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ON BOUNDEDNESS OF FRACTIONAL MAXIMAL OPERATOR IN WEIGHTED $L^{p(.)}$ SPACES

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Abstract. In this paper, we derive some sufficient conditions for the boundedness of the fractional maximal operator in the weighted variable exponent Lebesgue spaces $L^{p(.)}$, where Sawyer's type pair of modular conditions are proposed on a weight functions and it is assumed a local log-regularity and a decay condition on the exponent function $p(.)$.

1. Introduction

The aim of present paper is a study of two-weight inequality with respect to the norm of generalized Lebesgue space $L^{p(.)}$

$$\left\| v^{1/p(.)} M_\alpha f \right\|_{L^{p(.)}} \leqslant C \left\| \omega^{1/p(.)} f \right\|_{L^{p(.)}}; \quad f \geqslant 0, \tag{1.1}$$

for the fractional maximal operator $M_\alpha f(x) = \sup\limits_{Q \ni x} |Q|^{\frac{\alpha}{n}-1} \int\limits_Q |f(y)|dy$, $0 \leqslant \alpha < n$, where the *supremum* is taken over all cubes $Q \in E_n$ that contain x and whose sides are parallel to the coordinate axes. The M_α is Hardy-Littlewood maximal operator when $\alpha = 0$. If the exponents p, q are constant and $q \geqslant p > 1$, due to the well known Sawyer's result [30], the necessary and sufficient condition for the $L^p \to L^q$ boundedness of the fractional maximal operator M_α is

$$\int\limits_Q v \left(M_\alpha \chi_Q \sigma \right)^q dx \leqslant C \left(\int\limits_Q \sigma dx \right)^{q/p}. \tag{1.2}$$

for all cubes Q in $\mathbb{R}^n$. Also note that if $q > p$ and the weight function $\sigma = \omega^{-\frac{1}{p-1}}$ satisfies the reverse doubling condition the following "so called A_{pq}" condition suffices for the inequality (1.1) to hold

$$|Q|^{\alpha/n-1} \left(\int\limits_Q v(x)dx \right)^{1/q} \left(\int\limits_Q \sigma(x)dx \right)^{1/p} \leqslant C, \tag{1.3}$$

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see, e.g. in [5] for a new proof. For $q \geqslant p$, and σ satisfies A_∞ condition, in [29] Rakotondratsimba has shown that (1.3) implies (1.1).

Given a function $p: E_n \to [1,\infty)$, we say that $p(.)$ is locally log-Hölder continuous if there exists a constant C such that

$$|p(x)-p(y)| \leqslant \frac{C}{-\ln|x-y|} \quad \text{for} \quad x,y \in E_n \quad \text{and} \quad |x-y| < \frac{1}{2}. \tag{1.4}$$

Similarly, we say $p(.)$ is log-Hölder continuous at infinity if there exist constants C and $p(\infty)$ such that

$$|p(x)-p(\infty)| \leqslant \frac{C}{\ln(e+|x|)}, \quad x \in E_n. \tag{1.5}$$

The topic of present paper has been a subject of investigation in the recent papers [6, 8, 16]. The boundedness of a maximal operator on variable exponent Lebesgue space $L^{p(.)}(E_n)$ (the inequality (1.1) for $v=\omega=1$) was first proved by Diening [10] assuming that $p(.)$ satisfies (1.4) and is a constant outside a large ball. A corresponding result for the exponents satisfying (1.4) and (1.5) was obtained in [2, 3, 4]. Generally speaking, the log-Hölder continuity conditions are not necessary (see, [24]). However, if they are relaxed, it is possible to construct counterexamples (see, [28]); moreover, an example was constructed in [13] that shows Hardy operator $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$ is not bounded in $L^{p(.)}(0,1)$ if the log-Holder condition at zero is violated. In [27], (1.5) was relaxed to the condition: there exist $\beta > 1$, $p(\infty) > 1$ such that

