n \|Af_n - Af\|_D = 0$ as well. Also, we write $Af \geqslant 0$ if $Af(t) \geqslant 0$ holds for every $t \in D$, where f is a given function in the space $C([u,v])$.

The family of the polynomials of degree at most n is denoted by Π_n. The family of continuous n-convex functions on $[u,v]$ (i. e. right-continuous at a and left-continuous at b) is denoted by $K_n[u,v]$. Monomials are denoted by e_i, i. e. $e_i(x) = x^i$ for $i = 0,1,2,\dots$ The functions w_n and ρ_n, for $n \in \mathbb{N}$, $n \geqslant 2$, are

$$w_n(x,d) = (x-d)_+^{n-1}, \qquad \rho_n(x,d) = (x-d)_-^{n-1}, \tag{8}$$

where

$$(x-d)_+ = \begin{cases} x-d, & x \geqslant d, \\ 0, & x < d, \end{cases} \quad (x-d)_- = \begin{cases} 0, & x \geqslant d, \\ x-d, & x < d. \end{cases}$$

Specifically, for $n = 1$ we denote

$$w_1(x,d) = \begin{cases} 1, & x \geqslant d, \\ 0, & x < d, \end{cases} \qquad \rho_1(x,d) = 1 - w_1(x,d).$$

The following is a result of our interest (see [2], [6]).

THEOREM 1.3. *Let $A : C([a,b]) \to S(D)$ be a linear and continuous operator and $n \geqslant 2$. Then, the inequality*

$$Af \geqslant 0$$

holds for every function $f \in K_n([a,b])$ if and only if the operator A satisfies:

(a) $Ae_i = 0$ *for* $i = 0,1,\dots,n-1$,

(b) $Aw_n(\cdot,d) \geqslant 0$ *for every* $d \in [a,b]$.

The proof of Theorem 1.3 is based on the following representation of n-convex functions due to Popoviciu [8].

LEMMA 1.4. *Let the function F_m be of the form*

$$F_m(x) = P_{n-1}(x) + \sum_{i=1}^{m} \alpha_i w_n(x,x_i), \tag{9}$$

where $P_{n-1} \in \Pi_{n-1}$, α_i, $i = 1,\dots,m$, are real constants and $a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b$.

(a) *A necessary and sufficient condition for F_m to be n-convex is that $\alpha_i \geqslant 0$ ($i = 1,\dots,m$).*

(b) *Every continuous n-convex function on $[a,b]$ is the uniform limit of the sequence of functions F_m ($m = 1,2,\dots$) where the F_m's are of the form in (9) and $\alpha_i \geqslant 0$ ($i = 1,\dots,m$) are real constants.*

In this paper we will study necessary and sufficient conditions on linear operators A and B under which inequalities of type (6) and (7) hold for every function f that is $(n+1)$-convex at point c. Our main results will be proven in Section 2 and application of these results will be given in Section 3. For clarity of presentation and a reader's convenience, some technical proofs are moved to Appendix, which also contains a recollection of some well-known results on n-convex functions that are invoked throughout the paper.

2. Main results

We will first give necessary and sufficient conditions for inequalities of type (7) to hold. The result can be derived directly from Theorem 1.3, but we will derive it from Lemma 1.4 for an easier and more instructive comparison to Theorem 2.4.

THEOREM 2.1. *Let* $A : C([a,c]) \to S(D)$ *and* $B : C([c,b]) \to S(D)$ *be two linear and continuous operators,* $h : D \to \mathbb{R}$ *and* $n \geqslant 2$*. Then, the inequalities*

$$Af \leqslant \frac{K_f}{n!} h \leqslant Bf$$

hold for every continuous $f \in \mathscr{K}^c_{n+1}([a,b])$ *(and arbitrary constant* K_f *from Definition 1.1) if and only if the operators* A *and* B *satisfy:*

(*a*) $Ae_i = Be_i = 0$ *for* $i = 0,1,\dots,n-1$*, and* $Ae_n = Be_n = h$*,*

(*b*) $A\rho_n(\cdot,d) \leqslant 0$ *for every* $d \in [a,c]$*,*

(*c*) $Bw_n(\cdot,d) \geqslant 0$ *for every* $d \in [c,b]$*.*

Proof. Assume that (a)–(c) hold and let $F = f - K_f e_n/n!$ be as in Definition 1.1. Since F is n-concave on the segment $[a,c]$, by Lemma 1.4 it can be obtained as a uniform limit of functions F_m of the form

$$F_m(x) = P_{n-1}(x) - \sum_{i=1}^{m} \alpha_i w_n(x,x_i) = \tilde{P}_{n-1}(x) + \sum_{i=1}^{m} \alpha_i \rho_n(x,x_i),$$

where $P_{n-1} \in \Pi_{n-1}$, $\alpha_i \geqslant 0$, $a \leqslant x_1 < \cdots < x_m \leqslant c$ and $\tilde{P}_{n-1}(x) = P_{n-1}(x) - \sum_{i=1}^{m} \alpha_i (x - x_i)^{n-1}$. Due to the assumptions,

$$AF_m = A\tilde{P}_{n-1} + \sum_{i=1}^{m} \alpha_i A\rho_n(\cdot,x_i) \leqslant 0$$

and

$$Af - \frac{K_f}{n!} Ae_n = AF = \lim_{m\to\infty} AF_m \leqslant 0.$$

Similarly, F restricted to $[c,b]$ can be obtained as a uniform limit of the functions G_k of the form

$$G_k(y) = Q_{n-1}(y) + \sum_{i=1}^{k} \beta_i w_n(y,y_i),$$

where $Q_{n-1} \in \Pi_{n-1}$, $\beta_i \geqslant 0$ and $c \leqslant y_1 < \cdots < y_k \leqslant b$ and we conclude that

$$Bf - \frac{K_f}{n!} Be_n = BG = \lim_{k\to\infty} BG_k \geqslant 0.$$

On the other hand, suppose that (7) holds for every continuous $f \in \mathscr{K}^c_{n+1}([a,b])$. Then property (a) holds since both e_i and $-e_i$ for $i = 0,1,\dots,n-1$ belong to

