Mathematical
Inequalities
& Applications
Volume 19, Number 1 (2016), 101–115
doi:10.7153/mia-19-08

RELATIONS BETWEEN TWO CLASSES OF REAL FUNCTIONS AND APPLICATIONS TO BOUNDEDNESS AND COMPACTNESS OF OPERATORS BETWEEN ANALYTIC FUNCTION SPACES

BENOÎT F. SEHBA AND STEVO STEVIĆ

(*Communicated by I. Franjić*)

Abstract. Some relations between two classes of real functions, the class of positive lower type p, denoted by $\mathfrak{L}_p$, and the class of positive upper type q, denoted by $\mathfrak{U}^q$, which naturally appear in the definition of some Hardy-Orlicz and Bergman-Orlicz type spaces, are given. Some applications in characterizing the boundedness and compactness of an integral-type operator introduced by S. Stević, from Hardy-Orlicz and Bergman-Orlicz type spaces to a weighted-type space are given.

1. Introduction

We say that a function $\Phi \not\equiv 0$ is a *growth function*, if it is a continuous and non-decreasing function from the interval $[0,\infty)$ *onto* itself. Clearly, these conditions imply that $\Phi(0) = 0$. Such functions were used to extend classical Lebesgue spaces and their properties (see [13, 14, 15, 16, 17] and the references therein). They also appear in many other areas of mathematics, among others in the definition of Hardy-Orlicz and Bergman-Orlicz type spaces and related problems (see, e.g., [2, 3, 4, 5, 19, 20]).

It is said that function Φ is of *positive upper type* (resp. *negative upper type*) if there are $q > 0$ (resp. $q < 0$) and $C > 0$ such that

$$\Phi(st) \leqslant Ct^q\Phi(s), \tag{1}$$

for every $s > 0$ and $t \geqslant 1$.

By $\mathfrak{U}^q$ we denote the family of all growth functions Φ of *positive upper type* q, (for some $q \geqslant 1$), such that the function

$$F(t) := \frac{\Phi(t)}{t} \tag{2}$$

is nondecreasing on the interval $(0,\infty)$.

Mathematics subject classification (2010): 47B38, 47B33, 32A35, 32A36, 32A37.

Keywords and phrases: Classes of real functions, Hardy-Orlicz space, Bergman-Orlicz space, weighted space, boundedness, compactness, integral-type operator.

It is said that function Φ is of *positive lower type* (resp. *negative lower type*) if there are $p > 0$ (resp. $p < 0$) and $C > 0$ such that

$$\Phi(st) \leqslant Ct^p \Phi(s). \tag{3}$$

for every $s > 0$ and $0 < t \leqslant 1$.

By $\mathfrak{L}_p$ we denote the family of all growth functions Φ of *positive lower type* p, (for some $0 < p \leqslant 1$), such that function (2) is nonincreasing on $(0,\infty)$.

The notation $A \lesssim B$ means that A is less than or equal to a constant times B independently of involving variables. If $A \lesssim B$ and $B \lesssim A$, then we write $A \asymp B$.

Our aim here is to give some relations between classes of functions $\mathfrak{L}_p$ and $\mathfrak{U}^r$ for some values of p and r, and to apply obtained results in characterizing the boundedness and compactness of an integral-type operator introduced by S. Stević, from Hardy-Orlicz and Bergman-Orlicz type spaces to a weighted-type space. For more on functions in the above mentioned classes and results similar to some of those in Section 2 related to them, see [13, 14, 15, 16, 17].

2. Some results on classes of functions $\mathfrak{L}_p$ and $\mathfrak{U}^q$

Our first result deals with classes of functions $\mathfrak{L}_p$ and $\mathfrak{U}^q$ for the case $pq = 1$, when only the continuity is assumed.

PROPOSITION 1. *Assume that Φ is an increasing growth function and $p \in (0,1]$. Then the following assertion holds:*

$$\Phi \in \mathfrak{L}_p \text{ if and only if } \Phi^{-1} \in \mathfrak{U}^{1/p}.$$

Proof. Since Φ is an increasing growth function then there is Φ^{-1} which is also increasing and continuously maps $[0,\infty)$ onto itself, so that Φ^{-1} is also a growth function. Since for every $t \in (0,\infty)$ there is a unique $s \in (0,\infty)$ such that $t = \Phi^{-1}(s)$ and since Φ^{-1} is *onto*, the condition

$$\frac{\Phi(t_2)}{t_2} \leqslant \frac{\Phi(t_1)}{t_1},$$

for every $t_1, t_2 \in (0,\infty)$ such that $0 < t_1 < t_2$, is equivalent to

$$\frac{s_2}{\Phi^{-1}(s_2)} \leqslant \frac{s_1}{\Phi^{-1}(s_1)},$$

for every $s_1, s_2 \in (0,\infty)$ such that $0 < s_1 < s_2$ (here $t_i = \Phi^{-1}(s_i)$, $i = 1,2$), which is equivalent to

$$\frac{\Phi^{-1}(s_1)}{s_1} \leqslant \frac{\Phi^{-1}(s_2)}{s_2},$$

$0 < s_1 < s_2$, that is, $\frac{\Phi^{-1}(t)}{t}$ is nondecreasing.

Now we prove that Φ is of lower type p if and only if Φ^{-1} is of upper type $1/p$. Suppose that there is a $C \geqslant 1$ so that for every $s \leqslant 1$ and all $t > 0$, we have

$$\Phi(st) \leqslant Cs^p\Phi(t). \tag{4}$$

Let $x \geqslant 1$ and $y > 0$. Applying inequality (4) to

$$s = \frac{1}{(Cx)^{1/p}} \quad \text{and} \quad t = \Phi^{-1}(y)(Cx)^{1/p},$$

we obtain

$$y \leqslant \frac{1}{x}\Phi\left(\Phi^{-1}(y)(Cx)^{1/p}\right),$$

and consequently

$$\Phi^{-1}(xy) \leqslant C^{1/p}x^{1/p}\Phi^{-1}(y), \tag{5}$$

for every $x \geqslant 1$ and $y > 0$, that is, Φ^{-1} is of upper type $1/p$.

