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A DUAL MINKOWSKI TYPE INEQUALITY

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Abstract. The Orlicz Brunn-Minkowski theory was developed by Lutwak, Yang and Zhang. It is an extension of the L_p Brunn-Minkowski theory. In this paper, a new function on the set of star bodies is introduced. A dual Minkowski type inequality about this function is established. Some properties of the new function are presented.

1. Introduction

Recently, Lutwak, Yang and Zhang introduced the definitions of Orlicz projection body and Orlicz centroid body. It was shown (see [22, 23]) that a study of the Orlicz Petty projection inequality and Orlicz centroid inequality leads to the Orlicz Brunn-Minkowski theory which is a natural extension of the L_p Brunn-Minkowski theory (see [5, 7, 8, 9, 19, 20, 21, 25, 26]), and they define a corresponding Orlicz addition of convex bodies and provide a general framework for the Orlicz Brunn-Minkowski theory (see [3, 28]). Recently the Orlicz Brunn-Minkowski theory has developed rapidly. The Minkowski inequality is one of the centerpieces of the Brunn-Minkowski theory (cf. [1]). The work of C. Haberl, E. Lutwak, D. Yang and G. Zhang([6]) solved the even Orlicz Minkowski problem. The above cited work made it possible that the author of [29] introduced Orlicz mixed volumes and established the Orlicz John ellipsoids. For more information on these theories see, e.g., [12, 13, 14, 15, 16, 17, 24] and the references therein.

It is the aim of this paper to define a new function on the set of star bodies and to study the properties of it. We also establish a dual Minkowski type inequality about this function.

Let $\mathscr{C}$ be the class of functions $\psi:(0,\infty)\to(0,\infty)$ such that ψ is convex, and $\lim_{t\to 0}\psi(t)=\infty$, $\lim_{t\to\infty}\psi(t)=0$, and if ψ is strictly convex, then $\psi\in\mathscr{C}_0\subset\mathscr{C}$. Let $\mathscr{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$ and $\mathscr{S}_0^n$ denotes the set of star bodies that contain the origin in their interiors.

Define a function $V_\psi:\mathscr{S}_0^n\times\mathscr{S}_0^n\to(0,\infty)$ by

$$V_\psi(K,L)=\inf\left\{\lambda>0:\frac{1}{n|K|}\int_{S^{n-1}}\rho(K,u)^n\psi\left(\frac{\lambda\rho(L,u)}{\rho(K,u)}\right)dS(u)\leqslant 1\right\}, \tag{1}$$

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where $K,L \in \mathscr{S}_0^n$, $\rho(K,u)$, $\rho(L,u)$ are radial functions of K and L, $S(\cdot)$ is the spherical Lebesgue measure, and $|K|$ denotes the volume of K.

Note that the following homogeneity properties hold:

For an arbitrary constant $c > 0$,

$$V_\psi(cK,L) = cV_\psi(K,L),$$

and

$$V_\psi(K,cL) = \frac{1}{c}V_\psi(K,L).$$

In particular, when $\psi(t) = t^{-p}$, $t > 0$ and $p \geqslant 1$, the function $V_\psi(K,L)$ is the normalization of the L_p dual mixed volume, i.e.,

$$V_\psi(K,L) = \left(\frac{\tilde{V}_{-p}(K,L)}{|K|}\right)^{\frac{1}{p}}.$$

We study the properties of the function $V_\psi(K,L)$ and show that $V_\psi(K,L)$ is bounded and continuous about K,L and ψ. We also obtain a dual Minkowski type inequality:

THEOREM. *Let $\psi \in \mathscr{C}_0$ with $\psi(c_\psi) = 1$ and $K,L \in \mathscr{S}_0^n$. Then*

$$V_\psi(K,L)^n \geqslant |c_\psi K||L|^{-1},$$

with equality if and only if K and L are homothets.

The paper is organized as follows. In section 2, we will give various concepts and basic properties of the dual L_p Brunn-Minkowski theory. In section 3, we discuss some basic properties of the new function and give the proof of the main theorem.

2. Preliminaries

In this section we will give some necessary background materials. Good references are the books of Gardner ([2]), Gruber ([4]) and Schneider ([27]).

Let B denote the unit ball and S^{n-1} denote the unit sphere in Euclidean n-dimensional space $\mathbb{R}^n$. Let $GL(n)$ be the general linear group of $\mathbb{R}^n$.

