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ON A MULTIDIMENSIONAL HILBERT–TYPE INTEGRAL INEQUALITY WITH LOGARITHM FUNCTION

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Abstract. By the use of the transfer formula, the methods of weight functions and technique of Real Analysis, a multidimensional Hilbert-type integral inequality with a few parameters and a best possible constant factor related to the kernel of logarithm function is given. The equivalent forms and some reverses are obtained. The operator expressions and a few particular results related to the kernels of non-homogeneous and homogeneous are considered.

1. Introduction

If $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f(x),g(y)\geqslant 0,\ f\in L^p(\mathbf{R}_+),\ g\in L^q(\mathbf{R}_+),\ ||f||_p=(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}>0,\ ||g||_q>0$, then we have Hardy-Hilbert's integral inequality as follows (cf. [1]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m,b_n\geqslant 0$, $a=\{a_m\}_{m=1}^\infty\in l^p,\ b=\{b_n\}_{n=1}^\infty\in l^q,\ ||a||_p=(\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}}>0,\ ||b||_q>0$, we still have the following discrete variant of the above inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [1], [2], [3], [4], [5], [6]).

In 1998, by introducing an independent parameter $\lambda\in(0,1]$, Yang [7] gave an extension of (1) at $p=q=2$. In 2009 and 2011, Yang [3], [4] gave some extensions of (1) and (2) as follows:

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If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x), g(y) \geqslant 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := (\int_0^\infty \phi(x)|f(x)|^p dx)^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{3}$$

where, the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_{m,} b_n \geqslant 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := (\sum_{n=1}^\infty \phi(n)|a_n|^p)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}, \tag{4}$$

with the best possible constant factor $k(\lambda_1)$.

Clearly, for $\lambda = 1, k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2).

In 2006, by applying the transfer formula, Hong [8] first published a multidimensional Hilbert's integral inequality, which is an extension of (3). Some other results relating integral, discrete and half-discrete multidimensional Hilbert-type inequalities were given by [9]–[21], which provided some new ways to study these kinds of inequalities.

In this paper, by the use of the transfer formula, the methods of weight functions and technique of Real Analysis, we give a multidimensional Hilbert-type integral inequality with a few parameters and a best possible constant factor related to the kernel of logarithm function. The equivalent forms and some reverses are obtained. We also consider the operator expressions and some particular results related to the kernels of non-homogeneous and homogeneous.

2. Some lemmas

If $i_0, j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we put

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x_k|^\alpha \right)^{\frac{1}{\alpha}} \ (x = (x_1, \cdots, x_{i_0}) \in \mathbf{R}^{i_0}),$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y_k|^\beta\right)^{\frac{1}{\beta}} \; (y = (y_1, \cdots, y_{j_0}) \in \mathbf{R}^{j_0}).$$

LEMMA 1. *Assuming that* $s \in \mathbf{N}$, $\gamma, M > 0$, $\Psi(u)$ *is a non-negative measurable function in* $(0,1]$, *and*

$$D_M := \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leqslant 1 \right\},$$

we have the following transfer formula (cf. [6]*):*

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du, \tag{5}$$

where, $\Gamma(\cdot)$ *is the gamma function defined by*

$$\Gamma(t) := \int_0^\infty e^{-v} v^{t-1} dv (t > 0).$$

In view of (5) and the conditions, it follows that

(i) For $\mathbf{R}_+^s = \lim_{M\to\infty} D_M$, we have

$$\int \cdots \int_{\mathbf{R}_+^s} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \lim_{M\to\infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{6}$$

(ii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \geqslant 1\} = \lim_{M\to\infty} \left\{ x \in \mathbf{R}_+^s; \frac{1}{M^\gamma} \leqslant u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leqslant 1 \right\},$$

setting $\Psi(u) = 0 \;\; (u \in (0, \frac{1}{M^\gamma}))$, we have

$$\int \cdots \int_{\{x\in\mathbf{R}_+^s; ||x||_\gamma \geqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \lim_{M\to\infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\frac{1}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{7}$$

(iii) for

$$\{x \in \mathbf{R}_+^s; ||x||_\gamma \leqslant 1\} = \lim_{M\to\infty} \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leqslant \frac{1}{M^\gamma} (M > 1) \right\},$$

setting $\Psi(u) = 0 \;\; (u \in (\frac{1}{M^\gamma}, \infty))$, we have

$$\int \cdots \int_{\{||x||_\gamma \leqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \lim_{M\to\infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^{\frac{1}{M^\gamma}} \Psi(u) u^{\frac{s}{\gamma}-1} du.$$

NOTE 1. Setting $E_\delta := \{u > 0; u^\delta \geqslant \frac{1}{M^{\delta\gamma}}\}$ $(\delta \in \{-1,1\},\ M > 1)$, in view of (7) and the above result, it follows

$$\int \cdots \int_{\{x\in \mathbf{R}_+^s;||x||_\gamma^\delta \geqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s = \lim_{M\to\infty} \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})} \int_{E_\delta} \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{8}$$

LEMMA 2. *For* $s \in \mathbf{N}, \gamma,\ \varepsilon > 0$, *we have*

$$\int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \geqslant 1\}} ||x||_\gamma^{-s-\varepsilon} dx = \int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \leqslant 1\}} ||x||_\gamma^{-s+\varepsilon} dx = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon\gamma^{s-1}\Gamma(\frac{s}{\gamma})}. \tag{9}$$