$\mathscr{K}^c_{n+1}([a,b])$ with $K_{e_i} = K_{-e_i} = 0$ and both e_n and $-e_n$ belong to $\mathscr{K}^c_{n+1}([a,b])$ with $K_{e_n} = n! = -K_{-e_n}$. Moreover, since $\rho_n(\cdot,d)$ (resp. $w_n(\cdot,d)$) belongs to $\mathscr{K}^c_{n+1}([a,b])$ for $d \in [a,c]$ (resp. $d \in [c,b]$) and $B\rho_n(\cdot,d) = B0 = 0$ (resp. $Aw_n(\cdot,d) = A0 = 0$), we conclude that property (b) (resp. (c)) holds. $\square$

REMARK 2.2. Theorem 2.1 is an extension of Theorem 1.3. For a linear and continuous operator $B : C([c,b]) \to S(D)$ let us define the linear operator A with $Af = B(e_n)[x_0,x_1,\ldots,x_n]f$, where x_i, $i = 0,1,\ldots,n$, are some arbitrary distinct points in $[a,c]$. Notice that $Ae_i = 0$ for $i = 0,1,\ldots,n-1$ and $Ae_n = Be_n$, so A satisfies assumption (a) from Theorem 2.1. Moreover, if B satisfies the same assumption, then $BP_{n-1} = 0$ for every $P_{n-1} \in \Pi_{n-1}$ and if, additionally, B satisfies assumption (c), then using the representation of Lemma 1.4 for the n-convex function e_n, we conclude that $Be_n \geqslant 0$. Now, since $\rho_n(\cdot,d)$ is an n-concave function, we conclude that $A\rho_n(\cdot,d) \leqslant 0$, i. e. A satisfies assumption (b) as well. In conclusion, for the given A and B conditions (a)–(c) are equivalent to

(i) $Be_i = 0$ for $i = 0,1,\ldots,n-1$,

(ii) $Bw_n(\cdot,d) \geqslant 0$ for every $d \in [c,b]$,

i. e. the same conditions as for the linear operator A in Theorem 1.3. An arbitrary continuous n-convex function f on $[c,b]$ can be extended to a continuous function $f \in \mathscr{K}^c_{n+1}([a,b])$ with $K_f = 0$ by defining $f = g$ on $[a,c]$, where g is an arbitrary n-concave function such that $g(c) = f(c)$. Then (7) yields $Af \leqslant 0 \leqslant Bf$, which gives the "if" part of Theorem 1.3. The "only if" part is immediate since $w_n(\cdot,d)$, e_i and $-e_i$ for $i = 0,1,\ldots,n-1$ are all continuous n-convex functions.

As we can see from the proof of Theorem 2.1, the function F is approximated well by functions F_m on $[a,c]$ and by functions G_k on $[c,b]$. The polynomials $\tilde{P}_{n-1}$ and Q_{n-1} are different, but if F (and, hence, f as well) satisfies sufficiently strong regularity properties at c, then these two polynomials can be chosen equal, i. e. one polynomial can be used in approximation of F over the whole interval $[a,b]$. If this is the case, then we can obtain a result similar to Theorem 2.1, but without the middle part in (7).

The next lemma shows that it is enough to assume that $F^{(n-2)}$ is continuous at c. Since F is n-concave on $[a,c]$ and n-convex on $[c,b]$, $F^{(n-2)}$ exists and is continuous on the open intervals (a,c) and (c,b), so the additional requirement is that the same property holds at point c as well.

LEMMA 2.3. *Let $n \geqslant 2$ and let the function $F_{m,k}$ be of the form*

$$F_{m,k}(x) = P_{n-1}(x) + \sum_{i=1}^{m} \alpha_i \rho_n(x,x_i) + \sum_{j=1}^{k} \beta_j w_n(x,y_j), \tag{10}$$

where $P_{n-1} \in \Pi_{n-1}$, α_i $(i = 1,\ldots,m)$ and β_j $(j = 1,\ldots,k)$ are real constants and $a \leqslant x_1 < \cdots < x_m < c < y_1 < \cdots y_k \leqslant b$.

(a) *A necessary and sufficient condition for $F_{m,k}$ to be n-concave on $[a,c]$ and n-convex on $[c,b]$ is that $\alpha_i \geqslant 0$ ($i=1,\dots,m$) and $\beta_j \geqslant 0$ ($j=1,\dots,k$).*

(b) *Every function $F \in C([a,b]) \cap C^{n-2}((a,b))$ that is n-concave on $[a,c]$ and n-convex on $[c,b]$ is the uniform limit of a sequence of functions $F_{m,k}$ as $m \to \infty$ and $k \to \infty$, where the $F_{m,k}$'s are of the form (10) with real constants $\alpha_i \geqslant 0$ ($i=1,\dots,m$) and $\beta_j \geqslant 0$ ($j=1,\dots,k$).*

Proof. See Appendix. □

The following theorem gives necessary and sufficient conditions for inequality of type (6) to hold and it is based on Lemma 2.3.

