Mathematical
Inequalities
& Applications
Volume 18, Number 4 (2015), 1261–1273 doi:10.7153/mia-18-98

COMPACTNESS FOR THE COMMUTATOR OF THE PARAMETERIZED AREA INTEGRAL IN THE MORREY SPACE

YANPING CHEN AND HONG WANG

(*Communicated by B. Opic*)

Abstract. In this paper the authors prove that the commutator $[b,\mu_S^\rho]$ is a compact operator in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ for $1<p<\infty$ and $0<\lambda<n$, if and only if $b\in VMO(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$, where μ_S^ρ denotes the parameterized area integral.

1. Introduction

The parameterized area integral is defined by

$$\mu_S^\rho f(x) := \left(\int_0^\infty \int_{|x-y|<t} |F_{\Omega,t}(y)|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(y) := \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz,$$

where Ω satisfies the following conditions:

(a) Ω is homogeneous function of degree zero on $\mathbb{R}^n\setminus\{0\}$, i.e.

$$\Omega(tx)=\Omega(x) \ \text{ for any } t>0 \text{ and } x\in\mathbb{R}^n\setminus\{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, i.e.

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.2}$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. On the other hand, if taking

$$\varphi^\rho(x) := \Omega(x)|x|^{-n+\rho}\chi_{\{y:|y|\leqslant 1\}}(x), \quad 0<\rho<n,$$

$$\varphi_t(x) := t^{-n}\varphi(x/t), \quad t>0,$$

Mathematics subject classification (2010): 42B20, 42B25.
Keywords and phrases: Compactness, commutators, parameterized area integral, VMO, Morrey space.
The research was supported by NSF of China (Grant: 11471033), NCET of China (Grant: NCET-11-0574), the Fundamental Research Funds for the Central Universities (FRF-TP-12-006B).

then we can write

$$\mu_S^\rho f(x) := \left(\int_0^\infty \int_{|x-y|<t} |\varphi_t^\rho * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where

$$\varphi_t^\rho(x) := t^{-n}\varphi^\rho(x/t), \quad t>0.$$

Before stating some results, let us recall some definitions and notations.

DEFINITION 1.1. ([1]) Let X and Y be Banach spaces and V be a subset of X. Then operator $T : V \longmapsto Y$ is said to be a compact operator if T is continuous and maps bounded subsets of V into strongly pre-compact subsets of Y.

DEFINITION 1.2. ([2]) Suppose that $\Omega(x') \in L^q(S^{n-1})$ for $q \geqslant 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) := \sup_{||\tau||\leqslant\delta}\left(\int_{S^{n-1}} |\Omega(\tau x') - \Omega(x')|^q d\sigma(x')\right)^{\frac{1}{q}},$$

where τ denotes the rotation on $\mathbb{R}^n$ and $||\tau|| := \sup_{x'\in S^{n-1}} |\tau x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty.$$

Let $b \in L_{loc}(\mathbb{R}^n)$ and $0<\rho<n$, we define the commutator $[b,\mu_S^\rho]$ as follows:

$$[b,\mu_S^\rho]f(x) := \left(\int_0^\infty \int_{|x-y|<t} \left|\int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}.$$

In [2], the authors gave the L^p compactness of the commutator $[b,\mu_S^\rho]$.

THEOREM A. ([2]) *Let Ω satisfies (1.1), (1.2) and*

$$\left|\Omega(x') - \Omega(y')\right| \leqslant \frac{C_1}{(\log\frac{2}{|x'-y'|})^\gamma}, \quad C_1>0, \ \gamma>1, \ x',y' \in S^{n-1}. \tag{1.3}$$

If $n/2<\rho<n$ and $[b,\mu_S^\rho]$ is a compact operator in $L^p(\mathbb{R}^n)$ for $1<p<\infty$, then $b \in VMO(\mathbb{R}^n)$.

THEOREM B. ([2]) *Let $\Omega \in L^2(S^{n-1})$ satisfy (1.1), (1.2) and*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta < \infty, \quad \sigma>2. \tag{1.4}$$

If $n/2<\rho<n$, $b \in VMO(\mathbb{R}^n)$, then $[b,\mu_S^\rho]$ is a compact operator in $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

For $p \in (1,\infty)$, $\lambda \in (0,n)$, we give the definition of the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ as follows.

$$L^{p,\lambda}(\mathbb{R}^n) := \{f \in L_{loc}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty\},$$

where

$$\|f\|_{p,\lambda} := \sup_{y\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int_{B(y,r)} |f(x)|^p dx \right)^{\frac{1}{p}}$$

and $B(y,r)$ denotes the ball centered at y and with radius $r > 0$. The space becomes a Banach space with norm $\|\cdot\|_{p,\lambda}$. Moreover, if $\lambda = 0$ and $\lambda = n$, then $L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)$ coincide with the space $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively.