Proof. By (7), it follows

$$\begin{aligned}
&\int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \geqslant 1\}} ||x||_\gamma^{-s-\varepsilon} dx \\
&= \int \cdots \int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \geqslant 1\}} \left\{ M \left[\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right]^{\frac{1}{\gamma}} \right\}^{-s-\varepsilon} dx_1 \cdots dx_s \\
&= \lim_{M\to\infty} \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})} \int_{\frac{1}{M^\gamma}}^1 (Mu^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.
\end{aligned}$$

By (iii), we find

$$\begin{aligned}
&\int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \leqslant 1\}} ||x||_\gamma^{-s+\varepsilon} dx \\
&= \int \cdots \int_{\{x\in \mathbf{R}_+^s;||x||_\gamma \leqslant 1\}} \left\{ M \left[\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right]^{\frac{1}{\gamma}} \right\}^{-s+\varepsilon} dx_1 \cdots dx_s \\
&= \lim_{M\to\infty} \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})} \int_0^{\frac{1}{M^\gamma}} (Mu^{1/\gamma})^{-s+\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.
\end{aligned}$$

Hence, we have (9). □

NOTE 2. For $\varepsilon > 0,\ \delta \in \{-1,1\}$, in view of (9), we still have

$$\int_{\{x\in \mathbf{R}_+^s;||x||_\gamma^\delta \geqslant 1\}} ||x||_\gamma^{-s-\delta\varepsilon} dx = \int_{\{x\in \mathbf{R}_+^s;||x||_\gamma^\delta \leqslant 1\}} ||x||_\gamma^{-s+\delta\varepsilon} dx = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon\gamma^{s-1}\Gamma(\frac{s}{\gamma})}. \tag{10}$$

DEFINITION 1. Assuming that $i_0, j_0 \in \mathbf{N},\ \alpha, \beta, b > 0,\ 0 < \sigma < \eta,\ \delta_i \in \{-1,1\}$ $(i = 1,2)$, $x = (x_1, \cdots, x_{i_0}) \in \mathbf{R}_+^{i_0}$, $y = (y_1, \cdots, y_{j_0}) \in \mathbf{R}_+^{j_0}$, we define two weight functions $\omega(\sigma,y)$ and $\varpi(\sigma,x)$ as follows:

$$\omega(\sigma,y) := ||y||_\beta^{-\delta_2\sigma} \int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{1}{||x||_\alpha^{i_0-\delta_1\sigma}} dx, \tag{11}$$

$$\varpi(\sigma,x) := ||x||_\alpha^{\delta_1\sigma}\int_{\mathbf{R}_+^{j_0}}\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right]\frac{1}{||y||_\beta^{j_0+\delta_2\sigma}}dy. \tag{12}$$

By (6), we find

$$\begin{aligned}\omega(\sigma,y) &= ||y||_\beta^{-\delta_2\sigma}\int\cdots\int_{\mathbf{R}_+^{i_0}}\frac{\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{M^{\delta_1}[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{\delta_1}{\alpha}}}\right)^\eta\right]}{M^{i_0-\delta_1\sigma}\left[\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right]^{\frac{i_0-\delta_1\sigma}{\alpha}}}dx_1\cdots dx_{i_0}\\
&= ||y||_\beta^{-\delta_2\sigma}\lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\frac{\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{M^{\delta_1}u^{\delta_1/\alpha}}\right)^\eta\right]}{M^{i_0-\delta_1\sigma}u^{\frac{i_0-\delta_1\sigma}{\alpha}}}u^{\frac{i_0}{\alpha}-1}du\\
&= ||y||_\beta^{-\delta_2\sigma}\lim_{M\to\infty}\frac{M^{\delta_1\sigma}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{M^{\delta_1}u^{\delta_1/\alpha}}\right)^\eta\right]u^{\frac{\delta_1\sigma}{\alpha}-1}du.\end{aligned}$$

Setting $v=b\left(\frac{||y||_\beta^{\delta_2}}{M^{\delta_1}u^{\delta_1/\alpha}}\right)^\eta$ in the above, since $u=(b^{\frac{1}{\eta}}M^{-\delta_1}||y||_\beta^{\delta_2})^{\frac{\alpha}{\delta_1}}v^{\frac{-\alpha}{\delta_1\eta}}$,

$$du=\frac{-\alpha}{\delta_1\eta}(b^{\frac{1}{\eta}}M^{-\delta_1}||y||_\beta^{\delta_2})^{\frac{\alpha}{\delta_1}}v^{\frac{-\alpha}{\delta_1\eta}-1}dv,$$

for $\delta_1=\pm1$, by simplification, it follows that

$$\begin{aligned}\omega(\sigma,y) &= \frac{b^{\sigma/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\eta\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^\infty v^{-\frac{\sigma}{\eta}-1}\ln(1+v)dv\\
&= \frac{b^{\sigma/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\eta\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\left(\frac{-\eta}{\sigma}\right)\int_0^\infty\ln(1+v)dv^{-\frac{\sigma}{\eta}}\\
&= \frac{-b^{\sigma/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\sigma\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\left[\frac{\ln(1+v)}{v^{\frac{\sigma}{\eta}}}\Big|_0^\infty-\int_0^\infty\frac{v^{(1-\frac{\sigma}{\eta})-1}}{1+v}dv\right]\\
&= \frac{b^{\sigma/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\sigma\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\pi}{\sin\pi(1-\frac{\sigma}{\eta})}\\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{b^{\sigma/\eta}\pi}{\sigma\sin(\frac{\pi\sigma}{\eta})}.\end{aligned} \tag{13}$$