$$\int_{E_n} \beta^{-\frac{1}{|p(x)-p(\infty)|}}\, dx < \infty \tag{1.6}$$

In [17, 18, 19, 20, 21, 22], the power weight case $v=\omega=|x-x_0|^{\gamma p(.)}$ was considered; the necessary and sufficient condition $-\frac{n}{p(x_0)} < \gamma < \frac{n}{p'(x_0)}$ was obtained there for $\alpha = 0$. In case when $p(.)$ satisfies the conditions (1.4) and (1.5), it was proved in [6] that the inequality (1.1) holds for $v=\omega$ and $\alpha=0$ if and only if

$$\left\|\omega^{\frac{1}{p(.)}}\chi_Q\right\|_{p(.)}\left\|\omega^{-\frac{1}{p(.)}}\chi_Q\right\|_{p'(.)} \leqslant C|Q|. \tag{1.7}$$

In [8], under the same conditions on $p(.)$, it was proved that the inequality (1.1) holds for $v=\omega$ and $\alpha=0$ if and only if

$$\omega(Q)\left\|\omega^{-1}\chi_Q\right\|_{\frac{p'(.)}{p(.)}} \leqslant C|Q|^{\overline{p}_Q}, \tag{1.8}$$

where $\frac{1}{\overline{p}_Q} = \frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx$. Other authors have also considered weighted norm inequalities for the maximal operator on variable Lebesgue spaces (see, for instance, [1, 11, 15, 16, 25, 26]). In our previous works [25, 26], we have proved some easily verifiable sufficient conditions on pair of weights v and ω for the Hardy-Littlewood maximal operator M to be bounded in E_n.

Since the checking of conditions (1.7) and (1.8) is difficult (in view of a presence of variable exponent norm in their expressions), it is interesting to find a modular form sufficient conditions which is easy to verify. In the present paper, we derive a pair of Sawyer's type conditions that guarantees the boundedness of the fractional maximal operator from one weighted variable Lebesgue space to another one. They are also necessary in the class of uniformly bounded cubes (see, Remark 2.2).

In this paper, we use the following condition:

Local regularity condition: there exists $C>1$ such that for all $x,y\in Q$ with $\sigma(Q)<\frac{1}{2}$,

$$|p(x)-p(y)|\leqslant\frac{C}{-\ln\sigma(Q)} \tag{1.9}$$

to characterize the local regularity behavior of $p(.)$ (see, also [14]).

In place of condition (1.5), we put:

Decay condition: there exists a $\beta>1$ such that

$$\int_{E_n}\beta^{-r(x)}\,\sigma\,dx<\infty, \tag{1.10}$$

where $r(x)=\frac{p(x)p(\infty)}{|p(\infty)-p(x)|}$, $\sigma=\omega(x)^{-\frac{1}{p(x)-1}}$. In the unweighted case, this is condition (1.6).

The methods of our proof are based on the Calderon-Zygmund partitioning of E_n. We present the test function f as a sum of two summands f_1 and f_2 corresponding to its "biggest" and "smallest" parts and further get an estimation for every one. The sufficiency part essentially uses Sawyer's approach [30].

2. Main result

The following result may be considered as a variable exponent analogue of Sawyer's assertion.

THEOREM 1. (Main) *Let $0\leqslant\alpha<n$, $p:E_n\to[1,\infty)$ be a measurable function satisfying the condition $1<p^-\leqslant p(x)\leqslant p^+<\infty$. Let v,ω be measurable a.e. positive functions on E_n such that $v,\sigma=\omega(x)^{-1/(p(x)-1)}\in L^{1,loc}$ and together with p satisfy the conditions (1.9) and (1.10). Suppose that there exist $C>0$, $\beta>1$ such that for all $Q\subset E_n$ both conditions*

$$\int_Q M_\alpha(\sigma\chi_Q)^{p(x)}\;v(x)dx\leqslant C\int_Q\sigma\,dx; \tag{2.1}$$

$$\int_Q M_\alpha(\sigma\chi_Q)^{p(x)}\beta^{-r(x)}\;v(x)dx\leqslant C\int_Q\sigma(x)\beta^{-r(x)}\,dx \tag{2.2}$$

are satisfied; then the inequality (1.1) holds for any function $f\geqslant 0$.

Following corollary follows from Theorem 1 in the case $\omega=1$.