Now assume that $\Phi^{-1} \in \mathfrak{U}^{1/p}$. Then, for some $C > 0$, $p \in (0,1]$, and for every $x \geqslant 1$ and $y > 0$

$$\Phi^{-1}(xy) \leqslant Cx^{1/p}\Phi^{-1}(y). \tag{6}$$

Let $K = \max\{C,1\}$. Plugging

$$x = \frac{1}{s^p} \quad \text{and} \quad y = \Phi(t)s^p,$$

where $t > 0$, into (6) (note that $s \in (0,1]$), we obtain

$$t \leqslant \frac{C}{s}\Phi^{-1}\left(\Phi(t)s^p\right) \leqslant \frac{K}{s}\Phi^{-1}\left(\Phi(t)s^p\right), \tag{7}$$

for $s \in (0,1]$ and $t > 0$.

Since $\frac{\Phi^{-1}(u)}{u}$ is nondecreasing, from the first part of the proof we have that $\frac{\Phi(u)}{u}$ is nonincreasing. From this and since $K \geqslant 1$, we have that

$$\Phi(u) \leqslant K\Phi\left(\frac{u}{K}\right), \tag{8}$$

for every $u \in (0,\infty)$.

From (7) and (8) and some simple calculations, we obtain

$$\Phi(ts) \leqslant K\Phi\left(\frac{ts}{K}\right) \leqslant Ks^p\Phi(t), \tag{9}$$

which means that Φ is of lower type p, completing the proof of the proposition. □

REMARK 1. Note that from the proof of Proposition 1, we see that condition (4) implies directly (5) without using the monotonicity of function (2), unlike the part of the proof which shows that (6) implies (9). We also observe that Proposition 1 was stated in [20] but the proof there is not complete.

In what follows we will use the following condition:

$$c_1\frac{\Phi(t)}{t} \leqslant \Phi'(t) \leqslant c_2\frac{\Phi(t)}{t}, \tag{10}$$

for some positive constants c_1 and c_2 and every $t \in (0,\infty)$.

When Φ satisfies (10), we will sometimes write $c_1(\Phi)$ and $c_2(\Phi)$ to specify that the constants depend on the function. By using the functions $\Phi_a(t) = \Phi(at)$, $a \in (0,\infty)$, which satisfy the equalities

$$\frac{c_1}{a}\frac{\Phi_a(t)}{t} \leqslant \Phi_a'(t) \leqslant \frac{c_2}{a}\frac{\Phi_a(t)}{t},$$

we may always assume that $c_1 \leqslant 1 \leqslant c_2$. Note also that functions $\Phi_a(t)$ satisfy inequalities (1) and (3) if Φ satisfies them, and that the monotonicity of $\Phi_a(t)/t$ is also kept in the case when $\Phi(t)/t$ is monotone.

The next two results give relations between classes $\mathfrak{U}^q$ and $\mathfrak{L}_{1/q}$, under some additional assumptions connected to the inequalities in (10).

PROPOSITION 2. *Let $\Phi \in \mathfrak{U}^q \cap C^1[0,\infty)$, $q \geqslant 1$, and Φ satisfies inequalities (10). Then $\Phi^{-1} \in \mathfrak{L}_{1/q}$ and there are positive constants $\widehat{c}_1$ and $\widehat{c}_2$ such that for any $t \in (0,\infty)$,*

$$\widehat{c}_1\frac{\Phi^{-1}(t)}{t} \leqslant (\Phi^{-1})'(t) \leqslant \widehat{c}_2\frac{\Phi^{-1}(t)}{t}. \tag{11}$$

Proof. By Remark 1.2 in [19] we have that Φ is increasing so by Proposition 1, $\Phi^{-1} \in \mathfrak{L}_{1/q}$. Plugging $t = \Phi^{-1}(s)$, $s \in (0,\infty)$, in (10), and using the fact that Φ^{-1} is *onto*, we get

$$c_1\frac{s}{\Phi^{-1}(s)} \leqslant \Phi'(\Phi^{-1}(s)) \leqslant c_2\frac{s}{\Phi^{-1}(s)}, \tag{12}$$

for every $s \in (0,\infty)$.

From (12) and since

$$(\Phi^{-1})'(s) = \frac{1}{\Phi'(\Phi^{-1}(s))}$$

we obtain

$$\frac{1}{c_2}\frac{\Phi^{-1}(s)}{s} \leqslant (\Phi^{-1})'(s) \leqslant \frac{1}{c_1}\frac{\Phi^{-1}(s)}{s},$$

for every $s \in (0,\infty)$, from which (11) follows with $\widehat{c}_1 = 1/c_2$ and $\widehat{c}_2 = 1/c_1$. □

PROPOSITION 3. *Let $\Phi \in \mathfrak{L}_p \cap C^1[0,\infty)$, $p \in (0,1]$, and satisfies inequalities (10). Then $\Phi^{-1} \in \mathfrak{U}^{1/p}$ and*

$$\widehat{c}_1\frac{\Phi^{-1}(t)}{t} \leqslant (\Phi^{-1})'(t) \leqslant \widehat{c}_2\frac{\Phi^{-1}(t)}{t}, \tag{13}$$

for every $t \in (0,\infty)$ and $\widehat{c}_1$ and $\widehat{c}_2$ as in Proposition 2.