LEMMA 3. *Assuming that* $i_0,j_0\in\mathbf{N}$, $\alpha,\beta,b>0$, $0<\sigma,\widetilde{\sigma}<\eta$, $\frac{\sigma}{2}<\widetilde{\sigma}<\eta$, $\delta_i\in\{-1,1\}$ $(i=1,2)$, *we have*

$$\omega(\sigma,y) = K_\alpha(\sigma) := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{b^{\sigma/\eta}\pi}{\sigma\sin(\frac{\pi\sigma}{\eta})}(y\in\mathbf{R}_+^{j_0}), \tag{14}$$

$$\varpi(\sigma,x) = K_\beta(\sigma) := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{b^{\sigma/\eta}\pi}{\sigma\sin(\frac{\pi\sigma}{\eta})}(x\in\mathbf{R}_+^{i_0}), \tag{15}$$

$$\begin{aligned} w(\widetilde{\sigma},y) &:= ||y||_\beta^{-\delta_2\widetilde{\sigma}}\int_{\{x\in\mathbf{R}_+^{i_0};||x||_\alpha^{\delta_1}\geqslant 1\}}\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right]\frac{1}{||x||_\alpha^{i_0-\delta_1\widetilde{\sigma}}}dx \\ &= K_\alpha(\widetilde{\sigma})(1-\theta_{\widetilde{\sigma}}(y)), \end{aligned} \tag{16}$$

$$\theta_{\widetilde{\sigma}}(y) := \frac{\widetilde{\sigma}\sin(\frac{\pi\widetilde{\sigma}}{\eta})}{\eta\pi}\int_{b||y||_\beta^{\eta\delta_2}}^\infty \frac{\ln(1+v)}{v^{\frac{\widetilde{\sigma}}{\eta}+1}}dv = O(||y||_\beta^{-\frac{\delta_2\widetilde{\sigma}}{2}})(y\in\mathbf{R}_+^{j_0}). \tag{17}$$

Proof. By (13), we have (14). By the same way, we still can obtain (15).

By Note 1, setting $v = bM^{-\eta\delta_1}||y||_\beta^{\eta\delta_2}u^{\frac{-\eta\delta_1}{\alpha}}$, for $u^{\delta_1}\geqslant\frac{1}{M^{\delta_1\alpha}}$, we find $0<u^{\frac{-\eta\delta_1}{\alpha}}\leqslant M^{\eta\delta_1}$, and

$$\begin{aligned} w(\widetilde{\sigma},y) &= \lim_{M\to\infty}\frac{M^{\delta_1\widetilde{\sigma}}\Gamma^{i_0}(\frac{1}{\alpha})}{||y||_\beta^{\delta_2\widetilde{\sigma}}\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_{E_{\delta_1}}u^{\frac{\delta_1\widetilde{\sigma}}{\alpha}-1}\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{M^{\delta_1}u^{\delta_1/\alpha}}\right)^\eta\right]du \\ &= \frac{b^{\widetilde{\sigma}/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\eta\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^{b||y||_\beta^{\eta\delta_2}}v^{\frac{-\widetilde{\sigma}}{\eta}-1}\ln(1+v)dv \\ &= \frac{b^{\widetilde{\sigma}/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\eta\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\left[\int_0^\infty\frac{\ln(1+v)}{v^{\frac{\widetilde{\sigma}}{\eta}+1}}dv-\int_{b||y||_\beta^{\eta\delta_2}}^\infty\frac{\ln(1+v)}{v^{\frac{\widetilde{\sigma}}{\eta}+1}}dv\right] \\ &= \frac{b^{\widetilde{\sigma}/\eta}\Gamma^{i_0}(\frac{1}{\alpha})}{\eta\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\left[\frac{\eta\pi}{\widetilde{\sigma}\sin(\frac{\pi\widetilde{\sigma}}{\eta})}-\int_{b||y||_\beta^{\eta\delta_2}}^\infty\frac{\ln(1+v)}{v^{\frac{\widetilde{\sigma}}{\eta}+1}}dv\right] \\ &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{b^{\widetilde{\sigma}/\eta}\pi}{\widetilde{\sigma}\sin(\frac{\pi\widetilde{\sigma}}{\eta})}(1-\theta_{\widetilde{\sigma}}(y)). \end{aligned}$$