COROLLARY 1. *Let $0 \leqslant \alpha < n$, $p : E_n \to [1,\infty)$ be a measurable function satisfying the condition $1 < p^- \leqslant p(x) \leqslant p^+ < \infty$. Let v be measurable a.e. positive functions on E_n such that $v \in L^{1,loc}$ and together with p satisfy the conditions (1.4) and (1.6). Suppose that there exist $C > 0$, $\beta > 1$ such that for all $Q \subset E_n$ both conditions*

$$\int_Q M_\alpha (\chi_Q)^{p(x)} \; v(x)\, dx \leqslant C\, |Q|; \tag{2.3}$$

$$\int_Q M_\alpha (\chi_Q)^{p(x)} \beta^{-r(x)} \; v(x)\, dx \leqslant C \int_Q \beta^{-r(x)}\, dx \tag{2.4}$$

are satisfied; then the inequality

$$\left\| v^{1/p(.)} M_\alpha f \right\|_{L^{p(.)}} \leqslant C \, \|f\|_{L^{p(.)}}; \quad f \geqslant 0, \tag{2.5}$$

holds for any function $f \geqslant 0$.

As it is pointed to our knowledge by the referee, the condition (2.3) is necessary and sufficient for the inequality (1.1) to hold for the operator M_α restricted on a bounded domain. This assertion noted in [16].

3. Notations and auxiliary facts

Throughout the paper all notation are either standard or defined whenever necessary. E_n is an n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$. χ_E is the characteristic function of a measurable set E. We denote by Q an arbitrary cube in E_n whose sides are parallel to the coordinate axes. For a given measurable set E and measurable function ω, $|E|$ denotes the Lebesgue measure of E, and $\omega(E) = \int_E \omega dx$. For a given $1 < p < \infty$, $p' = p/(p-1)$ denotes the conjugate of p. Finally, C denotes a positive constant which may change its values at each appearance.

For the open set Ω let $p : \Omega \longrightarrow [1,\infty)$ be a measurable function. We define $L^{p(.)}(\Omega)$ as the class of measurable functions $f : \Omega \to E_n$, for which the modular

$$I_{p(.)}(f) := \int_\Omega |f(x)|^{p(x)}\, dx$$

is finite. If $p^+ = \sup\{p(x) : x \in E_n\} < \infty$ and $p^- = \inf\{p(x) : x \in E_n\} \geqslant 1$ hold, then the expression

$$\|f\|_{p(.)} := \inf\{\lambda > 0 : I_{p(.)}(f/\lambda) < 1\}$$

defines the norm in $L^{p(.)}(\Omega)$, thereby making $L^{p(.)}(\Omega)$ a Banach space. Moreover, the convergence in norm is equivalent to the convergence in modular, and $\|f\|_{p(.)} \leqslant 1$ if and only if $I_{p(.)}(f) \leqslant 1$. If $p^- > 1$ and $p^+ < \infty$, then the space $L^{p(.)}(\Omega)$ becomes reflexive. For the definition and other properties of the space $L^{p(.)}(\Omega)$ we refer the reader to [23] and [31].

In the proof of Theorem 3.1, the following lemma (the proof see e.g. in [5, 12]) is useful.

LEMMA 1. *Suppose that for a non-negative locally integrable function f over some cube Q and for some $t > 0$,*

$$|Q|^{\frac{\alpha}{n}-1}\int\limits_{Q} f dy > t$$

holds. Then, there exists a dyadic cube P such that $Q \subset 3P$ and a positive constant $C_{n,\alpha}$, depending only on n, α such that