Proof. By Remark 1.3 in [19] we have that Φ is increasing so by Proposition 1, $\Phi^{-1} \in \mathfrak{U}^{1/p}$. The rest of the proof is similar to the proof of Proposition 2 so is omitted. □

Now note that if $\Phi \in \mathfrak{L}_p$, then by using the monotonicity of function (2) we have that

$$\int_0^t \frac{\Phi(s)}{s} ds \geqslant \int_0^t \frac{\Phi(t)}{t} ds = \Phi(t) \tag{14}$$

and

$$\int_0^t \frac{\Phi(s)}{s} ds = \int_0^1 \frac{\Phi(tu)}{u} du \leqslant C \int_0^1 \frac{\Phi(t) u^p}{u} du = \frac{C}{p} \Phi(t), \tag{15}$$

for every $t \in (0, \infty)$.

From (14) and (15) it follows that the functions $\Phi(t)$ and

$$\widehat{F}(t) := \int_0^t \frac{\Phi(s)}{s} ds$$

are comparable.

From this reason, since $\widehat{F}$ is differentiable and

$$\widehat{F}'(t) = \frac{\Phi(t)}{t} \tag{16}$$

for $t \in (0, \infty)$, and by using (14) and (15) we see that

$$\frac{p}{C} \frac{\widehat{F}(t)}{t} \leqslant \widehat{F}'(t) \leqslant \frac{\widehat{F}(t)}{t}, \tag{17}$$

that is, function $\widehat{F}$ satisfies inequalities (10) with $c_1 = p/C$ and $c_2 = 1$.

Moreover, from (16) and since $\Phi(t)/t$ is nonincreasing we have that $\widehat{F}$ is concave. Hence, in the definitions of Hardy-Orlicz and Bergman-Orlicz type spaces ([2, 19]) that use functions from the class $\mathfrak{L}_p$ we may also assume that they are concave C^1 ones and satisfy inequalities (10) for some positive constants c_1 and c_2.

A natural question is: may we assume for $\Phi \in \mathfrak{U}^q$ that they are convex C^1 functions satisfying inequalities (10) for some positive constants c_1 and c_2?

In [19] the authors considered the subclass of $\mathfrak{L}_s$ of the form $\Phi(t^p)$, with $\Phi \in \mathfrak{U}^q$, but the subclass was not considered in detail. Note that if $\Phi(t) = t^q$, then $\Phi(t^p) = t^{pq}$, and clearly it belongs to class $\mathfrak{L}_{pq}$, when $pq \leqslant 1$. Since the inverse of t^q is $t^{1/q}$, and since $(t^{1/q})' = q^{-1} t^{1/q}/t$, we see that t^q satisfies inequalities (10) for $c_1 = c_2 = 1/q$. Hence, the condition $pq \leqslant 1$, can be written in the form $p \leqslant c_1(t^{1/q})$. In the next result we prove that the condition is in a way optimal. Namely, for $\Phi \in \mathfrak{U}^q$, we give a range of powers p for which the function $\Phi(t^p)$ is in $\mathfrak{L}_s$ as well as a concrete value of s.

PROPOSITION 4. *Assume that $\Phi \in \mathfrak{U}^q$ is such that $\Phi^{-1} \in \mathfrak{L}_{1/q} \cap C^1[0,\infty)$ and satisfies (10) with constants $c_1(\Phi^{-1}) \leqslant 1 \leqslant c_2(\Phi^{-1})$. Then for any $p \leqslant c_1(\Phi^{-1})$, the following assertions hold.*

(a) The function $t \mapsto \frac{\Phi(t^p)}{t}$ *is nonincreasing on* $(0,\infty)$.

(b) The function $t \mapsto \Phi(t^p)$ *is a growth function of lower type* $s = p/q \leqslant 1$.

Proof. (a) For simplicity, we write c_1 in place of $c_1(\Phi^{-1})$. To prove that $\frac{\Phi(t^p)}{t}$ is nonincreasing, we only have to check that the function $\frac{(\Phi^{-1}(t))^{1/p}}{t}$ is nondecreasing on $(0,\infty)$. Set $m = 1/p$ and observe from our hypothesis that $mc_1 \geqslant 1$. It follows that

$$\begin{aligned}\left(\frac{(\Phi^{-1}(t))^m}{t}\right)' &= \frac{mt\left(\Phi^{-1}(t)\right)'\left(\Phi^{-1}(t)\right)^{m-1} - \left(\Phi^{-1}(t)\right)^m}{t^2}\\ &\geqslant \frac{(mc_1-1)\left(\Phi^{-1}(t)\right)^m}{t^2}\\ &\geqslant 0.\end{aligned}$$

(b) As $\Phi^{-1}(t)$ is of lower-type $1/q$, we have by definition that there is a constant $c > 0$ such that

$$\Phi^{-1}(at) \leqslant ca^{1/q}\Phi^{-1}(t),$$

for any $0 < a \leqslant 1$, and any $t > 0$.

Hence from the fact that Φ and $\frac{\Phi(t)}{t}$ are nondecreasing, and by using (1), we obtain

$$\begin{aligned} at &\leqslant \Phi\left(ca^{1/q}\Phi^{-1}(t)\right)\\ &\leqslant \tilde{c}a^{1/q}t\end{aligned}$$

where

$$\tilde{c} = \begin{cases} 1 & \text{if } c \leqslant 1\\ Cc^q & \text{if } c > 1\end{cases},$$

C being the constant in (1).