Therefore, an interesting question arise naturally. That is, is true the conclusion in Theorem A and Theorem B if replacing the $L^p(\mathbb{R}^n)$ by $L^{p,\lambda}(\mathbb{R}^n)$? In this paper we will give a positive answer to above question. Now we state our results as follows.

THEOREM 1.1. *Let Ω satisfies (1.1), (1.2) and (1.3). If $\frac{n}{2} < \rho < n$ and $[b,\mu_S^\rho]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$ for $0 < \lambda < n$, $1 < p < \infty$, then $b \in VMO(\mathbb{R}^n)$.*

THEOREM 1.2. *Let $\Omega \in L^2(S^{n-1})$ satisfies (1.1), (1.2) and (1.4). If $\frac{n}{2} < \rho < n$, $b \in VMO(\mathbb{R}^n)$, then $[b,\mu_S^\rho]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$ for $0 < \lambda < n$, $1 < p < \infty$.*

It is easy to see that the condition (1.3) implies (1.4), so we may get the following corollary immediately.

COROLLARY 1.1. *Let Ω satisfy (1.1), (1.2) and (1.3). If $\frac{n}{2} < \rho < n$, $0 < \lambda < n$, $1 < p < \infty$, then $[b,\mu_S^\rho]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$ if and only if $b \in VMO(\mathbb{R}^n)$.*

Throughout this paper the letter C will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. Moreover, $|E|$ denotes the Lebesgue measure of the measurable set E in $\mathbb{R}^n$. As usual, for $p \geqslant 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

2. Lemmas

Let us begin with some lemmas, which will be used in the proof of the main results.

LEMMA 2.1. ([10]) *For $f \in BMO(\mathbb{R}^n)$, then $f \in VMO(\mathbb{R}^n)$ if and only if f satisfies the following three conditions:*

(i) $\lim_{a\to 0} \sup_{|Q|=a} M(f,Q) = 0$.

(ii) $\lim_{a\to\infty} \sup_{|Q|=a} M(f,Q) = 0$.

(iii) $\lim_{x\to\infty} M(f,x+Q) = 0$*, for each Q, where* $M(f,Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx$.

LEMMA 2.2. ([4]) *Let $1 \leqslant p < \infty$, $0 < \lambda < n$. If the subset G in $L^{p,\lambda}(\mathbb{R}^n)$ satisfies the following conditions:*

$$\sup_{f\in G} \|f\|_{p,\lambda} < \infty, \tag{2.1}$$

$$\lim_{|y|\to 0} \|f(\cdot+y) - f(\cdot)\|_{p,\lambda} = 0 \quad \text{uniformly in} \quad f \in G, \tag{2.2}$$

$$\lim_{\beta\to\infty} \left\| f\chi_{E_\beta} \right\|_{p,\lambda} = 0 \quad \text{uniformly in} \quad f \in G, \tag{2.3}$$

where $E_\beta = \{x \in \mathbb{R}^n : |x| > \beta\}$, *then* G *is a strongly pre-compact set in* $L^{p,\lambda}(\mathbb{R}^n)$.

3. The proof of Theorem 1.1

By Theorem 1.3 in [5], we get $b \in BMO(\mathbb{R}^n)$. Without loss of generality, we may assume $\|b\|_{BMO} = 1$. By Lemma 2.1, to prove that $b \in VMO(\mathbb{R}^n)$, it suffices to show that b must satisfy the conditions (i), (ii) and (iii) in Lemma 2.1. First, we show that if that b does not satisfy the condition (i) of Lemma 2.1, then $[b, \mu_S^\rho]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$. By the assumption, there exist a constant $\eta > 0$ and a sequence of cubes $\{Q_j(y_j, d_j)\}_{j=1}^\infty$ with $\lim_{j\to\infty} d_j = 0$, where d_j is the diameter of the cube $\{Q_j(y_j, d_j)\}$, such that for every j

$$M(b, Q_j) := \frac{1}{|Q_j|}\int_{Q_j} |b(y) - b_{Q_j}|dy > \eta. \tag{3.1}$$

Define the function sequence $\{f_j\}_{j=1}^\infty$ by

$$f_j(y) := |Q_j|^{-(n-\lambda)/np}[sgn(b(y) - b_{Q_j}) - c_0]\chi_{Q_j}(y), \quad j = 1, 2, ..., \tag{3.2}$$