THEOREM 2.4. *Let $n \geqslant 2$ and let $A : C([a,c]) \to S(D)$ and $B : C([c,b]) \to S(D)$ be two linear and continuous operators. Then, the inequality*

$$Af \leqslant Bf$$

holds for every continuous $f \in \mathscr{K}^c_{n+1}([a,b]) \cap C^{n-2}((a,b))$ if and only if the operators A and B satisfy:

($\tilde{a}$) *$Ae_i = Be_i$ for $i=0,1,\dots,n$,*

($\tilde{b}$) *$A\rho_n(\cdot,d) \leqslant 0$ for every $d \in [a,c]$,*

($\tilde{c}$) *$Bw_n(\cdot,d) \geqslant 0$ for every $d \in [c,b]$.*

Proof. Assume that ($\tilde{a}$)–($\tilde{c}$) hold and let $f \in \mathscr{K}^c_{n+1}([a,b]) \cap C^{n-2}((a,b))$ be continuous with $F = f - K_f e_n/n!$ as in Definition 1.1. By Lemma 2.3, the function F given by (4) can be obtained as a uniform limit of functions of the form (10) with $\alpha_i \geqslant 0$ and $\beta_j \geqslant 0$. Assumption ($\tilde{a}$) yields $AP_{n-1} = BP_{n-1}$. Moreover, since $Aw_n(\cdot,y_j) = A0 = 0$ for $y_j \in [c,b]$ and $B\rho_n(\cdot,x_i) = B0 = 0$ for $x_i \in [a,c]$, we have

$$\begin{aligned} AF_{m,k} &= AP_{n-1} + \sum_{i=1}^{m} \alpha_i A\rho_n(\cdot,x_i) \leqslant AP_{n-1} = BP_{n-1} \\ &\leqslant BP_{n-1} + \sum_{j=1}^{k} \beta_i Bw_n(\cdot,y_j) = BF_{m,k}. \end{aligned}$$

By taking limits we conclude $AF \leqslant BF$, so

$$Af = AF + \frac{K_f}{n!}Ae_n \leqslant BF + \frac{K_f}{n!}Be_n = Bf.$$

On the other hand, assume (6) holds for every continuous $f \in \mathscr{K}^c_{n+1}([a,b]) \cap C^{n-2}((a,b))$. Since both e_i and $-e_i$ for $i = 0,1,\dots,n-1$ belong to $\mathscr{K}^c_{n+1}([a,b]) \cap C^{n-2}((a,b))$, we conclude that both $Ae_i \leqslant Be_i$ and $A(-e_i) \leqslant B(-e_i)$, so ($\tilde{a}$) holds. Furthermore, $\rho_n(\cdot,x_i), w_n(\cdot,y_j) \in C^{n-2}((a,b))$ and analogously as in the proof of Theorem 2.1 we conclude that both ($\tilde{b}$) and ($\tilde{c}$) hold. □

REMARK 2.5. Condition (a) is stronger than condition $(\tilde{a})$, which is reflected in inequalities (7) being stronger than inequality (6) with the middle term squeezed in between in (7). On the other hand, Theorems 2.1 and 2.4 represent separate results since it is possible to construct linear operators A and B that satisfy conditions $(\tilde{a})$–$(\tilde{c})$ and such that there exists an i, $0 \leqslant i \leqslant n-2$, such that $Ae_i = Be_i \neq 0$.

For example, let $n=3$, $b=-a>0$, $c=0$, $x_i \in [a,0]$ $(i=1,\dots,m)$, $y_i=-x_i$ and let the operators A and B be given by

$$Af = \sum_{i=1}^{m} p_i f(x_i), \qquad Bf = \sum_{i=1}^{m} p_i f(y_i).$$

Notice that

$$Ae_0 = Be_0 = \sum_{i=1}^{m} p_i, \quad Ae_1 = -Be_1 = \sum_{i=1}^{m} p_i x_i, \quad Ae_2 = Be_2 = \sum_{i=1}^{m} p_i x_i^2.$$

If p_i's are such that $Ae_1 = 0$, then $(\tilde{a})$ holds. Furthermore, if p_i's and x_i's are such that the condition

$$\sum_{i=1}^{m} p_i (x_i - d)_- \leqslant 0 \qquad \text{for every } d \in [a,0]$$

holds, then also $(\tilde{b})$ and $(\tilde{c})$ hold. For example, all this holds for $m=2$, $p_1=1$, $x_1=-3$, $p_2=-3$ and $x_2=-1$. Therefore, the linear operators

$$Af = f(-3) - 3f(-1)$$
$$Bf = f(3) - 3f(1)$$

satisfy $(\tilde{a})$–$(\tilde{c})$, but $Ae_0 = Be_0 = -2 \neq 0$. Thus, $Af \leqslant Bf$ for every continuous $f \in \bar{\mathscr{K}}_1^{n,c}([a,b])$, but there exists such an f such that (2.1) doesn't hold. For example, the constant function $f(x)=u$, where $0 \neq u \in \mathbb{R}$, satisfies $K_f = f''(0) = 0$, so the middle term in (2.1) is zero, while $Af = Bf = -2u \neq 0$.

Similar results hold for $n=1$ as well, but with minor technical modifications. Firstly, the functions ρ_n and w_n for $n=1$ are not continuous, so we need to require that the linear operators A and B are defined on a larger class of functions that contains them. More importantly, we also loose the "only if" parts of Theorems 2.1 and 2.4. Secondly, the representation in Lemma 2.3 assumes that $F \in C^{n-2}((a,b))$, an assumption that is mute for $n=1$ and can, actually, be ignored. Therefore, for simplicity of presentation, we state the result for $n=1$ in a separate theorem. As for notation, let $\bar{C}([u,v])$ denote a linear space of functions such that $C([u,v]) \subset \bar{C}([u,v])$ and $w_1(\cdot,d) \in \bar{C}([u,v])$ for $d \in [u,v]$ (for example, $\bar{C}([u,v]) = \{f + \sum_{i=1}^{m} \alpha_i w_1(\cdot,x_i) : f \in C([u,v]), \alpha_i \in \mathbb{R}, x_i \in [u,v]\}$).