For $0<\widetilde{\sigma}<\eta$, since

$$\lim_{u\to\infty}\frac{\int_{bu}^\infty v^{\frac{-\widetilde{\sigma}}{\eta}-1}\ln(1+v)dv}{u^{-\widetilde{\sigma}/(2\eta)}} = \lim_{u\to\infty}\frac{b(bu)^{\frac{-\widetilde{\sigma}}{\eta}-1}\ln(1+bu)}{[\widetilde{\sigma}/(2\eta)]u^{-\widetilde{\sigma}/(2\eta)-1}} = 0,$$

$$\lim_{u\to 0^+}\frac{\int_{bu}^\infty v^{\frac{-\widetilde{\sigma}}{\eta}-1}\ln(1+v)dv}{u^{-\widetilde{\sigma}/(2\eta)}} = 0,$$

there exists a constant $L>0$, such that

$$0<\int_{bu}^\infty v^{\frac{-\widetilde{\sigma}}{\eta}-1}\ln(1+v)dv\leqslant Lu^{-\frac{\widetilde{\sigma}}{2\eta}},$$

and then for $u=||y||_\beta^{\eta\delta_2}$, it follows that

$$0<\theta_{\widetilde{\sigma}}(y)\leqslant\frac{\widetilde{\sigma}\sin(\frac{\pi\widetilde{\sigma}}{\eta})}{\eta\pi}L||y||_\beta^{-\frac{\delta_2\widetilde{\sigma}}{2}},$$

namely, $\theta_{\widetilde{\sigma}}(y) = O(||y||_\beta^{-\frac{\delta_2\widetilde{\sigma}}{2}})(y \in \mathbf{R}_+^{j_0})$. $\square$

LEMMA 4. *On the assumptions of Definition 1, if* $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x) = f(x_1,\cdots,x_{i_0}) \geqslant 0$, $g(y) = g(y_1,\cdots,y_{j_0}) \geqslant 0$, *then (i) for* $p>1$, *we have the following inequality:*

$$J_1 := \left\{ \int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{-p\delta_2\sigma-j_0}}{(\omega(\sigma,y))^{p-1}} \left[\int_{\mathbf{R}_+^{i_0}} \ln[1+b(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}})^\eta] f(x)dx \right]^p dy \right\}^{\frac{1}{p}}$$
$$\leqslant \left[\int_{\mathbf{R}_+^{i_0}} \varpi(\sigma,x)||x||_\alpha^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx \right]^{\frac{1}{p}}; \tag{18}$$

(ii) for $0<p<1$, *or* $p<0$, *we have the reverse of (18).*

Proof. (i) For $p>1$, by Hölder's inequality with weight (cf. [23]), it follows

$$\int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] f(x)dx$$
$$= \int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \left[\frac{||x||_\alpha^{(i_0-\delta_1\sigma)/q}}{||y||_\beta^{(j_0+\delta_2\sigma)/p}} f(x)\right] \left[\frac{||y||_\beta^{(j_0+\delta_2\sigma)/p}}{||x||_\alpha^{(i_0-\delta_1\sigma)/q}}\right] dx$$
$$\leqslant \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{||x||_\alpha^{(i_0-\delta_1\sigma)(p-1)}}{||y||_\beta^{j_0+\delta_2\sigma}} f^p(x)dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{||y||_\beta^{(j_0+\delta_2\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta_1\sigma}} dx \right\}^{\frac{1}{q}}$$
$$= (\omega(\sigma,y))^{\frac{1}{q}} ||y||_\beta^{\frac{j_0}{p}+\delta_2\sigma}$$
$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{||x||_\alpha^{(i_0-\delta_1\sigma)(p-1)}}{||y||_\beta^{j_0+\delta_2\sigma}} f^p(x)dx \right\}^{\frac{1}{p}}. \tag{19}$$

Then by Fubini theorem (cf. [22]), we have

$$
\begin{aligned}
J_1 &\leqslant \left\{ \int_{\mathbf{R}_+^{j_0}} \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{||x||_\alpha^{(i_0-\delta_1\sigma)(p-1)}}{||y||_\beta^{j_0+\delta_2\sigma}} f^p(x)dx \right\} dy \right\}^{\frac{1}{p}} \\
&= \left\{ \int_{\mathbf{R}_+^{i_0}} \left\{ \int_{\mathbf{R}_+^{j_0}} \ln\left[1 + b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] \frac{||x||_\alpha^{(i_0-\delta_1\sigma)(p-1)}}{||y||_\beta^{j_0+\delta_2\sigma}} dy \right\} f^p(x)dx \right\}^{\frac{1}{p}} \\
&= \left[\int_{\mathbf{R}_+^{i_0}} \varpi(\sigma,x)||x||_\alpha^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx \right]^{\frac{1}{p}}. \qquad (20)
\end{aligned}
$$

Hence, (18) follows.

(ii) For $0 < p < 1$, or $p < 0$, by the reverse Hölder's inequality with weight (cf. [23]), we obtain the reverse of (19). Then by Fubini theorem, we still can obtain the reverse of (18). □

LEMMA 5. *On the assumptions of Lemma 4, then*

(i) for $p > 1$, we have the following inequality equivalent to (18):

$$
\begin{aligned}
I &:= \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] f(x)g(y)dxdy \\
&\leqslant \left[\int_{\mathbf{R}_+^{i_0}} \varpi(\sigma,x)||x||_\alpha^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_\beta^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy \right]^{\frac{1}{q}}; \qquad (21)
\end{aligned}
$$

(ii) for $0 < p < 1$, or $p < 0$, we have the reverse of (21) equivalent to the reverse of (18).