$$|P|^{\frac{\alpha}{n}-1}\int\limits_{P} f dy > C_{n,\alpha} t.$$

4. Proof of Theorem 1

Let $f \geqslant 0$ be an arbitrary function such that

$$\left\| f\omega^{1/p(.)} \right\|_{L_{p(.)}} \leqslant 1, \tag{4.1}$$

then

$$I_{p(.)}\left(\omega^{1/p(.)} f\right) \leqslant 1. \tag{4.2}$$

To prove inequality (1.1) we have to show the inequality

$$I_{p(.)}\left(v^{1/p(.)} M_\alpha f\right) \leqslant C. \tag{4.3}$$

One can assume that f has a compact support. For the function f we denote

$$f_1 = f\chi_{f \geqslant \sigma}, \quad f_2 = f\chi_{f<\sigma}.$$

Then, $f = f_1 + f_2$. It is clear that

$$M_\alpha f(x) \leqslant M_\alpha f_1(x) + M_\alpha f_2(x). \tag{4.4}$$

Estimate of the function f_1. For every point $x \in E_n$ associate a cube Q_x. For a fixed cube Q_x, we have

$$|Q_x|^{\frac{\alpha}{n}-1}\int\limits_{Q_x} f_1 dy = \left(|Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x} \sigma dy\right)\left(\frac{1}{\sigma(3Q_x)}\int\limits_{Q_x}\left(\frac{f_1}{\sigma}\right)\sigma dx\right).$$

Denote $p_x^- = ess\inf\{p(y) : y \in Q_x\}$. Then

$$\frac{f_1(y)}{\sigma(y)} \leqslant \left(\frac{f_1(y)}{\sigma(y)}\right)^{p(y)/p_x^-}, \ y \in Q_x.$$

Hence, by virtue of Hölder's inequality,

$$|Q_x|^{\frac{\alpha}{n}-1}\int_{Q_x} f_1 dy = 3^{n-\alpha}\left(|3Q_x|^{\frac{\alpha}{n}-1}\int_{3Q_x}\sigma dy\right)\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p_x^-}\sigma dx\right)$$

$$\leqslant 3^{n-\alpha}\sigma(Q_x)^{1-p^-/p_x^-}\left(|3Q_x|^{\frac{\alpha}{n}-1}\int_{3Q_x}\sigma dy\right)\frac{1}{\sigma(3Q_x)}\left(\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-/p_x^-}.$$

Since $\sigma(Q_x)^{1-p^-/p_x^-} \leqslant \sigma(3Q_x)^{1-p^-/p_x^-}$, the term $\left(|Q_x|^{\frac{\alpha}{n}-1}\int_{Q_x} f_1 dy\right)^{p(x)}$ is exceeded by

$$3^{(n-\alpha)p^+}\left(|3Q_x|^{\frac{\alpha}{n}-1}\int_{3Q_x}\sigma dy\right)^{p(x)}\left[\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}\right]^{p(x)/p_x^-}.$$

Using Hölder's inequality and (4.3) we have

$$\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}$$

$$\leqslant\left(\frac{1}{\sigma(3Q_x)}\right)^{p^-}(\sigma(Q_x))^{p^--1}\int_{Q_x} f_1^{p(y)}\sigma^{1-p(y)}dy \leqslant (\sigma(3Q_x))^{-1}.$$

Then

$$\sigma(3Q_x)\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-} \leqslant 1.$$

Therefore, by using the previous inequality and $p(x) \geqslant p_x^-$, we have

$$\begin{aligned}&\left[\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}\right]^{p(x)/p_x^-}\\ &\leqslant (\sigma(3Q_x))^{1-p(x)/p_x^-}\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}.\end{aligned} \tag{4.5}$$

By virtue of condition (1.9) we see that $\sigma(3Q_x)^{1-p(x)/p_x^-} \leqslant C$ (the details see e.g. in [7, p. 95] or [14]). Hence the right hand side of (4.5) does not exceed

$$C\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}.$$

Therefore,

$$\left(|Q_x|^{\frac{\alpha}{n}-1}\int_{Q_x}f_1 dy\right)^{p(x)} \leqslant C\left(|3Q_x|^{\frac{\alpha}{n}-1}\int_{3Q_x}\sigma dy\right)^{p(x)}\left(\frac{1}{\sigma(3Q_x)}\int_{Q_x}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}. \tag{4.6}$$