where $c_0 = \frac{1}{|Q_j|}\int_{Q_j} \mathrm{sgn}(b(y) - b_{Q_j})dy$. Then $\{f_j\}$ satisfies the following properties:

$$supp f_j \subset Q_j, \tag{3.3}$$

$$f_j(y)(b(y) - b_{Q_j}) > 0, \tag{3.4}$$

$$\int_{\mathbb{R}^n} f_j(y)dy = 0, \tag{3.5}$$

$$|f_j(y)| \leqslant 2|Q_j|^{-(n-\lambda)/np}, \forall y \in Q_j. \tag{3.6}$$

In [4], the authors proved that $\{\|f_j\|_{p,\lambda}\}_{j=1}^\infty$ is bounded uniformly. Then by the $L^{p,\lambda}$ boundedness of $[b, \mu_S^\rho]$, if $\{[b, \mu_S^\rho]f_j\}_{j=1}^\infty$ is not a pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$, then $[b, \mu_S^\rho]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$ (see Definition 1.1). Consequently, it suffices to show that there exists a subsequence $\{[b, \mu_S^\rho]f_{j_k}\}_{k=1}^\infty$ which has no any convergence subsequence in $L^{p,\lambda}(\mathbb{R}^n)$.

From now on, for $1 \leqslant i \leqslant 8$, A_i denotes the positive constants depending only on $\Omega, p, n, \lambda, \gamma, \eta$ and A_k $(1 \leqslant k < i)$. Since Ω satisfies (1.2), then there exists an A_1 such that $0 < A_1 < 1$ and

$$\sigma\left(\left\{x' \in S^{n-1} : \Omega(x') \geqslant \frac{2C_1}{(\log \frac{2}{A_1})^\gamma}\right\}\right) > 0.$$

By the condition (1.3), it is easy to see that

$$\Lambda := \left\{x' \in S^{n-1} : \Omega(x') \geqslant \frac{2C_1}{(\log \frac{2}{A_1})^\gamma}\right\}$$

is a closed set. Then (see [3])

$$if\ x' \in \Lambda, y' \in S^{n-1} \text{ satisfying } |x'-y'| \leqslant A_1, \text{ then } \Omega(y') \geqslant \frac{C_1}{(\log \frac{2}{A_1})^{\gamma}}. \tag{3.7}$$

Taking $A_2 > 2/A_1$, for $x \in (A_2Q_j)^c \cap \{x : (x-y_j)' \in \Lambda\}$, by (3.1)–(3.4) and similar to the estimate of (4.4) in [2], we have that

$$\begin{aligned} &\left|[b,\mu_S^{\rho}]f_j(x)\right| \\ &\geqslant C\eta |Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} |x-y_j|^{-n} - C|Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} |b(x)-b_{Q_j}| |x-y_j|^{-n} \left(\log \frac{|x-y_j|}{d_j}\right)^{-\gamma}. \end{aligned} \tag{3.8}$$

On the other hand, by (3.6), and similar to the proof of (4.7) in [2], we get

$$\left|[b,\mu_S^{\rho}]f_j(x)\right| \leqslant C|x-y_j|^{-n} |b(x)-b_{Q_j}| |Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} + C|x-y_j|^{-n} |Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}}. \tag{3.9}$$

By $|b_{2Q}-b_Q| \leqslant C\|b\|_{BMO} = C$, we get

$$\left(\int_{2^k d_j < |x-y_j| \leqslant 2^{k+1} d_j} |b(x)-b_{Q_j}|^p dx\right)^{\frac{1}{p}} \leqslant C 2^{kn/p} k |Q_j|^{\frac{1}{p}}. \tag{3.10}$$

For $u > A_2$, by (3.9) and (3.10), we get

$$\begin{aligned} &\left(\int_{|x-y_j|>ud_j} \left|[b,\mu_S^{\rho}]f_j(x)\right|^p dx\right)^{\frac{1}{p}} \\ &\leqslant C|Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} \left(\int_{|x-y_j|>ud_j} \frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}} dx\right)^{\frac{1}{p}} \\ &\quad + C|Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} \left(\int_{|x-y_j|>ud_j} \frac{1}{|x-y_j|^{np}} dx\right)^{\frac{1}{p}} \\ &\leqslant A_3 |Q_j|^{\frac{\lambda}{np}} u^{-n(1-1/p)}. \end{aligned} \tag{3.11}$$