Proof. (i) For $p > 1$, by Hölder's inequality (cf. [23]), it follows

$$
\begin{aligned}
I &= \int_{\mathbf{R}_+^{j_0}} \frac{||y||_\beta^{\frac{j_0}{q}-(j_0+\delta_2\sigma)}}{(\omega(\sigma,y))^{\frac{1}{q}}} \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right] f(x)dx \right\} \\
&\quad \times \left[(\omega(\sigma,y))^{\frac{1}{q}} ||y||_\beta^{(j_0+\delta_2\sigma)-\frac{j_0}{q}} g(y) \right] dy \\
&\leqslant J_1 \left[\int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_\beta^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy \right]^{\frac{1}{q}}. \qquad (22)
\end{aligned}
$$

Then by (18), we have (21).

On the other hand, assuming that (21) is valid, we set

$$g(y) := \frac{||y||_{\beta}^{-p\delta_2\sigma-j_0}}{(\omega(\sigma,y))^{p-1}} \left\{ \int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||y||_{\beta}^{\delta_2}}{||x||_{\alpha}^{\delta_1}}\right)^{\eta}\right] f(x)dx \right\}^{p-1}, \quad y \in \mathbf{R}_+^{j_0}.$$

Then it follows

$$J_1^p = \int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy.$$

If $J_1 = 0,$ then (18) is trivially valid; if $J_1 = \infty,$ then by (20), (18) keeps the form of equality $(=\infty)$. Suppose that $0 < J_1 < \infty$. By (21), we have

$$\begin{aligned} 0 &< \int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy = J_1^p = I \\ &\leqslant \left[\int_{\mathbf{R}_+^{i_0}} \varpi(\sigma,x)||x||_{\alpha}^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx\right]^{\frac{1}{p}} \\ &\quad \times \left[\int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy\right]^{\frac{1}{q}} < \infty. \end{aligned}$$

It follows that

$$\begin{aligned} J_1 &= \left[\int_{\mathbf{R}_+^{j_0}} \omega(\sigma,y)||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy\right]^{\frac{1}{p}} \\ &\leqslant \left[\int_{\mathbf{R}_+^{i_0}} \varpi(\sigma,x)||x||_{\alpha}^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx\right]^{\frac{1}{p}}, \end{aligned}$$

and then (18) follows. Hence, (18) and (21) are equivalent.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we can obtain the reverse of (21) equivalent to the reverse of (18). □

3. Main results and operator expressions

Setting

$$\Phi(x) := ||x||_{\alpha}^{p(i_0-\delta_1\sigma)-i_0}, \Psi(y) := ||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} (x \in \mathbf{R}_+^{i_0}, y \in \mathbf{R}_+^{j_0}),$$

we have

THEOREM 1. *Suppose that* $i_0, j_0 \in \mathbf{N}$, $\alpha, \beta, b > 0$, $0 < \sigma < \eta$, $\delta_i \in \{-1,1\}$ $(i = 1,2)$, $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x) = f(x_1,\cdots,x_{i_0}) \geqslant 0$, $g(y) = g(y_1,\cdots,y_{j_0}) \geqslant 0$,

$$0 < ||f||_{p,\Phi} = \left[\int_{\mathbf{R}_+^{i_0}} ||x||_{\alpha}^{p(i_0-\delta_1\sigma)-i_0} f^p(x)dx\right]^{\frac{1}{p}} < \infty,$$

$$0 < ||g||_{q,\Psi} = \left[\int_{\mathbf{R}_+^{j_0}} ||y||_{\beta}^{q(j_0+\delta_2\sigma)-j_0} g^q(y)dy\right]^{\frac{1}{q}} < \infty.$$

(i) For $p>1$, we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:

$$I=\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}}\ln\left[1+b\left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}}\right)^\eta\right]f(x)g(y)dxdy<K(\sigma)||f||_{p,\Phi}||g||_{q,\Psi}, \tag{23}$$

$$J:=\left\{\int_{\mathbf{R}_+^{j_0}}\frac{1}{||y||_\beta^{p\delta_2\sigma+j_0}}\left[\int_{\mathbf{R}_+^{i_0}}\ln[1+b(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}})^\eta]f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\Phi}, \tag{24}$$

where,

$$K(\sigma):=(K_\beta(\sigma))^{\frac{1}{p}}(K_\alpha(\sigma))^{\frac{1}{q}} = \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{q}}\frac{b^{\sigma/\eta}\pi}{\sigma\sin(\frac{\pi\sigma}{\eta})}. \tag{25}$$

(ii) For $0<p<1$, or $p<0$, we still can obtain the equivalent reverses of (23) and (24) with the same best constant factor $K(\sigma)$.