Using again that Φ and $\frac{\Phi(t)}{t}$ are nondecreasing, and (1), it follows that

$$\Phi(a^p t^p) \leqslant \Phi(\tilde{c}^p a^{p/q} t^p) \leqslant \tilde{\tilde{c}} a^{p/q}\Phi(t^p)$$

with

$$\tilde{\tilde{c}} = \begin{cases} 1 & \text{if } \tilde{c} \leqslant 1\\ C\tilde{c}^{pq} & \text{if } \tilde{c} > 1\end{cases},$$

which means that $\Phi(t^p)$ is of lower-type $\frac{p}{q} \leqslant 1$, as claimed. □

The next result provides an analog of Proposition 4 when we interchange the classes $\mathfrak{L}_p$ and $\mathfrak{U}^q$.

PROPOSITION 5. *Assume that* $\Psi \in \mathfrak{L}_s \cap C^1[0,\infty)$ *and satisfies* (10) *with constants* $c_1(\Psi) \leqslant 1 \leqslant c_2(\Psi)$. *Then for any* $p \geqslant 1/c_1(\Psi)$, *the following assertions hold.*

(a) The function $t \mapsto \frac{\Psi(t^p)}{t}$ *is nondecreasing on* $(0,\infty)$.

(b) The function $t \mapsto \Psi(t^p)$ *is a growth function of upper-type* $p/s \geqslant 1$.

Proof. (a) Note that our hypotheses imply by Proposition 3 that $\Psi^{-1} \in \mathfrak{U}^{1/s}$ and satisfies (10) with constants

$$c_1(\Psi^{-1}) = \frac{1}{c_2(\Psi)} \quad \text{and} \quad c_2(\Psi^{-1}) = \frac{1}{c_1(\Psi)}.$$

Now let us prove that the function $\frac{(\Psi^{-1}(t))^{1/p}}{t}$ is nonincreasing which is enough to conclude that *(a)* holds. Set $m = 1/p$ and observe that our hypotheses imply $mc_2(\Psi^{-1}) = \frac{1}{pc_1(\Psi)} \leqslant 1$. We have

$$\begin{aligned}\left(\frac{(\Psi^{-1}(t))^m}{t}\right)' &= \frac{mt\left(\Psi^{-1}(t)\right)'\left(\Psi^{-1}(t)\right)^{m-1} - \left(\Psi^{-1}(t)\right)^m}{t^2} \\ &\leqslant \frac{\left(mc_2(\Psi^{-1}) - 1\right)\left(\Psi^{-1}(t)\right)^m}{t^2} \\ &\leqslant 0,\end{aligned}$$

from which the statement follows.

(b) Note that as $\Psi \in \mathfrak{L}_s$, we have from Proposition 1 that $\Psi^{-1} \in \mathfrak{U}^{1/s}$. Hence there is a constant $C > 0$ such that for any $a \geqslant 1$, and any $t > 0$,

$$\Psi^{-1}(at) \leqslant Ca^{1/s}\Psi^{-1}(t).$$

Since Ψ is nondecreasing and $\frac{\Psi(t)}{t}$ is nonincreasing, it follows that

$$at \leqslant \Psi\left(Ca^{1/s}\Psi^{-1}(t)\right) \leqslant \tilde{C}a^{1/s}t,$$

where $\tilde{C} = \max\{1, C\}$.

Using once more again the assumption that Ψ is nondecreasing and $\frac{\Psi(t)}{t}$ is nonincreasing, we obtain from the latter inequality that

$$\Psi(a^p t^p) \leqslant \Psi(\tilde{C}^p a^{p/s} t^p) \leqslant \tilde{C}a^{p/s}\Psi(t^p),$$

that is, $t \mapsto \Psi(t^p)$ is of upper-type $\frac{p}{s} \geqslant 1$, as desired. □

We can now prove that functions constructed in Proposition 5 are examples of functions in $\mathfrak{U}^q$ that answer positively the question above.

THEOREM 1. *Assume that* $\Psi \in \mathfrak{L}_s \cap C^1[0,\infty)$ *and satisfies* (10) *with constants* $c_1(\Psi) \leqslant 1 \leqslant c_2(\Psi)$. *Then for any* $p \geqslant 1/c_1(\Psi)$, *the function* $t \mapsto \Psi(t^p)$ *is comparable to a convex* C^1 *function* G *satisfying the inequalities in* (10) *with some constants* $c_1(G)$ *and* $c_2(G)$ *depending on* G.

Proof. Let

$$G(t)=\int_0^t\frac{\Psi(s^p)}{s}ds.$$

Then using that the map $s\mapsto\frac{\Psi(s^p)}{s}$ is nondecreasing, we obtain

$$G(t)=\int_0^t\frac{\Psi(s^p)}{s}ds\leqslant\frac{\Psi(t^p)}{t}\cdot t=\Psi(t^p). \tag{18}$$

Since $s\mapsto\frac{\Psi(s)}{s}$ is nonincreasing, we have that

$$G(t)=\int_0^t\frac{\Psi(s^p)}{s}ds=\int_0^t\frac{\Psi(s^p)}{s^p}s^{p-1}ds\geqslant\frac{\Psi(t^p)}{t^p}\int_0^t s^{p-1}ds=\frac{1}{p}\Psi(t^p). \tag{19}$$

From (18) and (19), we obtain

$$\frac{1}{p}\Psi(t^p)\leqslant G(t)\leqslant\Psi(t^p),\quad \text{for any } t>0,$$

which proves that the function $t\mapsto\Psi(t^p)$ is comparable to the function G.

Next, observing that G is differentiable and

$$G'(t)=\frac{\Psi(t^p)}{t},\quad \text{for } t\in(0,\infty),$$

we conclude with the help of (18) and (19) that

$$\frac{G(t)}{t}\leqslant G'(t)\leqslant p\frac{G(t)}{t}\quad \text{for } t\in(0,\infty),$$

that is, G satisfies (10) with constants $c_1(G)=1$ and $c_2(G)=p>1$.