A set K in $\mathbb{R}^n$ is called star-shaped with respect to a point $z \in K$ if the intersection of every line through z with K is a line segment. The radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$ of a star-shaped set K in $\mathbb{R}^n$ is defined by

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\}.$$

If the radial function is positive and continuous, then K is called a star body.

It follows from the definition of the radial function that the following facts hold (see [10]):

If $K,L \in \mathscr{S}_0^n$, then:

(i) $\rho(TK,u) = \rho(K,T^{-1}u)$, where $T \in GL(n)$, and T^{-1} is the inverse of T;

(ii) For $c > 0$, $\rho(K,cu) = 1/c\rho(K,u)$;

(iii) If $K \subset L$, then $\rho(K,u) \leqslant \rho(L,u)$.

For $K \in \mathscr{S}_0^n$, define r_K, R_K as follows:

$$r_K = \min_{u \in S^{n-1}} \rho(K,u) \quad \text{and} \quad R_K = \max_{u \in S^{n-1}} \rho(K,u).$$

For $p \geqslant 1$, $K,L \in \mathscr{S}_0^n$ and $\varepsilon > 0$, the L_p harmonic radial combination $K \tilde{+} \varepsilon \diamond L$, which is a star body, is defined by

$$\rho(K \tilde{+} \varepsilon \diamond L, \cdot)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon \rho(L,\cdot)^{-p}. \tag{2}$$

The dual L_p mixed volume $\tilde{V}_{-p}(K,L)$ of the star bodies $K,L \in \mathscr{S}_0^n$ was defined in [18]:

$$-\frac{n}{p}\tilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{|K \tilde{+} \varepsilon \diamond L| - |K|}{\varepsilon}.$$

If we let $K = L$, then the dual L_p mixed volume is the ordinary volume, that is,

$$\tilde{V}_{-p}(K,K) = |K|.$$

The polar coordinate formula for volume gives the following integral representation of the dual L_p mixed volume $\tilde{V}_{-p}(K,L)$ of the star bodies K,L:

$$\tilde{V}_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n+p} \rho(L,u)^{-p} dS(u). \tag{3}$$

The Hölder inequality (see [11]) and (3) yields the dual L_p Minkowski inequality: If $K,L \in \mathscr{S}_0^n$, $p \geqslant 1$, then

$$\tilde{V}_{-p}(K,L)^n \geqslant |K|^{n+p} |L|^{-p},$$

with equality if and only if K and L are dilates.

3. The main results

We will give some properties of the function V_ψ. At first we give a definition.

DEFINITION 1. For $K,L \in \mathscr{S}_0^n$ and $\varepsilon > 0$, $\psi \in \mathscr{C}$, define the *harmonic combination* $K \tilde{+}_\psi \varepsilon \diamond L \in \mathscr{S}_0^n$ by

$$\rho(K \tilde{+}_\psi \varepsilon \diamond L, \cdot) = \rho(K,\cdot)\psi^{-1}\left[\psi(1) + \varepsilon \psi\left(\frac{\rho(L,\cdot)}{\rho(K,\cdot)}\right)\right]. \tag{4}$$

Note that when $\psi(t) = t^{-p}$, $t > 0$ and $p \geqslant 1$, (4) is just (3).

Combined with the definition of the harmonic combination and the fact that $\rho(TK,u) = \rho(K, T^{-1}u)$ (where $T \in GL(n)$, and T^{-1} is the inverse of T), we have

LEMMA 3.1. *Suppose that* $\psi \in \mathscr{C}$, $K, L \in \mathscr{S}_0^n$ *and* $\varepsilon > 0$. *Then, for* $T \in GL(n)$,

$$T(K \tilde{+}_\psi \varepsilon \diamond L) = TK \tilde{+}_\psi \varepsilon \diamond TL.$$