Proof. (i) For $p>1$, by the conditions, we can prove that (19) takes the form of strict inequality. Otherwise, if (19) takes the form of equality for a $y\in\mathbf{R}_+^{j_0}$, then, there exist constants A and B, which are not all zero, such that (cf. [23])

$$A\frac{||x||_\alpha^{(i_0-\delta_1\sigma)(p-1)}}{||y||_\beta^{j_0+\delta_2\sigma}}f^p(x)=B\frac{||y||_\beta^{(j_0+\delta_2\sigma)(q-1)}}{||x||_\alpha^{i_0-\delta_1\sigma}} \text{ a.e. in } x\in\mathbf{R}_+^{i_0}. \tag{26}$$

If $A=0$, then $B=0$, which is impossible; if $A\neq 0$, then (26) reduces to

$$||x||_\alpha^{p(i_0-\delta_1\sigma)-i_0}f^p(x)=\frac{B||y||_\beta^{q(j_0+\delta_2\sigma)}}{A||x||_\alpha^{i_0}}\text{a.e. in } x\in\mathbf{R}_+^{i_0},$$

which contradicts the fact that $0<||f||_{p,\Phi}<\infty$. In fact, by (9), it follows $\int_{\mathbf{R}_+^{i_0}}||x||_\alpha^{-i_0}dx$ $=\infty$. Hence, (19) takes the form of strict inequality. So does (18). By (14) and (15), we have (24).

In view of (22) (for $\omega(\sigma,y)=1$), we still have

$$I\leqslant J\left[\int_{\mathbf{R}_+^{j_0}}||y||_\beta^{q(j_0+\delta_2\sigma)-j_0}g^q(y)dy\right]^{\frac{1}{q}}. \tag{27}$$

Then by (27) and (24), we have (23). It is evident that by Lemma 5 and the assumptions, (23) and (24) are also equivalent.

For $0 < \varepsilon < \frac{p\sigma}{2}$, we set $\widetilde{f}(x), \widetilde{g}(y)$ as follows:

$$\widetilde{f}(x) := \begin{cases} 0, 0 < ||x||_\alpha^{\delta_1} < 1, \\ ||x||_\alpha^{\delta_1(\sigma - \frac{\varepsilon}{p}) - i_0}, ||x||_\alpha^{\delta_1} \geqslant 1, \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} 0, 0 < ||y||_\beta^{\delta_2} < 1, \\ ||y||_\beta^{-\delta_2(\sigma + \frac{\varepsilon}{q}) - j_0}, ||y||_\beta^{\delta_2} \geqslant 1. \end{cases}$$

Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{p} \in (\frac{\sigma}{2}, \eta)$, by (10), we find

$$\begin{aligned} ||\widetilde{f}||_{p,\Phi} ||\widetilde{g}||_{q,\Psi} &= \left(\int_{\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^{\delta_1} \geqslant 1\}} ||x||_\alpha^{-i_0 - \delta_1 \varepsilon} dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta^{\delta_2} \geqslant 1\}} ||y||_\beta^{-j_0 - \delta_2 \varepsilon} dy \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \right)^{\frac{1}{q}}. \end{aligned}$$

In view of (17) and (10), it follows that

$$\begin{aligned} 0 &\leqslant \int_{\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta^{\delta_2} \geqslant 1\}} ||y||_\beta^{-j_0 - \delta_2 \varepsilon} O(||y||_\beta^{-\frac{\delta_2 \widetilde{\sigma}}{2}}) dy \\ &\leqslant L_1 \int_{\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta^{\delta_2} \geqslant 1\}} ||y||_\beta^{-j_0 - \delta_2(\varepsilon + \frac{\widetilde{\sigma}}{2})} dy \\ &= \frac{L_1 \Gamma^{j_0}(\frac{1}{\beta})}{(\varepsilon + \widetilde{\sigma}/2) \beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \leqslant \frac{L_1 \Gamma^{j_0}(\frac{1}{\beta})}{(\varepsilon + \sigma/4) \beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} < \infty, \end{aligned}$$

and

$$\begin{aligned} \widetilde{I} &:= \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} \ln \left[1 + b \left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}} \right)^\eta \right] \widetilde{f}(x) \widetilde{g}(y) dx dy \\ &= \int_{\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta^{\delta_2} \geqslant 1\}} ||y||_\beta^{-j_0 - \delta_2 \varepsilon} w(\widetilde{\sigma}, y) dy \\ &= K_\alpha(\widetilde{\sigma}) \int_{\{y \in \mathbf{R}_+^{j_0}; ||y||_\beta^{\delta_2} \geqslant 1\}} ||y||_\beta^{-j_0 - \delta_2 \varepsilon} \left(1 - O(||y||_\beta^{-\frac{\delta_2 \widetilde{\sigma}}{2}}) \right) dy \\ &= \frac{1}{\varepsilon} K_\alpha(\widetilde{\sigma}) \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} - \varepsilon O_\sigma(1) \right). \end{aligned}$$

If there exists a constant $K \leqslant K(\sigma)$, such that (23) is valid when replacing $K(\sigma)$ by K, then in particular, we have

$$\begin{aligned} &\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \frac{b^{\widetilde{\sigma}/\eta}\pi}{\widetilde{\sigma}\sin(\frac{\pi\widetilde{\sigma}}{\eta})} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} - \varepsilon O_\sigma(1) \right) \\ &\leqslant \varepsilon \widetilde{I} < \varepsilon K ||\widetilde{f}||_{p,\Phi} ||\widetilde{g}||_{q,\Psi} \\ &= K \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right)^{\frac{1}{q}}, \end{aligned}$$

and then $K(\sigma) \leqslant K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (23).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (24) is the best possible. Otherwise, we would reach a contradiction by (27) that the constant factor $K(\sigma)$ in (23) is not the best possible.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we still can obtain the equivalent reverses of (23) and (24) with the same best constant factor. □