Finally, since $\frac{\Psi(t^p)}{t}$ is nondecreasing, we have that G is convex, completing the proof of the theorem. □

3. Applications

In this section we extend some of the results in [19] to Hardy-Orlicz and Bergman-Orlicz spaces on the unit ball $\mathbb{B}^n$ of the complex vector space $\mathbb{C}^n$ defined for growth functions of the form $\Psi(t^p)$ where $\Psi\in\mathfrak{L}_s$ and $p\in(1,\infty)$ is large enough. We start by recalling some definitions.

By $d\nu$ we denote the Lebesgue measure on $\mathbb{B}^n$, $d\sigma$ the normalized measure on $\mathbb{S}^n=\partial\mathbb{B}^n$ (the boundary of $\mathbb{B}^n$), $H(\mathbb{B}^n)$ the space of all holomorphic functions on $\mathbb{B}^n$, and $S(\mathbb{B}^n)$ the class of all holomorphic self-maps of $\mathbb{B}^n$. Let $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ be points in $\mathbb{C}^n$ and $\langle z,w\rangle=z_1\overline{w}_1+\dots+z_n\overline{w}_n$, the standard scalar product on $\mathbb{C}^n$. By $d\nu_\alpha$, $\alpha>-1$, we denote the normalized Lebesgue measure $d\nu_\alpha(z)=c_\alpha(1-|z|^2)^\alpha d\nu(z)$ (i.e. $\nu_\alpha(\mathbb{B}^n)=1$).

Let Φ be a growth function. The weighted Bergman-Orlicz space $A^{\Phi}_{\alpha}(\mathbb{B}^n) = A^{\Phi}_{\alpha}$ is the space of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{A^{\Phi}_{\alpha}} := \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_{\alpha}(z) < \infty.$$

On $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ is defined the following quasi-norm

$$\|f\|^{lux}_{A^{\Phi}_{\alpha}} := \inf\left\{\lambda > 0 : \int_{\mathbb{B}^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_{\alpha}(z) \leqslant 1\right\}. \tag{20}$$

As observed in [19], if $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_q$, then the quantity in (20) is finite for every $f \in A^{\Phi}_{\alpha}(\mathbb{B}^n)$. The standard weighted Bergman space $A^p_{\alpha}(\mathbb{B}^n) = A^p_{\alpha}$, $p > 0$, $\alpha > -1$, corresponds to $\Phi(t) = t^p$, and for $p \geqslant 1$ it is a Banach space, while for $0 < p < 1$ it is a translation-invariant complete metric space.

Let Φ be a growth function. By $H^{\Phi}(\mathbb{B}^n) = H^{\Phi}$ we denote the Hardy-Orlicz space consisting of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{H^{\Phi}} := \sup_{0<r<1} \int_{\mathbb{S}^n} \Phi(|f(r\xi)|) d\sigma(\xi) < \infty.$$

On $H^{\Phi}(\mathbb{B}^n)$ is defined the next quasi-norm

$$\|f\|^{lux}_{H^{\Phi}} := \sup_{0<r<1} \|f_r\|^{lux}_{L^{\Phi}},$$

where $f_r(\xi) = f(r\xi)$, $0 \leqslant r < 1$, $\xi \in \mathbb{S}^n$, and $\|g\|^{lux}_{L^{\Phi}}$ is the Luxembourg quasi-norm defined by

$$\|g\|^{lux}_{L^{\Phi}} := \inf\left\{\lambda > 0 : \int_{\mathbb{S}^n} \Phi\left(\frac{|g(\xi)|}{\lambda}\right) d\sigma(\xi) \leqslant 1\right\}.$$

The quasi-norm is finite for every $f \in H^{\Phi}(\mathbb{B}^n)$. For $\Phi(t) = t^p$, $0 < p < \infty$, it becomes the Hardy space $H^p(\mathbb{B}^n) = H^p$. For some results on Hardy-Orlicz and Bergman-Orlicz spaces and operators on them, see, e.g., [2, 3, 4, 5, 19, 20] and the related references therein.

Motivated by the fact that $H^p(\mathbb{B}^n)$ space is the limit case of $A^p_{\alpha}(\mathbb{B}^n)$ as $\alpha \to -1+0$, if not specified otherwise, we will be using the notation $A^{\Phi}_{\alpha}(\mathbb{B}^n)$ for all $-1 \leqslant \alpha < \infty$, where for $\alpha = -1$ the space corresponds to $H^{\Phi}(\mathbb{B}^n)$.

DEFINITION 1. ([19]) We say that a function $\omega : (0,1] \to [0,\infty)$ belongs to class Ω_1 if ω is nonincreasing, $\frac{1}{\omega}$ is of some positive lower type and the function $t\omega(t)$ is increasing.

An $f \in H(\mathbb{B}^n)$ is said to be in $H^{\infty}_{\omega}(\mathbb{B}^n) = H^{\infty}_{\omega}$ if

$$\|f\|_{H^{\infty}_{\omega}} := \sup_{z\in\mathbb{B}^n} \frac{|f(z)|}{\omega(1-|z|)} < \infty. \tag{21}$$

It is easy to see that $H^{\infty}_{\omega}(\mathbb{B}^n)$ is a Banach space.