For $\omega > u > A_2$, by (3.8) and (3.10), we get

$$\begin{aligned} &\left(\int_{ud_j<|x-y_j|<\omega d_j} \left|[b,\mu_S^{\rho}]f_j(x)\right|^p dx\right)^{\frac{1}{p}} \\ &\geqslant C\eta |Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} \left(\int_{ud_j<|x-y_j|<\omega d_j} \frac{1}{|x-y_j|^{np}} dx\right)^{\frac{1}{p}} \\ &\quad - C|Q_j|^{\frac{1}{p'}+\frac{\lambda}{np}} \left(\int_{|x-y_j|>ud_j} \frac{|b(x)-b_{Q_j}|^p}{|x-y_j|^{np}\left(\log \frac{|x-y_j|}{d_j}\right)^{\gamma p}} dx\right)^{\frac{1}{p}} \\ &\geqslant A_4 \eta (u^{-np+n} - \omega^{-np+n})^{\frac{1}{p}} |Q_j|^{\frac{\lambda}{np}} - A_5 (\log u)^{1-\gamma} u^{-n+n/p} |Q_j|^{\frac{\lambda}{np}}. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we get there exist $A_6 > A_2$, $B = B(\Omega, p, n, \lambda, \eta, A_3, A_4, A_5) > 1$, A_8 such that

$$\left(\int_{A_6 d_j < |x-y_j| < BA_6 d_j} \left|\left[b, \mu_S^\rho\right] f_j(x)\right|^p dx\right)^{\frac{1}{p}} \geqslant A_8 \left|Q_j\right|^{\frac{\lambda}{np}}, \tag{3.13}$$

and

$$\left(\int_{|x-y_j| > BA_6 d_j} \left|\left[b, \mu_S^\rho\right] f_j(x)\right|^p dx\right)^{\frac{1}{p}} \leqslant \frac{A_8}{4} \left|Q_j\right|^{\frac{\lambda}{np}}. \tag{3.14}$$

Then using similar argumentations as in the proof of Theorem 1 in [3], we can show that $[b, \mu_S^\rho]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

Similarly, we may show that if b does not satisfy conditions (ii) or (iii) in Lemma 2.1, then $[b, \mu_S^\rho]$ is not a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

4. The proof of Theorem 1.2

Suppose that F is an arbitrary bounded set in $L^{p,\lambda}(\mathbb{R}^n)$, that is, there exists a constant $D > 0$ such that $\|f\|_{p,\lambda} \leqslant D$ for every $f \in F$. It is well known that (see [8], [11])

$$\|[b, \mu_S^\rho] f\|_{p,\lambda} \leqslant C \|b\|_{BMO} \|f\|_{p,\lambda}. \tag{4.1}$$

Thus $[b, \mu_S^\rho]$ is continuous in $L^{p,\lambda}(\mathbb{R}^n)$. Let $G = \{[b, \mu_S^\rho] f : f \in F\}$ if $b \in C_0^\infty(\mathbb{R}^n)$ and $\widetilde{G} = \{[b, \mu_S^\rho] f : f \in F\}$ if $b \in VMO(\mathbb{R}^n)$. So, by Definition 1.1, it suffices to prove that for any bounded set F in $L^{p,\lambda}(\mathbb{R}^n)$, $\widetilde{G}$ is a strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$. We first show that if (2.1)–(2.3) hold uniformly in G, then (2.1)–(2.3) hold also uniformly in $\widetilde{G}$ and thus $[b, \mu_S^\rho]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

Actually, assume that $b \in VMO(\mathbb{R}^n)$, then for any $\varepsilon > 0$ there exists $b^\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\|b - b^\varepsilon\|_{BMO} < \varepsilon$. By

$$\begin{aligned}
&|[b, \mu_S^\rho] f(x) - [b^\varepsilon, \mu_S^\rho] f(x)| \\
&\leqslant \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{|y-z| \leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} ((b - b^\varepsilon)(x) - (b - b^\varepsilon)(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}
\end{aligned}$$

and (4.1), we get

$$\|[b, \mu_S^\rho] - [b^\varepsilon, \mu_S^\rho]\|_{L^{p,\lambda} \to L^{p,\lambda}} \leqslant \|[b - b^\varepsilon, \mu_S^\rho]\|_{L^{p,\lambda} \to L^{p,\lambda}} \leqslant C\varepsilon. \tag{4.2}$$

By Minkowski's inequality and (4.2), for any fixed $f \in F$

$$\sup_{f \in F} \left\| \left[b, \mu_S^\rho\right] f \right\|_{p,\lambda} \leqslant \sup_{f \in F} \left\| \left[b^\varepsilon, \mu_S^\rho\right] f \right\|_{p,\lambda} + CD\varepsilon < \infty.$$