THEOREM 2.6. *Let $A : \bar{C}([a,c]) \to S(D)$ and $B : \bar{C}([c,b]) \to S(D)$ be two linear and continuous operators. If* (*i*) $Ae_0 = Be_0$ *and* $Ae_1 = Be_1$,

(*ii*) $A\rho_1(\cdot,d) \leqslant 0$ *for every* $d \in [a,c]$,

(*iii*) $Bw_1(\cdot,d) \geqslant 0$ *for every* $d \in (c,b]$,

then for every continuous $f \in \mathscr{K}_2^c([a,b])$ *the following inequality holds*

$$Af \leqslant Bf. \tag{11}$$

If, additionally,

(*iv*) $Ae_0 = Be_0 = 0$,

then for every constant K_f *from Definition 1.1 the following inequalities hold*

$$Af \leqslant K_f Ae_1 = K_f Be_1 \leqslant Bf.$$

Proof. The function $F = f - K_f e_1$ is continuous and it is non-increasing on $[a,c]$ and non-decreasing on $[c,b]$. Since a continuous function on a closed interval is uniformly continuous, for arbitrary $\varepsilon > 0$ there exist points $a \leqslant x_1 < x_2 < \ldots < x_m \leqslant c < y_1 < \ldots < y_k \leqslant b$ such that the step function

$$g(x) = F(c) + \sum_{i=1}^{m} \alpha_i \rho_1(x,x_i) + \sum_{j=1}^{k} \beta_j w_1(x,y_j)$$

satisfies

$$\max_{a \leqslant x \leqslant b} |F(x) - g(x)| \leqslant \varepsilon,$$

where

$$\begin{aligned}
\alpha_m &= F(x_m) - F(c) \geqslant 0,\\
\alpha_i &= F(x_{i-1}) - F(x_i) \geqslant 0, \quad i = m-1, m-2, \ldots, 2,\\
\beta_1 &= F(y_1) - F(c) \geqslant 0,\\
\beta_j &= F(y_j) - F(y_{j-1}) \geqslant 0, \quad j = 2,3,\ldots,k.
\end{aligned}$$

The rest of the proof follows the same lines as the proof of Theorem 2.4. □

REMARK 2.7. We, indeed, do not have the "only if" part in Theorem 2.6. For example, if A and B are the linear operators

$$Af = 0 \quad \text{and} \quad Bf = f_-(d) - f(d) \text{ for some fixed } d \in (c,b),$$

then $Af = Bf = 0$ for every continuous f (so (11) holds), but $Bw_1(\cdot,d) = -1 < 0$ (i. e. (iii) of Theorem 2.6 doesn't hold).

REMARK 2.8. If (i)–(iii) of Theorem 2.6 hold and $Aw_1(\cdot,\tilde{x}) = 0$ for some $a < \tilde{x} < c$ and $B\rho_1(\cdot,\tilde{y}) = 0$ for some $c < \tilde{y} < b$, then (iv) of Theorem 2.6 holds also. Indeed, for every d we have $w_1(\cdot,d) + \rho_1(\cdot,d) = e_0$, so

$$0 \geqslant A\rho_1(\cdot,\tilde{x}) = Ae_0 = Be_0 = Bw_1(\cdot,\tilde{y}) \geqslant 0.$$

3. Applications

We can obtain a probabilistic Levinson's type inequality as a consequence of Theorem 2.1.

COROLLARY 3.1. *Let* $X, Y : \Omega \to [a,c]$ *be two random variables such that* $\mathrm{Var}[X] = \mathrm{Var}[Y] = C$. *Then, for every continuous* $f \in \mathscr{K}_3^c([a,b])$ *the inequalities*

$$\mathrm{E}[f(X)] - f(\mathrm{E}[X]) \leqslant \frac{K_f}{2} C \leqslant \mathrm{E}[f(Y)] - f(\mathrm{E}[Y])$$

hold.

Proof. Apply Theorem 2.1 to the linear operators

$$Af = \mathrm{E}[f(X)] - f(\mathrm{E}[X]),$$
$$Bf = \mathrm{E}[f(Y)] - f(\mathrm{E}[Y]).$$

Since continuous functions on a segment are bounded, by the dominated convergence theorem the linear operators A and B are continuous. Condition (a) holds since $Ae_0 = Be_0 = \mathrm{E}[1] - 1 = 0$, $Ae_1 = \mathrm{E}[X] - \mathrm{E}[X] = 0 = \mathrm{E}[Y] - \mathrm{E}[Y] = Bf$, $Ae_2 = \mathrm{Var}[X]$ and $Be_2 = \mathrm{Var}[Y]$. Furthermore, the functions $w_2(\cdot,d)$ (resp. $\rho_2(\cdot,d)$) for $d \in [a,c]$ (resp. $d \in [c,b]$) are convex (resp. concave), so (b) (resp. (c)) hold due to Jensen's inequality. $\square$

We can get a generalization of the probabilistic Levinson type inequality from Corollary 3.1 without the middle term as a corollary of Theorem 2.4.

COROLLARY 3.2. *Let* $\lambda : [a,c] \to \mathbb{R}$ *and* $\mu : [c,b] \to \mathbb{R}$ *be two functions of bounded variation such that*

$$\bar{x}_\lambda = \int_a^c x\, d\lambda(x) \in [a,c] \quad \text{and} \quad \bar{x}_\mu = \int_c^b x\, d\mu(x) \in [c,b].$$

Then, the inequality

$$\int_a^c f(x)\, d\lambda(x) - f(\bar{x}_\lambda) \leqslant \int_c^b f(x)\, d\mu(x) - f(\bar{x}_\mu)$$

holds for every continuous $f \in \mathscr{K}_3^c([a,b])$ *if and only if* λ *and* μ *satisfy:*

$$(i) \int_a^c d\lambda(x) = \int_c^b d\mu(x) \text{ and } \int_a^c x^2\, d\lambda(x) - \bar{x}_\lambda^2 = \int_c^b x^2\, d\mu(x) - \bar{x}_\mu^2,$$

$$(ii) \int_a^d (x-d)\, d\lambda(x) \leqslant (\bar{x}_\lambda - d)_- = \min\{\bar{x}_\lambda - d, 0\} \text{ for every } d \in [a,c],$$

$$(iii) \int_d^b (x-d)\, d\mu(x) \geqslant (\bar{x}_\mu - d)_+ = \max\{\bar{x}_\mu - d, 0\} \text{ for every } d \in [c,b].$$