Let $f \in H(\mathbb{B}^n)$. The radial derivative $\mathscr{R}f$ of f is given by

$$\mathscr{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

An $f \in H(\mathbb{B}^n)$ belongs to $\Lambda_\omega(\mathbb{B}^n) = \Lambda_\omega$ if $\mathscr{R}f \in H^\infty_{\frac{\omega}{t}}(\mathbb{B}^n)$ (here ω/t is an abbreviation for the function $\omega(t)/t$), that is

$$b_{\Lambda_\omega}(f) := \sup_{z\in\mathbb{B}^n} \frac{(1-|z|)|\mathscr{R}f(z)|}{\omega(1-|z|)} < \infty.$$

It is also easy to see that Λ_ω is a Banach space under the following norm

$$\|f\|_{\Lambda_\omega} := |f(0)| + \sup_{z\in\mathbb{B}^n} \frac{(1-|z|)|\mathscr{R}f(z)|}{\omega(1-|z|)}.$$

The following result was proved in [19].

LEMMA 1. *Suppose that* $\omega \in \Omega_1$. *Then* $H^\infty_\omega(\mathbb{B}^n) = \Lambda_\omega$ *with equivalent norms.*

For $\varphi \in S(\mathbb{B}^n)$, and $g \in H(\mathbb{B}^n)$ with $g(0) = 0$, we consider the following integral-type operator defined on $H(\mathbb{B}^n)$ by

$$P^g_\varphi f(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}. \tag{22}$$

Operator P^g_φ was introduced by S. Stević in [21] and later studied between various spaces of holomorphic functions, for example, in [7, 19, 22, 23, 24, 25, 27, 28, 29, 36]. For $\varphi(z) = z$ and $g = \mathscr{R}h$ where $h \in H(\mathbb{B}^n)$, it is reduced to operator $T_h := P^{\mathscr{R}h}_z$, so-called, extended Cesàro operator, introduced in [6] and later studied, for example, in [1, 8, 10, 11, 12] (see also the references therein). Operator theoretic properties of related operators between various spaces of holomorphic functions on several domains have been recently considerably studied (see, for example, [9, 26, 30, 31, 32, 33, 34, 35] and the related references therein).

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively. It is said that a linear operator $T : X \to Y$ is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Tf, 0) \leqslant K d_X(f, 0) \quad \text{for all} \quad f \in X.$$

When X and Y are Banach spaces, the metrical boundedness coincides with the boundedness of operators between Banach spaces.

Operator $T : X \to Y$ is said to be *metrically compact* if it takes every metric ball in X into a relatively compact set in Y.

Here we will study the metrical boundedness and compactness of operator (22) from the Hardy-Orlicz space $H^{\Psi_p}(\mathbb{B}^n)$ and the weighted Bergman-Orlicz space $A^{\Psi_p}_\alpha(\mathbb{B}^n)$

to the weighted-type space $H^\infty_\omega(\mathbb{B}^n)$, when $\Psi \in \mathfrak{L}_s$, $p \in (1,\infty)$ is large enough, and $\omega \in \Omega_1$. We use the following notation

$$\Psi_p(t) := \Psi(t^p). \tag{23}$$

Note that due to Theorem 1, if $\Psi \in \mathfrak{L}_s$ we may also assume that the function given by (23) is convex, belongs to the class $C^1(0,\infty)$, and satisfy (10). Then, $\Psi(t^p)/t = \Psi_p(t)/t$ is nondecreasing, and $A^{\Psi_p}_\alpha(\mathbb{B}^n)$ $(-1 < \alpha < \infty)$ embeds continuously into $A^1_\alpha(\mathbb{B}^n)$. We also have that $H^{\Psi_p}(\mathbb{B}^n)$ embeds continuously into $H^1(\mathbb{B}^n)$ and consequently that any $f \in H^{\Psi_p}(\mathbb{B}^n)$ admits radial limits $f^*(\xi) = \lim_{r\to 1} f(r\xi)$ for almost every $\xi \in \mathbb{S}^n$. As a consequence, for $-1 \leqslant \alpha < \infty$, if $\|f\|^{lux}_{A^{\Psi_p}_\alpha(\mathbb{B}^n)} = 0$, then $f \equiv 0$ a.e on $\mathbb{S}^n$ (see also [19] for more comments).

Now we give several auxiliary results which are incorporated into the lemmas which follow.

Using that $\Psi_p(t)$ is convex, we obtain as in [4, Proposition 1.9] the following lemma.

LEMMA 2. *Let $\Psi \in \mathfrak{L}_s$, $p \in (1,\infty)$ is large enough, and $-1 < \alpha < \infty$. Then there is a constant $C > 0$ such that for every $f \in A^{\Psi_p}_\alpha(\mathbb{B}^n)$ and $a \in \mathbb{B}^n$*

$$|f(a)| \leqslant C\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)\|f\|^{lux}_{A^{\Psi_p}_\alpha}. \tag{24}$$

The next lemma is obtained similarly to [3, Proposition 1.6].

LEMMA 3. *Let $\Psi \in \mathfrak{L}_s$, and $p \in (1,\infty)$ large enough. Then there is a constant $C > 0$ such that for every $f \in H^{\Psi_p}(\mathbb{B}^n)$ and $a \in \mathbb{B}^n$*

$$|f(a)| \leqslant C\Psi_p^{-1}\left(\frac{4}{(1-|a|^2)^n}\right)\|f\|^{lux}_{H^{\Psi_p}}. \tag{25}$$

The next lemma provides a useful class of test functions in $A^{\Psi_p}_\alpha(\mathbb{B}^n)$.