By Minkowski's inequality, (4,1) and (4.2), for any fixed $f \in F$

$$\begin{aligned}
&\lim_{|z|\to 0} \left\| \left[b,\mu_S^\rho\right] f(\cdot+z) - \left[b,\mu_S^\rho\right] f(\cdot) \right\|_{p,\lambda} \\
\leqslant &\lim_{|z|\to 0} \left\| \left[b-b^\varepsilon,\mu_S^\rho\right] f(\cdot+z) \right\|_{p,\lambda} \\
&+ \lim_{|z|\to 0} \left\| \left[b-b^\varepsilon,\mu_S^\rho\right] f(\cdot) \right\|_{p,\lambda} \\
&+ \lim_{|z|\to 0} \left\| \left[b^\varepsilon,\mu_S^\rho\right] f(\cdot+z) - \left[b^\varepsilon,\mu_S^\rho\right] f(\cdot) \right\|_{p,\lambda} \\
\leqslant &\, 2CD\varepsilon.
\end{aligned}$$

Similarly, by Minkowski's inequality, (4.1) and (4.2), for any fixed $f \in F$

$$\begin{aligned}
\lim_{\beta\to\infty} \left\| \left[b,\mu_S^\rho\right] f\chi_{E_\beta} \right\|_{p,\lambda} &\leqslant \lim_{\beta\to\infty} \left\| \left[b^\varepsilon,\mu_S^\rho\right] f\chi_{E_\beta} \right\|_{p,\lambda} + \lim_{\beta\to\infty} \left\| \left[b-b^\varepsilon,\mu_S^\rho\right] f\chi_{E_\beta} \right\|_{p,\lambda} \\
&\leqslant CD\varepsilon.
\end{aligned}$$

Thus (2.1)–(2.3) hold uniformly for $\tilde{G}$. Therefore, by Lemma 2.2, we know $\tilde{G}$ is a strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$ and then $[b,\mu_S^\rho]$ is a compact operator in $L^{p,\lambda}(\mathbb{R}^n)$.

So, it suffices to prove that (2.1)–(2.3) hold uniformly in G. Without loss of the generality, we can assume $\|b\|_{BMO} = 1$. By (4.1), we have

$$\sup_{f\in F} \|[b,\mu_S^\rho]f\|_{p,\lambda} \leqslant C\|b\|_{BMO}\|f\|_{p,\lambda} \leqslant CD < \infty. \tag{4.3}$$

The estimate (4.3) shows that (2.1) holds for the commutator $[b,\mu_S^\rho]$ in G uniformly. Now we discuss (2.3). For any $\varepsilon > 0$, $q > 1$ there exists $A > 0$ such that

$$\left(\int_A^\infty \frac{dr}{r^{nq-n+1}} \right)^{\frac{1}{q}} < \varepsilon. \tag{4.4}$$

Suppose that $\mathrm{supp}\, b \subset \{z : |z| < \eta\}$ for some $\eta > 0$ and $\beta = \max\{2A, 4\eta\}$. Thus, for any x satisfying $|x| > \beta$ and every $f \in F$,

$$\begin{aligned}
&|[b,\mu_S^\rho]f(x)| \\
&= \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|<t \\ |z|<\eta}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} (b(x)-b(z)) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2} \\
&= \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|<t \\ |z|<\eta}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b(z) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1+2\rho}} \right)^{1/2}.
\end{aligned}$$

Since $2(n-\rho) < n$ and $|x-z| \leqslant |x-y| + |y-z| < 2t$. For $1 < q < p < \infty$, by Minkowski's inequality and $\Omega \in L^2(S^{n-1})$, we have

$$
\begin{aligned}
&|[b,\mu_S^\rho]f(x)| \\
&\leqslant C\int_{|z|<\eta}|b(z)||f(z)|\bigg(\int_{|x-z|/2}^{\infty}\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+1+2\rho}}\bigg)^{1/2}dz \\
&\leqslant C\int_{|z|<\eta}|x-z|^{-n}|f(z)|\,dz \\
&\leqslant C\bigg(\int_{|z|<\eta}|x-z|^{-nq}|f(z)|^q\,dz\bigg)^{1/q} \\
&\leqslant C\bigg(\int_{|z|>\beta-\eta}|z|^{-nq}|f(x-z)|^q\,dz\bigg)^{1/q}.
\end{aligned}
\tag{4.5}
$$