Proof. Apply Theorem 2.4 to the linear operators A and B given by

$$Af = \int_a^c f(x)\,d\lambda(x) - f(\bar{x}_\lambda),$$
$$Bf = \int_c^b f(x)\,d\mu(x) - f(\bar{x}_\mu).$$

By the same argument as in the proof of Corollary 3.1, the operators A and B are continuous. Conditions $(\tilde{a})$–$(\tilde{c})$ for these particular operators correspond to conditions (i)–(iii). □

Since the functions λ and μ in Corollary 3.3 do not need to generate probability measures, that corollary is, indeed, a generalization of Corollary 3.1. For example, if λ and μ satisfy the Jensen-Steffensen conditions (i. e. $\lambda(a) \leqslant \lambda(x) \leqslant \lambda(c)$ for every $x \in [a,c]$ and $\lambda(a) < \lambda(c)$; $\mu(c) \leqslant \mu(x) \leqslant \mu(b)$ for every $x \in [c,b]$ and $\mu(c) < \mu(b)$), then the Jensen-type inequality still holds for convex functions (see, e. g., [6]). Hence, the convex functions $w_2(\cdot,d)$ and $-\rho_2(\cdot,d)$ satisfy the inequalities in (ii) and (iii) of Corollary 3.3.

The following is another corollary of Theorem 2.4.

COROLLARY 3.3. *Let $\lambda : [a,c] \to \mathbb{R}$ and $\mu : [c,b] \to \mathbb{R}$ be two functions of bounded variation and $n \geqslant 2$. Then, the inequality*

$$\int_a^c f(x)\,d\lambda(x) \leqslant \int_c^b f(x)\,d\mu(x)$$

holds for every continuous $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$ if and only if λ and μ satisfy:

$$(i)\ \int_a^c x^i\,d\lambda(x) = \int_c^b x^i\,d\mu(x),\ \textit{for every}\ i = 0,1,\dots,n,$$
$$(ii)\ \int_a^d (x-d)^{n-1}\,d\lambda(x) \leqslant 0\ \textit{for every}\ d \in [a,c],$$
$$(iii)\ \int_d^b (x-d)^{n-1}\,d\mu(x) \geqslant 0\ \textit{for every}\ d \in [c,b].$$

Proof. Apply Theorem 2.4 to the linear operators A and B given by

$$Af = \int_a^c f(x)\,d\lambda(x),$$
$$Bf = \int_c^b f(x)\,d\mu(x).$$

By the same argument as in the proof of Corollary 3.1, the operators A and B are continuous. Conditions $(\tilde{a})$–$(\tilde{c})$ for these particular operators correspond to conditions (i)–(iii). □

The following corollary is the discrete version of Corollary 3.3.

COROLLARY 3.4. *Let $n \in \mathbb{N}$, $n \geqslant 2$, and $a \leqslant x_1 < \cdots < x_m \leqslant c \leqslant y_1 < \cdots < y_k \leqslant b$. Then, the inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \leqslant \sum_{j=1}^{k} q_j f(y_j)$$

holds for every continuous $f \in \mathscr{K}_{n+1}^{c}([a,b]) \cap C^{n-2}((a,b))$ if and only if the sequences p and q satisfy:

(*i*) $\sum_{j=1}^{m} p_j x_j^i = \sum_{j=1}^{k} q_j y_j^i$ *for every* $i = 0, 1, \ldots, n$,

(*ii*) $\sum_{i=1}^{m} p_i (x_i - d)_-^{n-1} \leqslant 0$ *for every* $d \in [a,c]$,

(*iii*) $\sum_{j=1}^{k} q_j (y_j - d)_+^{n-1} \geqslant 0$ *for every* $d \in [c,b]$.

Proof. Apply Theorem 2.4 to the linear operators

$$Af = \sum_{i=1}^{m} p_i f(x_i) \qquad \text{and} \qquad Bf = \sum_{j=1}^{k} q_j f(y_j). \tag{12}$$

□

REMARK 3.5. Popoviciu studied necessary and sufficient conditions on points x_i and weights p_i for inequality $\sum_{i=1}^{m} p_m f(x_m) \geqslant 0$ to be valid for every n-convex function f (see [7], [6]). In light of Remark 2.2, Corollary 3.4 is an extension of Popoviciu's results.

The version of Corollary 3.4 for $n = 1$ can be obtained as a corollary of Theorem 2.6.

COROLLARY 3.6. *Let $a \leqslant x_1 \leqslant \cdots < x_m < c < y_1 < \cdots < y_k \leqslant b$, $\bar{x}_p = \sum_{i=1}^{m} p_i x_i$, $\bar{y}_q = \sum_{j=1}^{k} q_j y_j$ be such that:*

(*i*) $\bar{x}_p = \bar{y}_q$,

(*ii*) $\sum_{i=1}^{m_1} p_i \leqslant 0$ *for every* $m_1 = 1, \ldots, m-1$, *and* $\sum_{i=1}^{m} p_i = 0$,

(*iii*) $\sum_{j=k_1}^{k} q_j \geqslant 0$ *for every* $k_1 = 2, \ldots, k$, *and* $\sum_{j=1}^{k} q_j = 0$.

Then, the inequality

$$\sum_{i=1}^{m} p_i f(x_i) \leqslant K_f \bar{x}_p = K_f \bar{y}_q \leqslant \sum_{j=1}^{k} q_j f(y_j) \tag{13}$$

holds for every continuous $f \in \mathscr{K}_2^{c}([a,b])$.