For each $k\in\mathbb{Z}$, let

$$\Omega_k=\left\{x\in E_n : 2^k<M_\alpha f\leqslant 2^{k+1}\right\}.$$

For each $x\in\Omega_k$ there exists a cube Q_x^k containing x such that $\left|Q_x^k\right|^{\frac{\alpha}{n}-1}\int_{Q_x^k} f_1(x)dx\geqslant 2^k$. By Lemma 1, there exists a dyadic cube P_x^k such that $Q_x^k\subset 3P_x^k$ and $\left|P_x^k\right|^{\frac{\alpha}{n}-1}\int_{P_x^k} f_1(x)\,dx \geqslant C_{n,\alpha}2^k$. Since the P_x^k's are dyadic and bounded in size (since f_1 has compact support) we can pass to a maximal disjoint subcollection $\left\{P_j^k\right\}$ such that for each x, $Q_x^k\subset 3P_x^k$, for some j. Hence $\Omega_k\subset\bigcup_j 3P_j^k$. Define the sets E_j^k inductively: $E_1^k=3P_1^k\cap\Omega_k$, $E_2^k=\left(3P_2^k\backslash 3P_1^k\right)\cap\Omega_k$, $E_3^k=\left(3P_3^k\backslash\left(3P_2^k\cup 3P_1^k\right)\right)\cap\Omega_k$, etc. We have $\Omega_k\subset\bigcup_{j=1}^{\infty}E_j^k$ (up to a set of measure 0). By applying the above estimates and taking instead of the cube Q_x the cubes $\left\{P_j^k\right\}$, we now have:

$$\int_{E_n}(M_\alpha f_1(x))^{p(x)}v(x)dx=\sum_k\int_{\Omega_k}M_\alpha f_1(x)^{p(x)}vdx\leqslant\sum_k\int_{\Omega_k}\left(2^{k+1}\right)^{p(x)}vdx$$

$$\leqslant 2^{p^+}C_n\sum_{k,j}\int_{E_j^k}\left(\left|P_j^k\right|^{\frac{\alpha}{n}-1}\int_{P_j^k}f_1 dy\right)^{p(x)}v(x)dx.$$

Now we shall apply the method of [5, 12], constructed by the procedure which takes place in Sawyer's type statement [30].

By virtue of (4.6) for the cube P_j^k in place of Q_x, we have

$$\leqslant C\sum_{k,j}\left(\int_{E_j^k} v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)} dx\right)\left(\frac{1}{\sigma(3Q_x)}\int_{P_j^k}(f_1/\sigma)^{p(y)/p^-}\sigma dx\right)^{p^-}$$

$$=\int_X T\left[\left(\frac{f(y)}{\sigma(y)}\right)^{p(y)/p^-}\right]^{p^-} d\zeta, \tag{4.7}$$

where $X=\mathbb{N}\times\mathbb{Z}$ and the measure ζ on X is given by

$$\zeta(j,k)=\int_{E_j^k}\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)} v(x)dx$$

and for all nonnegative, measurable functions h, the operator T is defined by

$$Th(j,k)=\frac{1}{\sigma\left(3P_j^k\right)}\int_{P_j^k} h\sigma dx.$$

If $T:L^{p^-}(E_n,\sigma)\to L^{p^-}(X,\zeta)$ is a bounded operator, then for the first summand in (4.4) we get the estimate

$$\int_X\left(T\left(\frac{f}{\sigma}\right)^{\frac{p}{p^-}}\right)^{p^-} d\zeta\leqslant C\int_{E_n}\left(\left|\frac{f}{\sigma}\right|^{\frac{p}{p^-}}\right)^{p^-}\sigma dx=C\int_{E_n}\left|\frac{f}{\sigma}\right|^{p(x)}\sigma dx=C \tag{4.8}$$

by (4.2). It remains to show that the operator T is bounded. Since T is bounded on L^∞, by Marcinkiewicz interpolation theorem it will suffice to show that T is of weak type (1,1). Take h which is bounded and with compact support, fix $\lambda>0$ and let $F_\lambda=\{(j,k)\in X: Th(j,k)>\lambda\}$. Since $E_j^k\subset 3P_j^k$,

$$\zeta(F_\lambda)=\sum_{(j,k)\in F_\lambda}\int_{E_j^k} v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)} dx\leqslant\sum_{(j,k)\in F_\lambda}\int_{E_j^k} v(x)M_\alpha\left(\sigma\chi_{3P_j^k}\right)^{p(x)} dx. \tag{4.9}$$