Proof. From the definition of the harmonic combination (4), we have

$$\begin{aligned}
\rho(TK \tilde{+}_\psi \varepsilon \diamond TL, u) &= \rho(TK, u)\psi^{-1}\left[\psi(1) + \varepsilon\psi\left(\frac{\rho(TL, u)}{\rho(TK, u)}\right)\right] \\
&= \rho(K, T^{-1}u)\psi^{-1}\left[\psi(1) + \varepsilon\psi\left(\frac{\rho(L, T^{-1}u)}{\rho(K, T^{-1}u)}\right)\right] \\
&= \rho(K \tilde{+}_\psi \varepsilon \diamond L, T^{-1}u) \\
&= \rho(T(K \tilde{+}_\psi \varepsilon \diamond L), u),
\end{aligned}$$

which completes the proof. □

LEMMA 3.2. *Suppose that* $\psi \in \mathscr{C}$, $K, L \in \mathscr{S}_0^n$ *and* $\varepsilon > 0$. *Then*

$$\lim_{\varepsilon \to 0} \frac{V(K \tilde{+}_\psi \varepsilon \diamond L) - V(K)}{\varepsilon} = \alpha \int_{S^{n-1}} \rho(K, u)^n \psi\left(\frac{\rho(L, u)}{\rho(K, u)}\right) dS(u),$$

where $\alpha = (\psi^{-1})'_r(\psi(1))$.

Proof. From (4) and the polar coordinate formula for volume, we have

$$\begin{aligned}
V(K \tilde{+}_\psi \varepsilon \diamond L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K \tilde{+}_\psi \varepsilon \diamond L, u)^n dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^n \left[\psi^{-1}\left(\psi(1) + \varepsilon\psi\left(\frac{\rho(L, u)}{\rho(K, u)}\right)\right)\right]^n dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^n \left[1 + \varepsilon(\psi^{-1})'_r(\psi(1))\psi\left(\frac{\rho(L, u)}{\rho(K, u)}\right) + o(\varepsilon^2)\right]^n dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^n \left[1 + n\varepsilon(\psi^{-1})'_r(\psi(1))\psi\left(\frac{\rho(L, u)}{\rho(K, u)}\right) + o(\varepsilon^2)\right] dS(u),
\end{aligned}$$

which completes the proof. □

From our assumption that ψ is strictly decreasing on $(0, \infty)$, it follows that the function

$$\lambda \to \int_{S^{n-1}} \rho(K, u)^n \psi\left(\frac{\lambda\rho(L, u)}{\rho(K, u)}\right) dS(u)$$

is strictly decreasing on $(0, \infty)$. Then from (1) we have

LEMMA 3.3. *Suppose that* $\psi \in \mathscr{C}$ *and* $K, L \in \mathscr{S}_0^n$, $\lambda > 0$. *Then*

$$\frac{1}{n|K|}\int_{S^{n-1}} \rho(K, u)^n \psi\left(\frac{\lambda\rho(L, u)}{\rho(K, u)}\right) dS(u) = 1$$

if and only if

$$V_\psi(K,L)=\lambda.$$

Since ψ is strictly decreasing and convex on $(0,\infty)$, and $\lim_{t\to 0}\psi(t)=\infty$, $\lim_{t\to\infty}\psi(t)=0$, there exists a real number $0<c_\psi<\infty$ such that $\psi(c_\psi)=1$.

The next lemma shows that V_ψ is bounded.

LEMMA 3.4. *Let $\psi\in\mathscr{C}$ and $K,L\in\mathscr{S}_0^n$. We have*

$$\frac{c_\psi r_K}{R_L}\leqslant V_\psi(K,L)\leqslant\frac{c_\psi R_K}{r_L}.$$

Proof. Let $\psi(c_\psi)=1$. By Lemma 3.3, the monotonicity of ψ and the polar coordinate formula for volume, we have

$$\begin{aligned}\psi(c_\psi)=1&=\frac{1}{n|K|}\int_{S^{n-1}}\rho(K,u)^n\psi\left(\frac{V_\psi(K,L)\rho(L,u)}{\rho(K,u)}\right)dS(u)\\&\geqslant\frac{1}{n|K|}\int_{S^{n-1}}\rho(K,u)^n\psi\left(\frac{V_\psi(K,L)\max\limits_{u\in S^{n-1}}\rho(L,u)}{\min\limits_{u\in S^{n-1}}\rho(K,u)}\right)dS(u)\\&=\frac{1}{n|K|}\int_{S^{n-1}}\rho(K,u)^n\psi\left(\frac{V_\psi(K,L)R_L}{r_K}\right)dS(u)\\&=\psi\left(\frac{V_\psi(K,L)R_L}{r_K}\right)\frac{1}{n|K|}\int_{S^{n-1}}\rho(K,u)^n dS(u)\\&=\psi\left(\frac{V_\psi(K,L)R_L}{r_K}\right),\end{aligned}$$

and then

$$V_\psi(K,L)\geqslant\frac{c_\psi r_K}{R_L}.$$

Using the analogous method, we obtain the upper bound on $V_\psi(K,L)$ by

$$V_\psi(K,L)\leqslant\frac{c_\psi R_K}{r_L}.\quad\square$$

Next, we will show that $V_\psi(K,L)$ is continuous with respect to K and L.