LEMMA 4. *Let $\Psi \in \mathfrak{L}_s$, and $p \in (1,\infty)$ large enough, $-1 < \alpha < \infty$. Then the following function is in $A^{\Psi_p}_\alpha(\mathbb{B}^n)$*

$$f_a(z) = \Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)\left(\frac{1-|a|^2}{1-\langle z,a\rangle}\right)^{2(n+1+\alpha)}. \tag{26}$$

Moreover

$$\sup_{a\in\mathbb{B}^n}\|f_a\|^{lux}_{A^{\Psi_p}_\alpha} \lesssim 1.$$

Proof. Let

$$g_a(z)=\left(\frac{1-|a|^2}{1-\langle z,a\rangle}\right)^{2(n+1+\alpha)}.$$

Then, we have

$$\begin{aligned}\int_{\mathbb{B}^n}\Psi_p(|f_a(z)|)d\nu_\alpha(z)&\leqslant\int_{\mathbb{B}^n}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)|g_a(z)|\right)d\nu_\alpha(z)\\&=I+J\end{aligned}$$

with

$$I=\int_{\{z\in\mathbb{B}^n:|g_a(z)|\leqslant 1\}}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)|g_a(z)|\right)d\nu_\alpha(z)$$

and

$$J=\int_{\{z\in\mathbb{B}^n:|g_a(z)|>1\}}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)|g_a(z)|\right)d\nu_\alpha(z).$$

Since the function $\Psi_p(t)/t$ is nondecreasing on $[0,\infty)$, we have

$$\frac{\Psi_p(t|g_a(z)|)}{t|g_a(z)|}\leqslant\frac{\Psi_p(t)}{t},\quad\text{when}\quad|g_a(z)|\leqslant 1,$$

which along with [18, Proposition 1.4.10] gives

$$\begin{aligned}I&=\int_{\{z\in\mathbb{B}^n:|g_a(z)|\leqslant 1\}}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)|g_a(z)|\right)d\nu_\alpha(z)\\&\leqslant\int_{\{z\in\mathbb{B}^n:|g_a(z)|\leqslant 1\}}|g_a(z)|\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)\right)d\nu_\alpha(z)\\&\leqslant 4^{n+1+\alpha}\int_{\mathbb{B}^n}\frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}}d\nu_\alpha(z)\\&\lesssim 1.\end{aligned}$$

Using that Ψ_p is of upper-type $p/s\geqslant 1$, inequality (1), and [18, Proposition 1.4.10], we have that

$$\begin{aligned}J&=\int_{\{z\in\mathbb{B}^n:|g_a(z)|\geqslant 1\}}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)|g_a(z)|\right)d\nu_\alpha(z)\\&\lesssim\int_{\{z\in\mathbb{B}^n:|g_a(z)|\geqslant 1\}}|g_a(z)|^{p/s}\Psi_p\left(\Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\right)\right)d\nu_\alpha(z)\\&=\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha}\int_{\mathbb{B}^n}\frac{(1-|a|^2)^{2\frac{p}{s}(n+1+\alpha)}}{|1-\langle z,a\rangle|^{2\frac{p}{s}(n+1+\alpha)}}d\nu_\alpha(z)\\&\lesssim 1.\end{aligned}$$

From this and the inequality $\|f\|^{lux}_{A_\alpha^{\Psi_p}} \leqslant \|f\|_{A_\alpha^{\Psi_p}}$ (see [19]) the lemma follows. □

In a similar way, is obtained the following auxiliary result.

LEMMA 5. *Suppose that* $\Psi \in \mathfrak{L}_s$ *and* $p \in (1,\infty)$ *is large enough. Then for each* $a \in \mathbb{B}^n$ *the function*

$$g_a(z) = \Psi_p^{-1}\left(\left(\frac{4}{1-|a|^2}\right)^n\right)\left(\frac{1-|a|}{1-\langle z,a\rangle}\right)^{2n}, \tag{27}$$

belongs to H^{Ψ_p}. *Moreover*

$$\sup_{a\in\mathbb{B}^n} \|g_a\|^{lux}_{H^{\Psi_p}} \lesssim 1.$$

Now we formulate the main consequence of our results in Section 2.

THEOREM 2. *Let* $\varphi \in S(\mathbb{B}^n)$, $g \in H(\mathbb{B}^n)$, $g(0)=0$, $\omega \in \Omega_1, -1 \leqslant \alpha < \infty$, $\Psi \in \mathfrak{L}_s$, *and let* $p \in (1,\infty)$ *be large enough. Then*

(a) $P_\varphi^g : A_\alpha^{\Psi_p}(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ *is metrically bounded if and only if*

$$\sup_{z\in\mathbb{B}^n} \frac{(1-|z|)|g(z)|}{\omega(1-|z|)} \Psi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{n+1+\alpha}\right) < \infty.$$

(b) $P_\varphi^g : A_\alpha^{\Psi_p}(\mathbb{B}^n) \to H_\omega^\infty(\mathbb{B}^n)$ *is metrically compact if and only if* $g \in H^\infty_{\frac{\omega}{t}}$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|)|g(z)|}{\omega(1-|z|)} \Psi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{n+1+\alpha}\right) = 0.$$

The proof of Theorem 2 follows the lines of the proofs of Theorems 3.3, 3.6 and 3.7 in [19], and uses Lemmas 2–5 so we leave it to the interested reader.

REFERENCES

[1] K. AVETISYAN AND S. STEVIĆ, *Extended Cesàro operators between different Hardy spaces*, Appl. Math. Comput. **207** (2009), 346–350.

[2] A. BONAMI AND B. F. SEHBA, *Hankel operators between Hardy-Orlicz spaces and products of holomorphic functions*, Rev. Un. Mat. Argentina **50** (2) (2009), 187–199.

[3] S. CHARPENTIER, *Composition operators on Hardy-Orlicz spaces on the ball*, Integral Equations Operator Theory **70** (2011), no. 3, 429–450.