Proof. Follows by applying Theorem 2.6 to the linear operators A and B given by (12). Notice that $Aw_1(\cdot,d)=0$ for $d\in(x_m,c)$ and $B\rho_1(\cdot,d)=0$ for $d\in(c,y_1)$, so, by Remark 2.8, $Ae_0=\sum_{i=1}^{m}p_i=0$ and $Be_0=\sum_{j=1}^{k}q_j=0$ are implied by the other assumptions. □

A simple example when p_i's and q_j's satisfy assumptions (ii) and (iii) of Corollary 3.6 is when $m=2m'$ and $k=2k'$ are even and $p_i=(-1)^i$, $q_j=(-1)^j$. Furthermore, if x_i's and y_j's are such that $\bar{x}_p=\sum_{i=1}^{m'}(x_{2i}-x_{2i-1})=\sum_{j=1}^{k'}\left(y_{2j}-y_{2j-1}\right)=\bar{y}_q$, then (i) holds as well and inequality (13) states

$$\sum_{i=1}^{m'}(f(x_{2i})-f(x_{2i-1}))\leqslant K_f\bar{x}_p=K_f\bar{y}_q\leqslant\sum_{j=1}^{k'}\left(f(y_{2j})-f(y_{2j-1})\right).$$

4. Appendix

For the sake of clarity and completeness, in this section we will recall some of the well known results that are frequently used in the paper, as well as give a rather technical proof of Lemma 2.3. The stated results without explicitly cited source can be found in, e. g., [6].

A kth order divided difference of a function $f:I\to\mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0,\ldots,x_k\in I$ is defined recursively by

$$[x_i]f=f(x_i),\qquad \text{for } i=0,\ldots,k$$

and

$$[x_0,\ldots,x_k]f=\frac{[x_1,\ldots,x_k]f-[x_0,\ldots,x_{k-1}]f}{x_k-x_0}.$$

The value $[x_0,\ldots,x_k]f$ is independent of the order of the points $x_0,\ldots,x_k$. If $P_k\in\Pi_k$, then $[x_0,\ldots,x_k]P_k$ is equal to the coefficient of P_k in front of x^k.

A function $f:I\to\mathbb{R}$ is called n-convex if $[x_0,\ldots,x_n]f\geqslant 0$ for all choices of $n+1$ distinct points $x_0,\ldots,x_n\in I$. A function f is said to be n-concave if the function $-f$ is n-convex. If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)}\geqslant 0$. More precisely, differentiability properties of n-convex functions and connection between various orders of convexity are described in the following lemma.

LEMMA 4.1. *Let $f:(a,b)\to\mathbb{R}$ and $n\geqslant 2$. Then, the following statements are equivalent:*

(i) *f is n-convex,*

(ii) *$f^{(k)}$ exists and is $(n-k)$-convex for $0\leqslant k\leqslant n-2$,*

(iii) *$f_+^{(n-1)}$ exists on (a,b) and it is right-continuous and non-decreasing,*

(iv) *$f_-^{(n-1)}$ exists on (a,b) and it is left-continuous and non-decreasing.*

If f is n-convex, then $f_-^{(n-1)} \leqslant f_+^{(n-1)}$ and $f_-^{(n-1)}(x) \neq f_+^{(n-1)}(x)$ for at most countably many points x. Moreover, for any point d outside this countable set and any g such that $f_-^{(n-1)} \leqslant g \leqslant f_+^{(n-1)}$ it holds $f^{(n-2)}(x) = f^{(n-2)}(d) + \int_d^x g(t)\,dt$.

The class of 3-convex functions at a point was introduced in [1]. There it was proven that a function is 3-convex on an interval if and only if it is 3-convex at every point of its interior, which justifies the name. From statement (ii) of Lemma 4.1 we can deduce that this property transfers to $(n+1)$-convex functions, i. e. a function is $(n+1)$-convex on an interval if and only if it is $(n+1)$-convex at every point of its interior. In [1] it was also shown that if $f''_-(c)$ and $f''_+(c)$ exist for $f \in \mathscr{K}_3^c(I)$, then K_f from Definiton 1.1 satisfies $f''_-(c) \leqslant K_f \leqslant f''_+(c)$ (moreover, any constant from the interval $[f''_-(c), f''_+(c)]$ can be taken for K_f). Again, due to Lemma 4.1 (ii), we can see that this property transfers to $f \in \mathscr{K}_{n+1}^c(I)$, i. e. if $f_-^{(n)}(c)$ and $f_+^{(n)}(c)$ exist, then $f_-^{(n)}(c) \leqslant K_f \leqslant f_+^{(n)}(c)$.

Let us also mention that an n-convex function f, $n \geqslant 2$, on the closed interval $[a,b]$ can have discontinuities only at the edges, a and b, and only in a certain direction. More precisely, it holds $(-1)^n(f_+(a) - f(a)) \leqslant 0$ and $f_-(b) \leqslant f(b)$. Consequently, $f \in \mathscr{K}_{n+1}^c([a,b])$ can have discontinuities only at a, c and b and their directions can be derived from the aforementioned discontinuity properties of n-convex functions.

Proof of Lemma 2.3. The intuitive idea of the proof is simple – the goal is to construct a step function that approximates $F_+^{(n-1)}$ well enough so that, after integrating it $n-1$ times, we get a uniformly good approximation of F.

Firstly, due to the assumptions, $F_+^{(n-1)}$ exists on (a,c), where it is non-increasing, and on (c,b), where it is non-decreasing. Furthermore, for every $x, x' \in (a,c)$ and $y, y' \in (c,b)$ it holds

$$\int_x^{x'} F_+^{(n-1)}(t)\,dt = F^{(n-2)}(x') - F^{(n-2)}(x), \tag{14}$$

$$\int_{y'}^{y} F_+^{(n-1)}(t)\,dt = F^{(n-2)}(y) - F^{(n-2)}(y'). \tag{15}$$

Since $F^{(n-2)}$ is continuous at c, the limits $x' \to c$ in (14) and $y' \to c$ in (15) exist. The limit $\lim_{t \nearrow c} F_+^{(n-1)}(t)$ (resp. $\lim_{t \searrow c} F_+^{(n-1)}(t)$) can be $-\infty$, but then the integral (14) (resp. (15)) with $x' = c$ (resp. $y' = c$) exists and is finite as an improper integral. In conclusion,