Then by Minkowski's inequality and (4.5), we have

$$
\begin{aligned}
&\bigg(\frac{1}{r^\lambda}\int_{B(s,r)}|[b,\mu_S^\rho]f(x)\chi_{\{y:|y|>\beta\}}(x)|^p\,dx\bigg)^{1/p} \\
&\leqslant C\|f\|_{p,\lambda}\bigg(\int_{|x|>\beta/2}|x|^{-nq}\,dx\bigg)^{1/q} \\
&\leqslant CD\varepsilon.
\end{aligned}
\tag{4.6}
$$

(4.6) shows that (2.3) holds for the commutator $[b,\mu_S^\rho]$ in G uniformly. Finally, to finish the proof of Theorem 1.2, it remains to show (2.2) holds for the the commutator $[b,\mu_S^\rho]$ in G uniformly. We need to prove that for any fixed $\varepsilon>0$, if $|v|$ is sufficiently small depended only on ε, then for every $f\in F$,

$$\|[b,\mu_S^\rho]f(\cdot+v)-[b,\mu_S^\rho]f(\cdot)\|_{p,\lambda}\leqslant C\varepsilon.$$

To do this, for any $v\in\mathbb{R}^n$, by the proof of Theorem 4 in [2], we have

$$|[b,\mu_S^\rho]f(x+v)-[b,\mu_S^\rho]f(x)|\leqslant\bigg(\int_0^\infty\int_{|x-y|<t}|I(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\bigg)^{1/2},$$

where

$$
\begin{aligned}
I(x,v,y,t)=&\int_{|z-y|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)\,dz \\
&-\int_{|y+v-z|<t}\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}(b(x+v)-b(z))f(z)\,dz.
\end{aligned}
$$

For any $0<\varepsilon<\frac{1}{3}$ and $v\in\mathbb{R}^n$, write $I(x,v,y,t)$ as

$$
\begin{aligned}
I(x,v,y,t)=&\int_{\substack{|x-z|>2^{\frac{1}{\varepsilon}}|v| \\ |y-z|<t,\,|y+v-z|\geqslant t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x+v)-b(z))f(z)\,dz \\
&+\int_{\substack{|x-z|>2^{\frac{1}{\varepsilon}}|v| \\ |y-z|\geqslant t,\,|y+v-z|<t}}\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}(b(z)-b(x+v))f(z)\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|,\,|y-z|<t,\,|y+v-z|<t}\left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}\right)(b(x+v)-b(z))f(z)\,dz\\
&+\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|,\,|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(x+v))f(z)\,dz\\
&+\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|,\,|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)dz\\
&+\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|,\,|y+v-z|<t}\frac{\Omega(y+v-z)}{|y+v-z|^{n-\rho}}(b(z)-b(x+v))f(z)dz\\
=:&\sum_{i=1}^{6}I_i(x,v,y,t).
\end{aligned}
$$

Further we use the estimates for $J_i, 1\leqslant i\leqslant 6$ from [2]. Since

$$
\left(\int_0^\infty\int_{|x-y|<t}|I_1(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\leqslant C\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|v|^{\frac{1}{2}}|f(z)|}{|x-z|^{n+\frac{1}{2}}}dz,
$$

(see, [2] p. 291), by Minkowski's inequality, for any $s\in\mathbb{R}^n$, $r>0$, we have

$$
\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_0^\infty\int_{|x-y|<t}|I_1(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right|^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
\leqslant{}& C|v|^{\frac{1}{2}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^{n+\frac{1}{2}}}dz\right|^p dx\right)^{\frac{1}{p}}\\
\leqslant{}& C|v|^{\frac{1}{2}}\int_{|z|>2^{\frac{1}{\varepsilon}}|v|}\frac{1}{|z|^{n+\frac{1}{2}}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f(x-z)|^p dx\right)^{\frac{1}{p}}dz\\
\leqslant{}& C\|f\|_{p,\lambda}2^{-\frac{1}{2\varepsilon}}\leqslant CD\varepsilon.
\end{aligned}
$$

Thus we get

$$
\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_1(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda}\leqslant CD\varepsilon. \tag{4.7}
$$

Similarly, we can get

$$
\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_2(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda}\leqslant CD\varepsilon. \tag{4.8}
$$