Let $\{P_i\}$ be the maximal disjoint subcollection of $\left\{P_j^k:(j,k)\in F_\lambda\right\}$. Then, since the E_j^k 's are pairwise disjoint,

$$\leqslant\sum_i\sum_{P_j^k\subset P_i}\int_{E_j^k} v(x)M_\alpha\left(\sigma\chi_{3P_j^k}\right)^{p(x)} dx\leqslant\sum_i\int_{3P_i} v(x)M_\alpha\left(\sigma\chi_{3P_i}\right)^{p(x)} dx$$

by inequality (2.3)

$$\leqslant \sum_i C \int_{3P_i} \sigma dx.$$

For each i, there exists $(j,k) \in F_\lambda$ such that $P_i = P_j^k$. Hence by definition of F_λ and since the P_i's are disjoint and $p^- > 1$,

$$\leqslant C \frac{1}{\lambda} \sum_i \int_{P_i} h\sigma dx \leqslant C \frac{1}{\lambda} \int_{E_n} h\, \sigma dx.$$

Therefore, T is weak (1,1) and (4.8) has been shown. Hence

$$\int_{E_n} (M_\alpha f_1)^{p(x)} v(x)\, dx \leqslant C. \tag{4.10}$$

Estimate of the function f_2. Let Q_x be a fixed cube. Denote $r(x) = \frac{p(\infty)p(x)}{|p(\infty)-p(x)|}$. If $\frac{1}{\sigma(3Q_x)} \int_{Q_x} (f_2/\sigma)\, \sigma dx \leqslant \beta^{-\frac{r(x)}{p(x)}}$, we have:

$$\begin{aligned}\left(|Q_x|^{\frac{\alpha}{n}-1} \int_{Q_x} f_2 dy\right)^{p(x)} &= \left(|Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} \sigma dy\right)^{p(x)} \left(\frac{1}{\sigma(3Q_x)} \int_{Q_x} \frac{f_2}{\sigma} \sigma dx\right)^{p(x)} \\ &\leqslant \left(|3Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} \sigma dy\right)^{p(x)} \beta^{-r(x)},\end{aligned}$$

hence,

$$\left(|Q_x|^{\frac{\alpha}{n}-1} \int_{Q_x} f_2 dy\right)^{p(x)} \leqslant C\beta^{-r(x)} \left(|3Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} \sigma dy\right)^{p(x)}. \tag{4.11}$$

If $\frac{1}{\sigma(3Q_x)} \int_{Q_x} (f_2/\sigma)\, \sigma dx > \beta^{-\frac{r(x)}{p(x)}}$ and $p(x) < p(\infty)$, then

$$\begin{aligned}\left(|3Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} f_2 dy\right)^{p(x)} &= \left(|3Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} \sigma dy\right)^{p(x)} \left(\frac{1}{\sigma(3Q_x)} \int_{Q_x} \frac{f_2}{\sigma} d\mu\right)^{p(\infty)} \\ &\quad \times \left(\frac{1}{\sigma(3Q_x)} \int_{Q_x} \frac{f_2}{\sigma} \sigma dx\right)^{p(x)-p(\infty)} \qquad (4.12) \\ &\leqslant C\beta^{p(\infty)} \left(|3Q_x|^{\frac{\alpha}{n}-1} \int_{3Q_x} \sigma dy\right)^{p(x)} \left(\frac{1}{\sigma(3Q_x)} \int_{Q_x} \frac{f_2}{\sigma} \sigma dx\right)^{p(\infty)}.\end{aligned}$$

Thus,

$$\left(|3Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x} f_2 dy\right)^{p(x)} \leqslant C\beta^{p(\infty)}\left(|3Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x}\sigma dy\right)^{p(x)}\left(\frac{1}{\sigma(3Q_x)}\int\limits_{Q_x}\frac{f_2}{\sigma}\sigma dx\right)^{p(\infty)}. \tag{4.13}$$