$$\int_x^{y} F_+^{(n-1)}(t)\,dt = F^{(n-2)}(y) - F^{(n-2)}(x), \quad \text{for every } x, y \in (a,b), \tag{16}$$

with, potentially, improper integral(s) at c. Furthermore, due to the properties of $F_+^{(n-1)}$ mentioned above, it is easy to see that for arbitrary $\varepsilon_1 > 0$ there exist a constant γ and points $\tilde{x} < c$ and $\tilde{y} > c$ sufficiently close to c such that

$$\int_{\tilde{x}}^{\tilde{y}} \left| F_+^{(n-1)}(t) - \gamma \right| dt < \varepsilon_1, \tag{17}$$

where $\gamma = \min\{F_+^{(n-1)}(\tilde{x}), F_+^{(n-1)}(\tilde{y})\}$. Let us now define the step function

$$g_{n-1}(x) = \gamma + \sum_{i=1}^{m} \tilde{\alpha}_i \rho_1(x, x_i) + \sum_{j=1}^{k} \tilde{\beta}_j w_1(x, y_j), \tag{18}$$

where

$$\begin{aligned}
\tilde{\alpha}_i &= F_+^{(n-1)}(x_i) - F_+^{(n-1)}(x_{i+1}) \geqslant 0, \quad i = 1, \ldots, m-1 \\
\tilde{\alpha}_m &= F_+^{(n-1)}(x_m) - \gamma \geqslant 0 \\
\tilde{\beta}_1 &= F_+^{(n-1)}(y_1) - \gamma \geqslant 0 \\
\tilde{\beta}_j &= F_+^{(n-1)}(y_j) - F_+^{(n-1)}(y_{j-1}) \geqslant 0, \quad j = 2, \ldots, k.
\end{aligned}$$

The points x_i's and y_j's will suitably be chosen later (so that g_{n-1} will be a "good" approximation of $F_+^{(n-1)}$). Furthermore, let us define, recursively, for $l = n-2, \ldots, 1, 0$:

$$\begin{aligned}
g_l(x) &= \int_c^x g_{l+1}(t)\,dt + F^{(l)}(c) \\
&= P_{n-1-l}(x) + \frac{1}{(n-l)!}\left(\sum_{i=1}^{m} \tilde{\alpha}_i \rho_{n-l}(x, x_i) + \sum_{j=1}^{k} \tilde{\beta}_j w_{n-l}(x, y_j)\right).
\end{aligned} \tag{19}$$

Since g_{n-1} will be a "good" approximation of $F_+^{(n-1)}$, by construction (19) the function g_l will be a "good" approximation of $F^{(l)}$. Notice, also, that $g_l^{(j)} = g_{l+j}$ and g_0 is a function of the form (10) with $\alpha_i = \tilde{\alpha}_i/n!$ and $\beta_i = \tilde{\beta}_i/n!$.

Let $\varepsilon_2 > 0$ be arbitrary and let us now choose the points $y_1, y_2, \ldots$ recursively by the following algorithm: set $y_1 = \tilde{y}$, where $\tilde{y}$ is from (17). If y_j is chosen, let

$$y_{j+1} = \inf_{y_i < y < b} \{y : F_+^{(n-1)}(y) - F_+^{(n-1)}(y_j) \geqslant \varepsilon_2\}. \tag{20}$$

Since $F_+^{(n-1)}$ is right-continuous and non-increasing on (c, b) we have

$$|F_+^{(n-1)}(y_j) - F_+^{(n-1)}(y)| \leqslant \varepsilon_2 \text{ for all } y \in [y_j, y_{j+1}), \tag{21}$$

$$F_+^{(n-1)}(y_{j+1}) - F_+^{(n-1)}(y_j) \geqslant \varepsilon_2. \tag{22}$$

Due to (22), if $\lim_{t \nearrow b} F_+^{(n-1)}(t)$ is finite, then the procedure (20) will stop after finitely many steps at some y_{k-1} and, in that case, set $y_k = b$. Otherwise, if $\lim_{t \nearrow b} F_+^{(n-1)}(t) = \infty$, then for sufficiently large k the point y_k can be arbitrarily close to b.

If $\lim_{t \searrow a} F_+^{(n-1)}(t)$ is finite, then set $x_1 = a$. Otherwise, if $\lim_{t \searrow a} F_+^{(n-1)}(t) = \infty$, the point x_1 will suitably be chosen later (and such to be sufficiently close to a). Let us now choose the points $x_2, x_3, \ldots$ recursively by the following rule: if x_i is chosen, let

$$x_{i+1} = \inf_{x_i < x < c} \{x : F_+^{(n-1)}(x_i) - F_+^{(n-1)}(x) \geqslant \varepsilon_2\}. \tag{23}$$

Again, the following holds

$$|F_+^{(n-1)}(x_i) - F_+^{(n-1)}(x)| \leqslant \varepsilon_2 \text{ for all } x \in [x_i, x_{i+1}), \tag{24}$$

$$F_+^{(n-1)}(x_i) - F_+^{(n-1)}(x_{i+1}) \geqslant \varepsilon_2. \tag{25}$$

Due to (25), after finitely many steps of the form (23) we will reach a point x_{m-1} such that $\tilde{x}$ from (17) satisfies $F_+^{(n-1)}(x_{m-1}) - F_+^{(n-1)}(\tilde{x}) \leqslant \varepsilon_2$. Set $x_m = \tilde{x}$ and stop. Notice now that, due to (17), (21) and (24), for every $x \in [x_1, y_k]$ one has

$$\begin{aligned}
|F^{(n-2)}(x) - g_{n-2}(x)| &= \left| \int_c^x \left(F_+^{(n-1)}(t) - g_{n-1}(t) \right) dt \right| \\
&\leqslant \int_{x_1}^{x_m} \left| F_+^{(n-1)}(t) - g_{n-1}(t) \right| dt + \int_{x_m}^{y_1} \left| F_+^{(n-1)}(t) - g_{n-1}(t) \right| dt \\
&\quad + \int_{y_1}^{y_k} \left| F_+^{(n-1)}(t) - g_{n-1}(t) \right| dt \\
&\leqslant \varepsilon_2 (x_m - x_1 + y_k - y_1) + \varepsilon_1 \leqslant \varepsilon_3,
\end{aligned}$$

where the last inequality holds for arbitrary $\varepsilon_3 > 0$ when we choose ε_1 and ε_2 sufficiently small. In a similar way, it can now be shown by induction that for every $x \in [x_1, y_k]$ and $i = n-3, n-4, \ldots, 1, 0$,

$$|F^{(i)}(x) - g_i(x)| = \left| \int_c^x \left(F^{(i+1)}(t) - g_{i+1}(t) \right) dt \right| \leqslant \varepsilon_3 (b-a)^{n-2-i}. \tag{26}$$