On the assumptions of Theorem 1, for $p > 1$, in view of $J < K(\sigma)||f||_{p,\Phi}$, we give the following definition:

DEFINITION 2. We define a multidimensional Hilbert-type integral operator

$$T : \mathbf{L}_{p,\Phi}(\mathbf{R}_+^{i_0}) \to \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0})$$

as follows: For $f \in \mathbf{L}_{p,\Phi}(\mathbf{R}_+^{i_0})$, there exists a unique representation $Tf \in \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^{j_0})$, satisfying

$$(Tf)(y) := \int_{\mathbf{R}_+^{i_0}} \ln \left[1 + b \left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}} \right)^\eta \right] f(x) dx (y \in \mathbf{R}_+^{j_0}), \tag{28}$$

and $||Tf||_{p,\Psi^{1-p}} = J$. For $g \in \mathbf{L}_{q,\Psi}(\mathbf{R}_+^{j_0})$, we define the following formal inner product of Tf and g as follows:

$$(Tf, g) := \int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} \ln \left[1 + b \left(\frac{||y||_\beta^{\delta_2}}{||x||_\alpha^{\delta_1}} \right)^\eta \right] f(x) g(y) dx dy. \tag{29}$$

Then by Theorem 1, for $p > 1$, $0 < ||f||_{p,\Phi}, ||g||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(Tf, g) < K(\sigma) ||f||_{p,\Phi} ||g||_{q,\Psi}, \tag{30}$$

$$||Tf||_{p,\Psi^{1-p}} < K(\sigma) ||f||_{p,\Phi}. \tag{31}$$

It follows that T is bounded with

$$||T|| := \sup_{f(\neq\theta) \in \mathbf{L}_{p,\Phi}(\mathbf{R}_+^{i_0})} \frac{||Tf||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}} \leqslant K(\sigma).$$

Since the constant factor $K(\sigma)$ in (31) is the best possible, we have

$$||T|| = K(\sigma) = \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right)^{\frac{1}{p}} \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{q}} \frac{b^{\sigma/\eta}\pi}{\sigma\sin(\frac{\pi\sigma}{\eta})}. \tag{32}$$

4. Some corollaries

We also set

$$\widehat{\Phi}(x) := ||x||_\alpha^{p(i_0+\sigma)-i_0}, \qquad \widetilde{\Phi}(x) := ||x||_\alpha^{p(i_0-\sigma)-i_0} (x \in \mathbf{R}_+^{i_0}),$$
$$\widehat{\Psi}(y) := ||y||_\beta^{q(j_0+\sigma)-j_0}, \qquad \widetilde{\Psi}(y) := ||y||_\beta^{q(j_0-\sigma)-j_0} (y \in \mathbf{R}_+^{j_0}).$$

For $\delta_1 = -1,\ \delta_2 = 1$ in Theorem 1, we have

COROLLARY 1. *Suppose that* $i_0, j_0 \in \mathbf{N},\ \alpha, \beta, b > 0,\ 0 < \sigma < \eta, p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p} + \frac{1}{q} = 1,\ f(x) = f(x_1, \cdots, x_{i_0}) \geqslant 0,\ g(y) = g(y_1, \cdots, y_{j_0}) \geqslant 0$,

$$0 < ||f||_{p,\widehat{\Phi}} = \left[\int_{\mathbf{R}_+^{i_0}} ||x||_\alpha^{p(i_0+\sigma)-i_0} f^p(x)dx\right]^{\frac{1}{p}} < \infty,$$
$$0 < ||g||_{q,\widehat{\Psi}} = \left[\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{q(j_0+\sigma)-j_0} g^q(y)dy\right]^{\frac{1}{q}} < \infty.$$

(i) For $p > 1$, we have the equivalent inequalities with the non-homogeneous kernel and the best possible constant factor $K(\sigma)$ as follows:

$$\int_{\mathbf{R}_+^{j_0}} \int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(||x||_\alpha ||y||_\beta\right)^\eta\right] f(x)g(y)dxdy < K(\sigma)||f||_{p,\widehat{\Phi}}||g||_{q,\widehat{\Psi}}, \tag{33}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}} \frac{1}{||y||_\beta^{p\sigma+j_0}} \left[\int_{\mathbf{R}_+^{i_0}} \ln\left[1 + b\left(||x||_\alpha ||y||_\beta\right)^\eta\right] f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\widehat{\Phi}}. \tag{34}$$

(ii) For $0 < p < 1$, or $p < 0$, we still have the equivalent reverses of (33) and (34) with the same best constant factor $K(\sigma)$.