If $p(x) \geqslant p(\infty)$, by virtue of $\frac{1}{\sigma(3Q_x)}\int\limits_{Q_x}(f_2/\sigma)\,\sigma dx \leqslant 1$, it follows that

$$\left(|Q_x|^{\frac{\alpha}{n}-1}\int\limits_{Q_x} f_2 dy\right)^{p(x)} \leqslant C\left(|3Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x}\sigma dy\right)^{p(x)}\left(\frac{1}{\sigma(3Q_x)}\int\limits_{Q_x}\frac{f_2}{\sigma}\sigma dx\right)^{p(\infty)}. \tag{4.14}$$

Combining (4.11) and (4.14) we have the estimate

$$\begin{aligned}\left(|Q_x|^{\frac{\alpha}{n}-1}\int\limits_{Q_x} f_2 dy\right)^{p(x)} &\leqslant C\beta^{-r(x)}\left(|3Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x}\sigma dy\right)^{p(x)} \\ &\quad + C\left(|3Q_x|^{\frac{\alpha}{n}-1}\int\limits_{3Q_x}\sigma dy\right)^{p(x)}\left(\frac{1}{\sigma(3Q_x)}\int\limits_{Q_x}\frac{f_2}{\sigma}\sigma dx\right)^{p(\infty)}.\end{aligned} \tag{4.15}$$

Now, repeating our previous reasoning for the function f_2, we obtain (having replaced the cubes Q_x by the balls P_j^k, where $k \in \mathbb{Z},\ j \in \mathbb{N}$):

$$\begin{aligned}\int\limits_{E_n}(M_\alpha f_2(x))^{p(x)}v(x)dx &= \sum_k\int\limits_{\Omega_k} M_\alpha f_2(x)^{p(x)}v(x)dx \leqslant \sum_k\int\limits_{\Omega_k}\left(2^{k+1}\right)^{p(x)}v(x)dx \\ &\leqslant 2^{p^+}3^{p^+n}C_{n,\alpha}\sum_{k,j}\int\limits_{E_j^k}\left(\left|P_j^k\right|^{\frac{\alpha}{n}-1}\int\limits_{P_j^k} f_2 dy\right)^{p(x)}v(x)dx.\end{aligned}$$

By virtue of (4.15),

$$\begin{aligned}&\int\limits_{E_n}(M_\alpha f_2)^{p(x)}v(x)\,dx \\ &\leqslant C\sum_{k,j}\left(\int\limits_{E_j^k}v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int\limits_{3P_j^k}\sigma dy\right)^{p(x)}\beta^{-r(x)}\,dx\right) \\ &\quad + C\sum_{k,j}\left[\int\limits_{E_j^k}v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int\limits_{3P_j^k}\sigma dy\right)^{p(x)}dx\right]\left(\frac{1}{\sigma\left(3P_j^k\right)}\int\limits_{P_j^k}\frac{f_2}{\sigma}\sigma dx\right)^{p(\infty)} \\ &:= I_1 + I_2\end{aligned}$$

By applying the procedure above, as in the obtaining process of inequality (4.8), in this case we have:

$$\overline{\zeta}(j,k)=\int_{E_j^k} v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)}\beta^{-r(x)}\,dx;$$

$$\overline{T}h(j,k)=\frac{1}{\eta\left(3P_j^k\right)}\int_{P_j^k} h\,\eta dx,\ \text{ where }\ \eta=\sigma\beta^{-r(x)}.$$

Further, I_1 does not exceed

$$\begin{aligned}&\leqslant C\sum_{k,j}\left(\int_{E_i^k} v\beta^{-r(x)}\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_i^k}\sigma dy\right)^{p(x)}dx\right)\left(\frac{1}{\eta\left(3P_j^k\right)}\int_{P_j^k}1\cdot\eta dx\right)^{p(\infty)}\\&=\int_X\overline{T}(1)^{p(\infty)}\,d\overline{\zeta}\leqslant C\int_{R^n}\frac{\sigma}{\beta^{r(x)}}dx<\infty.\end{aligned}\tag{4.16}$$