Since

$$
\begin{aligned}
&\left(\int_0^\infty\int_{|x-y|<t}|I_3(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\\
\leqslant{}& C|v|^{\rho-\frac{n}{2}}\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^{n/2+\rho}}dz\\
&+C\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^n(\log\frac{|x-z|}{|v|})^{2+\theta}}dz,
\end{aligned}
$$

where $\theta < \min\{1, n(\lambda-2), 2\rho - n, \sigma - 2\}$ ($\sigma > 2$, see Lemma 2.2 in [6] and [11]), (see, [2] p. 293). Then for any fixed $s \in \mathbb{R}^n$, $r > 0$, we have

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_0^\infty\int_{|x-y|<t}|I_3(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right|^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\leqslant C|v|^{\rho-n/2}\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^{\rho+n/2}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&\quad+C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|x-z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^n(\log\frac{|x-z|}{|v|})^{2+\theta}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&= C|v|^{\rho-n/2}\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(x-z)|}{|z|^{n/2+\rho}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&\quad+C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|z|>2^{\frac{1}{\varepsilon}}|v|}\frac{|f(x-z)|}{|z|^n\left(\log\frac{|z|}{|v|}\right)^{2+\theta}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&\leqslant C|v|^{\rho-n/2}\int_{|z|>2^{\frac{1}{\varepsilon}}|v|}\frac{1}{|z|^{n/2+\rho}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f(x-z)|^p dx\right)^{\frac{1}{p}}dz\\
&\quad+C\int_{|z|>2^{\frac{1}{\varepsilon}}|v|}\frac{1}{|z|^n\left(\log\frac{|z|}{|v|}\right)^{2+\theta}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f(x-z)|^p dx\right)^{\frac{1}{p}}dz\\
&\leqslant C\left(\frac{1}{2^{\frac{1}{\varepsilon}(\rho-\frac{n}{2})}}+\frac{\varepsilon^{1+\theta}}{(1+\theta)(\log 2)^{1+\theta}}\right)\|f\|_{p,\lambda}\leqslant CD\varepsilon.
\end{aligned}$$

Therefore we get

$$\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_3(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda}\leqslant CD\varepsilon. \tag{4.9}$$

For I_4,

$$\left(\int_0^\infty\int_{|x-y|<t}|I_4(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\leqslant|b(x)-b(x+v)|S^\rho_{2^{\frac{1}{\varepsilon}}|v|}f(x),$$

where

$$S^\rho_{2^{\frac{1}{\varepsilon}}|v|}f(x)=\left(\int_0^\infty\int_{|x-y|<t}\left|\int_{\substack{|x-z|>2^{\frac{1}{\varepsilon}}|v|\\|y-z|<t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}},$$

(see, [2] p. 296). For any fixed $1<q<p<\infty$, by the estimate of $S^\rho_{2^{\frac{1}{\varepsilon}}|v|}f(x)$ in [2], we get

$$S^\rho_{2^{\frac{1}{\varepsilon}}|v|}f(x)\leqslant CM(\mu^\rho_S f)(x)+C(M|f|^q(x))^{\frac{1}{q}}+CMf(x).$$

By the $L^{p,\lambda}$ boundedness of μ_S^ρ (see [7] and [11]) and $\|(M(|f|^q))^{1/q}\|_{p,\lambda} \leqslant C\|f\|_{p,\lambda}$ (see [9]), we get

$$\|S^\rho_{2^{\frac{1}{\varepsilon}}|v|} f\|_{p,\lambda} \leqslant C\|f\|_{p,\lambda}.$$

Now we give the estimate of I_4. Since $b \in C_0^\infty$, we have $|b(x+v)-b(x)| \leqslant C|v|$, then for any $s \in \mathbb{R}^n$, $r>0$, we have

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_0^\infty\int_{|x-y|<t}|I_4(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right|^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\leqslant C|v|\left\|S^\rho_{2^{\frac{1}{\varepsilon}}|v|}f\right\|_{p,\lambda} \leqslant C|v|\|f\|_{p,\lambda} \leqslant CD|v|.
\end{aligned}$$

That is

$$\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_4(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda} \leqslant CD|v|. \tag{4.10}$$

About I_5, since $|b(x)-b(z)| \leqslant C|x-z|$, $t>|x-z|/2$, $2n-2\rho<n$, $\Omega \in L^2(S^{n-1})$, by Minkowski's inequality and the estimate of J_5 in [2], we get

$$\left(\int_0^\infty\int_{|x-y|<t}|I_5(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}} \leqslant C\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^{n-1}}dz,$$

(see, [2] p. 298). Then for any fixed $s \in \mathbb{R}^n$, $r>0$, we get

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_0^\infty\int_{|x-y|<t}|I_5(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right|^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\leqslant C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x-z|^{n-1}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&= C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(x-z)|}{|z|^{n-1}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&\leqslant C\int_{|z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{1}{|z|^{n-1}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f(x-z)|^p dx\right)^{\frac{1}{p}}dz\\
&\leqslant CD2^{\frac{1}{\varepsilon}}|v|.
\end{aligned}$$