LEMMA 3.5. *If $\psi\in\mathscr{C}$, $K_i,L_i\in\mathscr{S}_0^n$, and*

$$K_i\to K\in\mathscr{S}_0^n,\quad L_i\to L\in\mathscr{S}_0^n,$$

then

$$V_\psi(K_i,L_i)\to V_\psi(K,L).$$

Proof. Suppose that $V_\psi(K_i,L_i)=\lambda_i$. Then Lemma 3.4 shows that

$$\frac{c_\psi r_{K_i}}{R_{L_i}}\leqslant V_\psi(K_i,L_i)\leqslant\frac{c_\psi R_{K_i}}{r_{L_i}}.$$

Since $K_i \to K \in \mathscr{S}_0^n$ and $L_i \to L \in \mathscr{S}_0^n$, then $r_{K_i} \to r_K > 0$, $r_{L_i} \to r_L > 0$ and $R_{K_i} \to R_K < \infty$, $R_{L_i} \to R_L < \infty$. Then there exist a and b such that

$$0 < a \leqslant \lambda_i \leqslant b < \infty,$$

for all i.

To prove that the bounded sequence $\{\lambda_i\}$ converges to $V_\psi(K,L)$, we show that every convergent subsequence converges to $V_\psi(K,L)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well. Suppose that for this subsequence $\lambda_i \to \lambda_0$, then we have $0 < a \leqslant \lambda_0 \leqslant b < \infty$. Let $\bar{L_i} = \lambda_i L_i$. Then we have $\bar{L_i} \to \lambda_0 L$. From the definition of the function V_ψ, we know that

$$\lambda_i^{-1} V_\psi(K_i, L_i) = V_\psi(K_i, \lambda_i L_i) = V_\psi(K_i, \bar{L_i}) = 1.$$

Then, by Lemma 3.3, we have

$$\begin{aligned} 1 &= \frac{1}{n|K_i|} \int_{S^{n-1}} \rho(K_i,u)^n \psi\left(\frac{\rho(\bar{L_i},u)}{\rho(K_i,u)}\right) dS(u) \\ &= \frac{1}{n|K_i|} \int_{S^{n-1}} \rho(K_i,u)^n \psi\left(\frac{\rho(\lambda_i L_i,u)}{\rho(K_i,u)}\right) dS(u), \end{aligned}$$

for all i.

While $K_i \to K$ and $\bar{L_i} \to \lambda_0 L$ imply that $\rho_{K_i} \to \rho_K$, $\rho_{\bar{L_i}} \to \rho_{\lambda_0 L}$ uniformly on S^{n-1} to use the continuity of ψ and $\lambda_i \to \lambda_0$, we get

$$\frac{1}{n|K|} \int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{\rho(\lambda_0 L,u)}{\rho(K,u)}\right) dS(u) = 1,$$

which implies that $V_\psi(K, \lambda_0 L) = 1$. Then we have $V_\psi(K,L) = \lambda_0$ from the definition of V_ψ. This completes the proof. □

The following lemma shows that V_ψ is also continuous with respect to ψ. We say that the sequence $\psi_i \to \psi \in \mathscr{C}$ for $\psi_i \in \mathscr{C}$ provided that

$$\max_{t \in I} |\psi_i(t) - \psi(t)| \to 0,$$

for every compact interval $I \subset \mathbb{R}$.

LEMMA 3.6. *If* $\psi \in \mathscr{C}$, $K, L \in \mathscr{S}_0^n$, *then*

$$\psi_i \to \psi \in \mathscr{C} \quad \Rightarrow \quad V_{\psi_i}(K,L) \to V_\psi(K,L).$$

Proof. Let $V_{\psi_i}(K,L) = \lambda_i$. Lemma 3.4 gives

$$\frac{c_{\psi_i} r_K}{R_L} \leqslant \lambda_i \leqslant \frac{c_{\psi_i} R_K}{r_L}.$$

Since $\psi_i \to \psi$, we have $c(\psi_i) \to c(\psi) \in (0,\infty)$. Thus there exist a and b such that

$$0 < a \leqslant \lambda_i \leqslant b < \infty, \quad \text{for all } i.$$

To prove that the bounded sequence $\{\lambda_i\}$ converges to $V_\psi(K,L)$, we show that every subsequence converges to $V_\psi(K,L)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well. Suppose that for this subsequence $\lambda_i \to \lambda_0$.