We had used that the operator $\overline{T}$ is bounded from $L^{p(\infty)}(E_n,\eta)$ to $L^{p(\infty)}\left(X,\overline{\zeta}\right)$. This condition is verified similarly to (4.9):

$$\begin{aligned}\overline{\zeta}(F_\lambda)&=\sum_{(j,k)\in F_\lambda}\int_{E_j^k} v(x)\,\beta^{-r(x)}\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)}dx\\&\leqslant\sum_{(j,k)\in F_\lambda}\int_{E_j^k} v(x)\beta^{-r(x)}M_\alpha\left(\sigma\chi_{3P_j^k}\right)^{p(x)}dx\\&\leqslant\sum_i\sum_{P_j^k\subset P_i}\int_{E_j^k} v(x)\beta^{-r(x)}M_\alpha\left(\sigma\chi_{3P_j^k}\right)^{p(x)}dx\\&\leqslant\sum_i\int_{3P_i} v(x)\beta^{-r(x)}M_\alpha\left(\sigma\chi_{3P_i}\right)^{p(x)}dx,\end{aligned}$$

by virtue of condition (2.3),

$$\begin{aligned}&\leqslant\sum_i C\int_{3P_i}\beta^{-r(x)}\,\sigma dx=C\sum_i\int_{3P_i}\eta\,dx\\&\leqslant\sum_i\frac{C}{\lambda}\int_{P_i} h\,\eta dx\leqslant\frac{C}{\lambda}\int_{E_n} h\,\eta dx.\end{aligned}$$

The operator $\overline{T}$ is bounded, so, the estimate in (4.16) is legitimate.

Using the condition (2.3) we can get the estimation of I_2 as above:

$$I_2 = C\sum_{k,j}\left(\int_{E_j^k} v(x)\left(\left|3P_j^k\right|^{\frac{\alpha}{n}-1}\int_{3P_j^k}\sigma dy\right)^{p(x)} dx\right)\left(\frac{1}{\mu\left(3P_j^k\right)}\int_{P_j^k}(f_2/\sigma)\,\sigma dx\right)^{p(\infty)}$$
$$= \int_X \left(T\left(\frac{f_2}{\sigma}\right)\right)^{p(\ \infty)} d\zeta \leqslant C\int_{E_n} f_2^{p(\infty)}\sigma^{1-p(\infty)}dx = C\int_{E_n}\left(\frac{f_2}{\sigma}\right)^{p(\infty)}\sigma dx.$$

Furthermore,

$$I_2 \leqslant C\int_{E_n}\left(\frac{f_2}{\sigma}\right)^{p(\infty)}\sigma dx$$
$$= C\int_{p(x)\leqslant p(\infty)}\left(\frac{f_2}{\sigma}\right)^{p(\infty)}\sigma dx + C\int_{p(x)>p(\infty)}\left(\frac{f_2}{\sigma}\right)^{p(\infty)}\sigma dx$$
$$= J_1 + J_2.$$

Since $p(x) < p(\infty)$ and $f_2 \leqslant \sigma$ in the first integral by (4.2) we have

$$J_1 \leqslant C\int_{E_n}\left(\frac{f_2}{\sigma}\right)^{p(x)}\sigma \leqslant C. \tag{4.17}$$

$$J_2 = C\int_{\left\{x:\ p(x)>p(\infty),\ f_2/\sigma>\beta^{-\frac{r(x)}{p(\infty)}}\right\}}\left(\frac{f_2}{\sigma}\right)^{p(x)}\left(\frac{f_2}{\sigma}\right)^{p(\infty)-p(x)}\sigma dx$$
$$+C\int_{\left\{x:\ p(x)>p(\infty),\ f_2/\sigma\leqslant\beta^{-\frac{r(x)}{p(\infty)}}\right\}}\left(\frac{f_2}{\sigma}\right)^{p(\infty)}\sigma dx$$
$$\leqslant C_1\int_{E_n}\beta^{-r(x)}\ \sigma dx + C\beta^{p^+}\int_{E_n}\left(\frac{f_2}{\sigma}\right)^{p(x)}\sigma dx \leqslant C_2$$

by virtue of (1.5) and (4.2).

This completes the proof of Theorem 3.1.

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