That is

$$\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_5(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda} \leqslant CD2^{\frac{1}{\varepsilon}}|v|. \tag{4.11}$$

Similar to the estimate of I_5, using the estimates $|b(x+v)-b(z)| \leqslant C|x+v-z|$ and $2t > |x-y|+|y+v-z| \geqslant |x+v-z|$ we get

$$\left(\int_0^\infty\int_{|x-y|<t}|I_6(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}} \leqslant C\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x+v-z|^{n-1}}dz.$$

Then for any $s \in \mathbb{R}^n$, $r > 0$, we get

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_0^\infty\int_{|x-y|<t}|I_6(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right|^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\leqslant C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|x-z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(z)|}{|x+v-z|^{n-1}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&= C\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left|\int_{|z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{|f(x-z)|}{|z+v|^{n-1}}dz\right|^p dx\right)^{\frac{1}{p}}\\
&\leqslant C\int_{|z|\leqslant 2^{\frac{1}{\varepsilon}}|v|}\frac{1}{|z+v|^{n-1}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f(x-z)|^p dx\right)^{\frac{1}{p}}dz\\
&\leqslant CD(2^{\frac{1}{\varepsilon}}|v|+|v|).
\end{aligned}$$

That is

$$\left\|\left(\int_0^\infty\int_{|x-y|<t}|I_6(x,v,y,t)|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}\right\|_{p,\lambda}\leqslant CD(2^{\frac{1}{\varepsilon}}|v|+|v|). \tag{4.12}$$

In (4.7)–(4.12), taking $|v|$ sufficiently small depending on ε, we can get

$$\lim_{|v|\to 0}\|[b,\mu_S^\rho]f(\cdot+v)-[b,\mu_S^\rho]f(\cdot)\|_{p,\lambda}=0 \text{ uniformly in } f\in F.$$

This says (2.2) holds for the commutator $[b,\mu_S^\rho]$ in G uniformly and therefore we complete the proof of Theorem 1.2.

Acknowledgement. The authors would like to express their deep gratitude to the referee for giving many valuable suggestions.

REFERENCES

[1] M. S. BERGER, *Nonlinearity and Functional Ananysis*, Academic Press, New York, 1977, 64–107.
[2] Y. CHEN AND Y. DING, *Compactness Characterization of Commutators for Littlewood-Paley Operators*, Kodai Math. J. **32** (2009), 257–324.
[3] Y. CHEN, Y. DING AND X. WANG, *Compactness for Commutators of Marcinkiewicz Integral in Morrey Spaces*, Taiwanese J. Math. **15** (2011), 633–658.
[4] Y. CHEN, Y. DING AND X. WANG, *Compactness of commutators for singular integrals on Morrey Spaces*, Canad. J. Math. **64** (2012), 257–281.
[5] Y. CHEN, Y. DING AND X. WANG, *Commutators of Littlewood-Paley Operators on the Generalized Morrey Space*, J. Ineq. Appl. Article ID 961502, 20 pages, 2010.
[6] Y. DING, S. LU AND Q. XUE, *Parametrized area integrals on Hardy spaces and weak Hardy spaces*, Acta Math. Sinica (English Series), **23** (2007), 1537–1552.
[7] Q. XUE, Y. DING, *Weighted L^p boundedness for parameterized Littlewood-Paley operators*, Taiwanese J. Math. **11** (2007), no. 4, 1143–1165.
[8] Y. DING, Q. XUE, *Endpoint estimates for commutators of a class of Littlewood-Paley operators*, Hokkaido Math. J. **36** (2007), no. 2, 245–282.

[9] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis, ICM-90 Satellite Conf. Proc., S. Igari (ed.), Springer-Verlag, Tokyo, 1991, pp. 183–189.
[10] A. UCHIYAMA, *On the compactness of operators of Hankel type*, Tôhoku Math. J. **30** (1978), 163–171.
[11] Q. XUE, *Parameterized area integral and Littlewood-Paley operators*, PhD Thesis. 2004.

(Received February 18, 2014)

Yanping Chen
Department of Applied Mathematics
University of Science and Technology Beijing
Beijing 100083, China
e-mail: yanpingch@126.com

Hong Wang
Department of Applied Mathematics
University of Science and Technology Beijing
Beijing 100083, China
e-mail: wanghong0229@126.com