Since $0 < a \leqslant \lambda_i \leqslant b < \infty$ and $V_{\psi_i}(K,L) = \lambda_i$, from Lemma 3.3 it follows that

$$\frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi_i\left(\frac{\lambda_i \rho(L,u)}{\rho(K,u)}\right) dS(u) = 1.$$

Together with the fact that $\psi_i \to \psi \in \mathscr{C}$ and $\lambda_i \to \lambda_0 \in (0,\infty)$, we have

$$\frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{\lambda_0 \rho(L,u)}{\rho(K,u)}\right) dS(u) = 1.$$

This shows

$$V_\psi(K,L) = \lambda_0.$$

Then we obtain the conclusion

$$V_{\psi_i}(K,L) \to V_\psi(K,L). \quad \square$$

Let $O(n)$ denote the group of rotations. Our next lemma shows that $V_\psi(K,L)$ is invariant under rotation.

LEMMA 3.7. *Let $\psi \in \mathscr{C}$, $K_i, L_i \in \mathscr{S}_0^n$ and $T \in O(n)$. Then*

$$V_\psi(TK,L) = V_\psi(K,T^{-1}L).$$

Proof. From (1) and the fact that the spherical Lebesgue measure is rotation invariant, we have

$$\begin{aligned}
V_\psi(TK,L) &= \inf\left\{\lambda > 0 : \frac{1}{n|TK|}\int_{S^{n-1}} \rho(TK,u)^n \psi\left(\frac{\lambda \rho(L,u)}{\rho(TK,u)}\right) dS(u) \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{n|K|}\int_{S^{n-1}} \rho(K,T^{-1}u)^n \psi\left(\frac{\lambda \rho(L,u)}{\rho(K,T^{-1}u)}\right) dS(u) \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{\lambda \rho(L,Tu)}{\rho(K,u)}\right) dS(Tu) \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{\lambda \rho(L,Tu)}{\rho(K,u)}\right) dS(u) \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{\lambda \rho(T^{-1}L,u)}{\rho(K,u)}\right) dS(u) \leqslant 1\right\} \\
&= V_\psi(K,T^{-1}L),
\end{aligned}$$

which completes the proof. $\square$

In the following, we will establish a dual Minkowski type inequality.

THEOREM. *Let $\psi \in \mathscr{C}_0$ with $\psi(c_\psi) = 1$ and $K, L \in \mathscr{S}_0^n$. Then*

$$V_\psi(K,L)^n \geqslant |c_\psi K||L|^{-1},$$

with equality if and only if K and L are homothets.

Proof. By Lemma 3.3, Jessen's inequality and Hölder's inequality (see [11]), we have

$$\begin{aligned}
\psi(c_\psi) = 1 &= \frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \psi\left(\frac{V_\psi(K,L)\rho(L,u)}{\rho(K,u)}\right) dS(u) \\
&\geqslant \psi\left(\frac{1}{n|K|}\int_{S^{n-1}} \rho(K,u)^n \left(\frac{V_\psi(K,L)\rho(L,u)}{\rho(K,u)}\right) dS(u)\right) \\
&= \psi\left(\frac{V_\psi(K,L)}{n|K|}\int_{S^{n-1}} \rho(K,u)^{n-1}\rho(L,u) dS(u)\right) \\
&\geqslant \psi\left(\frac{V_\psi(K,L)|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}}{|K|}\right).
\end{aligned}$$

Then the monotonicity of ψ implies that

$$V_\psi(K,L)^n \geqslant |c_\psi K||L|^{-1}.$$

The equality condition follows from the equality conditions for Jessen's inequality and Hölder's inequality. Note that if ψ is strictly convex, then Jessen's inequality implies that the first inequality is strict unless ψ is constant, that is, $\rho(L,u)/\rho(K,u)$ is constant for all u. This implies that K and L are homothets.

An immediate consequence of the condition that equality holds in Hölder's inequality is that K and L are homothets. □

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