For $\delta_1 = 1, \delta_2 = -1$ in Theorem 1, we have

COROLLARY 2. *Suppose that* $i_0, j_0 \in \mathbf{N},\ \alpha, \beta, b > 0,\ 0 < \sigma < \eta,\ p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p} + \frac{1}{q} = 1,\ f(x) = f(x_1, \cdots, x_{i_0}) \geqslant 0,\ g(y) = g(y_1, \cdots, y_{j_0}) \geqslant 0$,

$$0 < ||f||_{p,\widetilde{\Phi}} = \left[\int_{\mathbf{R}_+^{i_0}} ||x||_\alpha^{p(i_0-\sigma)-i_0} f^p(x)dx\right]^{\frac{1}{p}} < \infty,$$
$$0 < ||g||_{q,\widetilde{\Psi}} = \left[\int_{\mathbf{R}_+^{j_0}} ||y||_\beta^{q(j_0-\sigma)-j_0} g^q(y)dy\right]^{\frac{1}{q}} < \infty.$$

(i) For $p>1,$ we have the equivalent inequalities with the non-homogeneous kernel and the best possible constant factor $K(\sigma)$ as follows:

$$\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}}\ln\left[1+\frac{b}{(||x||_\alpha||y||_\beta)^\eta}\right]f(x)g(y)dxdy<K(\sigma)||f||_{p,\widetilde{\Phi}}||g||_{q,\widetilde{\Psi}}, \tag{35}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}}\frac{1}{||y||_\beta^{-p\sigma+j_0}}\left[\int_{\mathbf{R}_+^{i_0}}\ln[1+\frac{b}{(||x||_\alpha||y||_\beta)^\eta}]f(x)dx\right]^p dy\right\}^{\frac{1}{p}}<K(\sigma)||f||_{p,\widetilde{\Phi}}. \tag{36}$$

(ii) For $0<p<1,$ or $p<0,$ we still have the equivalent reverses of (35) and (36) with the same best constant factor $K(\sigma)$.

For $\delta_1=\delta_2=1$ in Theorem 1, we have

COROLLARY 3. *Suppose that* $i_0,\ j_0\in\mathbf{N},\ \alpha,\beta,b>0,\ 0<\sigma<\eta,\ p\in\mathbf{R}\backslash\{0,1\},$ $\frac{1}{p}+\frac{1}{q}=1,\ f(x)=f(x_1,\cdots,x_{i_0})\geqslant 0,\ g(y)=g(y_1,\cdots,y_{j_0})\geqslant 0,$

$$0<||f||_{p,\widetilde{\Phi}}=\left[\int_{\mathbf{R}_+^{i_0}}||x||_\alpha^{p(i_0-\sigma)-i_0}f^p(x)dx\right]^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\widehat{\Psi}}=\left[\int_{\mathbf{R}_+^{j_0}}||y||_\beta^{q(j_0+\sigma)-j_0}g^q(y)dy\right]^{\frac{1}{q}}<\infty.$$

(i) For $p>1,$ we have the following equivalent inequalities with the homogeneous kernel of degree 0 and the best possible constant factor $K(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}}\ln\left[1+b\left(\frac{||y||_\beta}{||x||_\alpha}\right)^\eta\right]f(x)g(y)dxdy<K(\sigma)||f||_{p,\widetilde{\Phi}}||g||_{q,\widehat{\Psi}}, \tag{37}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}}\frac{1}{||y||_\beta^{p\sigma+j_0}}\left[\int_{\mathbf{R}_+^{i_0}}\ln[1+b(\frac{||y||_\beta}{||x||_\alpha})^\eta]f(x)dx\right]^p dy\right\}^{\frac{1}{p}}<K(\sigma)||f||_{p,\widetilde{\Phi}}. \tag{38}$$

(ii) For $0<p<1,$ or $p<0,$ we still have the equivalent reverses of (37) and (38) with the same best constant factor $K(\sigma)$.

For $\delta_1=\delta_2=-1$ in Theorem 1, we have

COROLLARY 4. *Suppose that* $i_0,\ j_0\in\mathbf{N},\ \alpha,\beta,b>0,\ 0<\sigma<\eta,\ p\in\mathbf{R}\backslash\{0,1\},$ $\frac{1}{p}+\frac{1}{q}=1,\ f(x)=f(x_1,\cdots,x_{i_0})\geqslant 0,\ g(y)=g(y_1,\cdots,y_{j_0})\geqslant 0,$

$$0<||f||_{p,\widehat{\Phi}}=\left[\int_{\mathbf{R}_+^{i_0}}||x||_\alpha^{p(i_0+\sigma)-i_0}f^p(x)dx\right]^{\frac{1}{p}}<\infty,$$

$$0<||g||_{q,\widetilde{\Psi}}=\left[\int_{\mathbf{R}_+^{j_0}}||y||_\beta^{q(j_0-\sigma)-j_0}g^q(y)dy\right]^{\frac{1}{q}}<\infty.$$

(i) For $p>1$, we have the following equivalent inequalities with the homogeneous kernel of degree 0 and the best possible constant factor $K(\sigma)$:

$$\int_{\mathbf{R}_+^{j_0}}\int_{\mathbf{R}_+^{i_0}} \ln\left[1+b\left(\frac{||x||_\alpha}{||y||_\beta}\right)^\eta\right] f(x)g(y)dxdy < K(\sigma)||f||_{p,\widehat{\Phi}}||g||_{q,\widetilde{\Psi}}, \tag{39}$$

$$\left\{\int_{\mathbf{R}_+^{j_0}} \frac{1}{||y||_\beta^{-p\sigma+j_0}}\left[\int_{\mathbf{R}_+^{i_0}} \ln[1+b(\frac{||x||_\alpha}{||y||_\beta})^\eta]f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\widehat{\Phi}}. \tag{40}$$

(ii) For $0<p<1$, or $p<0$, we still have the equivalent reverses of (39) and (40) with the same best constant factor $K(\sigma)$.

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