If $x_1 = a$ and $y_k = b$, then $\|F - g_0\| \leqslant \varepsilon_3 (b-a)^{n-2}$ by (26) and this finishes the proof. Otherwise, if $\lim_{t \nearrow b} F_+^{(n-1)}(t) = \infty$ or $\lim_{t \searrow a} F_+^{(n-1)}(t)$, we will use some properties of Taylor's expansion and polynomials.

Let $P_{y_k, n-2} \in \Pi_{n-2}$ denote Taylor's polynomial of F at y_k of degree $n-2$, i. e. $P_{y_k,n-2}^{(i)}(y_k) = F^{(i)}(y_k)$ for $i = 0, 1, \ldots, n-2$. Due to (16), the remainder in Taylor's expansion can be written in the integral form, i. e. for every $x \in (a, b)$ it holds

$$F(x) = P_{y_k,n-2}(x) + \frac{1}{(n-2)!} \int_{y_k}^x (x-t)^{n-2} F_+^{(n-1)}(t)\, dt.$$

Let us also denote by $\bar{P}_{n-1}$ the polynomial

$$\begin{aligned}
\bar{P}_{n-1}(x) &= P_{y_k,n-2}(x) + F_+^{(n-2)}(y_k)(x - y_k)^{n-1} \\
&= \sum_{i=0}^{n-2} \frac{F^{(i)}(y_k)}{i!} (x - y_k)^i + F_+^{(n-2)}(y_k)(x - y_k)^{n-1}.
\end{aligned}$$

We have

$$F(x) = \bar{P}_{n-1}(x) + \frac{1}{(n-2)!} \int_{y_k}^x (x-t)^{n-2} (F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\, dt \tag{27}$$

since $\int_{y_k}^{x}(x-t)^{n-2}\,dt=(n-2)!(x-y_k)^{n-1}$. It is easy to see that the mapping $h_{y_k}(x)=\int_{y_k}^{x}(x-t)^{n-2}(F_+^{(n-1)}(t)-F_+^{(n-1)}(y_k))\,dt$ is monotone on $[y_k,b]$ with $h_{y_k}(y_k)=0$. Since F is continuous at b, the limit $x\to b$ in (27) exists and the integral

$$\int_{y_k}^{b}(x-t)^{n-2}(F_+^{(n-1)}(t)-F_+^{(n-1)}(y_k))\,dt \tag{28}$$

is finite. Moreover, since we can choose y_k arbitrarily close to b, by the dominated convergence theorem integral (28) can be arbitrarily small. Therefore, for arbitrary $\varepsilon_4>0$, we can choose y_k such that for every $x\in[y_k,b]$ it holds

$$\begin{aligned}|F(x)-\bar{P}_{n-1}(x)|&=\left|\frac{1}{(n-2)!}\int_{y_k}^{x}(x-t)^{n-2}(F_+^{(n-1)}(t)-F_+^{(n-1)}(y_k))\,dt\right|\\&\leqslant\left|\frac{1}{(n-2)!}\int_{y_k}^{b}(b-t)^{n-2}(F_+^{(n-1)}(t)-F_+^{(n-1)}(y_k))\,dt\right|<\varepsilon_4.\end{aligned}$$

By construction, for $x\in[y_k,b]$ we have $g_0(x)=P_{n-1}(x)+\sum_{j=1}^{k}\beta_j(x-y_j)^{n-1}$, i. e. g_0 on the interval $[y_k,b]$ is a polynomial in Π_{n-1}. Furthermore, by construction $g_0\in C^{(n-2)}([a,b])$ and $g_0^{(n-1)}(y_k)=g_{n-1}(y_k)=F_+^{(n-1)}(y_k)$. Therefore, for $x\in[y_k,b]$ it holds

$$g_0(x)=\sum_{i=0}^{n-2}\frac{g_0^{(i)}(y_k)}{i!}(x-y_k)^i+F_+^{(n-1)}(y_k)(x-y_k)^{n-1}.$$

From (26) we conclude $|F^{(i)}(y_k)-g_0^{(i)}(y_k)|\leqslant\varepsilon_3(b-a)^{n-2-i}$. Therefore, for every $x\in[y_k,b]$ we have

$$|\bar{P}(x)-g_0(x)|\leqslant\varepsilon_3(b-a)^{n-2}\sum_{i=0}^{n-2}\frac{1}{i!}$$

and

$$|F(x)-g_0(x)|\leqslant|F(x)-\bar{P}(x)|+|\bar{P}(x)-g_0(x)|\leqslant\varepsilon_4+\varepsilon_3(b-a)^{n-2}\sum_{i=0}^{n-2}\frac{1}{i!}. \tag{29}$$

In the same way we can show that we can choose x_1 sufficiently close to a such that (29) holds for every $x\in[a,x_1]$. Finally, for sufficiently small ε_3 and ε_4, from (26) and (29) we conclude that for arbitrary $\varepsilon>0$ we can construct g_0 of the form (10) such that

$$|F(x)-g_0(x)|\leqslant\varepsilon$$

for every $x\in[a,b]$. □

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