[4] S. CHARPENTIER, *Composition operators on weighted Bergman-Orlicz spaces on the ball*, Complex Anal. Oper. Theory **7** (1) (2013), 43–68.

[5] S. CHARPENTIER AND B. F. SEHBA, *Carleson measure theorems for large Hardy-Orlicz and Bergman-Orlicz spaces*, J. Funct. Spaces Appl. Vol. 2012, Article ID 792763, (2012), 21 pages.

[6] Z. HU, *Extended Cesàro operators on the Bloch space in the unit ball of* $\mathbb{C}^n$, Acta Math. Sci. Ser. B Engl. Ed. **23** (4) (2003), 561–566.

[7] S. KRANTZ AND S. STEVIĆ, *On the iterated logarithmic Bloch space on the unit ball*, Nonlinear Anal. TMA **71** (2009), 1772–1795.
[8] S. LI AND S. STEVIĆ, *Integral type operators from mixed-norm spaces to α-Bloch spaces*, Integral Transform. Spec. Funct. **18** (7) (2007), 485–493.
[9] S. LI AND S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338** (2008), 1282–1295.
[10] S. LI AND S. STEVIĆ, *Riemann-Stieltjes operators between different weighted Bergman spaces*, Bull. Belg. Math. Soc. Simon Stevin **15** (4) (2008), 677–686.
[11] S. LI AND S. STEVIĆ, *Cesàro type operators on some spaces of analytic functions on the unit ball*, Appl. Math. Comput. **208** (2009), 378–388.
[12] S. LI AND S. STEVIĆ, *Integral-type operators from Bloch-type spaces to Zygmund-type spaces*, Appl. Math. Comput. **215** (2009), 464–473.
[13] L. MALIGRANDA, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.) **234** (1985), 49 pp.
[14] L. MALIGRANDA, *Orlicz Spaces and Interpolation*, Sem. Math. **5**, Dep. Mat., Universidade Estadual de Campinas, Campinas SP, Brazil, 1989.
[15] W. MATUSZEWSKA AND W. ORLICZ, *On certain properties of Φ-functions*, Bull. Acad. Polon. Sci. **8** (1960), 439–443.
[16] W. MATUSZEWSKA AND W. ORLICZ, *On some classes of functions with regard to their orders of growth*, Studia Math. **26** (1965), 11–24.
[17] W. ORLICZ, *On spaces of φ-integrable functions*, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, Jerusalem Academic Press, Jerusalem 1961, 357–365.
[18] W. RUDIN, *Function theory in the unit ball of $\mathbb{C}^n$*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York-Berlin (1980).
[19] B. SEHBA AND S. STEVIĆ, *On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces*, Appl. Math. Comput. **233** (2014), 565–581.
[20] B. SEHBA AND E. TCHOUNDJA, *Hankel operators on holomorphic Hardy-Orlicz spaces*, Integr. Equ. Oper. Theory, Vol. **73**, No. 3 (2012), 331–349.
[21] S. STEVIĆ, *On a new operator from H^∞ to the Bloch-type space on the unit ball*, Util. Math. **77** (2008), 257–263.
[22] S. STEVIĆ, *On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball*, Discrete Dyn. Nat. Soc. Vol. 2008, Article ID 154263, (2008), 14 pages.
[23] S. STEVIĆ, *On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball*, Appl. Math. Comput. **206** (2008), 313–320.
[24] S. STEVIĆ, *On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball*, J. Math. Anal. Appl. **354** (2009), 426–434.
[25] S. STEVIĆ, *On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces*, Nonlinear Anal. TMA **71** (2009), 6323–6342.
[26] S. STEVIĆ, *On an integral operator between Bloch-type spaces on the unit ball*, Bull. Sci. Math. **134** (2010), 329–339.
[27] S. STEVIĆ, *On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on unit ball*, Appl. Math. Comput. **215** (2010), 4248–4255.
[28] S. STEVIĆ, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstr. Appl. Anal. Vol. 2010, Article ID 198608, (2010), 7 pages.
[29] S. STEVIĆ, *On some integral-type operators between a general space and Bloch-type spaces*, Appl. Math. Comput. **218** (2011), 2600–2618.
[30] S. STEVIĆ AND S. I. UEKI, *On an integral-type operator acting between Bloch-type spaces on the unit ball*, Abstr. Appl. Anal. Vol. 2010, Article ID 214762, (2010), 14 pages.
[31] S. UEKI, *On the Li-Stević integral type operators from weighted Bergman Spaces into β-Zygmund spaces*, Integral Equations Operator Theory **74** (1) (2012), 137–150.
[32] W. YANG, Y. LUO AND X. ZHU, *Differences of generalized composition operators between Bloch type spaces*, Math. Inequal. Appl. **17** (3) (2014), 977–987.
[33] W. YANG AND X. MENG, *Generalized composition operators from $F(p,q,s)$ spaces to Bloch-type spaces*, Appl. Math. Comput. **217** (6) (2010), 2513–2519.
[34] Y. YU AND Y. LIU, *On Li-Stević integral-type operators between different weighted Bloch-type spaces*, J. Inequal. Appl. Vol. 2008, Article ID 780845, (2008), 14 pages.

[35] X. ZHU, *Integral-type operators from iterated logarithmic Bloch spaces to Zygmund-type spaces*, Appl. Math. Comput. **215** (3) (2009), 1170–1175.

[36] X. ZHU, *Volterra composition operators from weighted-type spaces to Bloch-type spaces and mixed norm spaces*, Math. Inequal. Appl. **14** (1) (2011), 223–233.

(Received December 8, 2014)

Benoît F. Sehba
Department of Mathematics, University of Ghana
P. O. Box LG 62, Legon, Accra Ghana
e-mail: bfsehba@ug.edu.gh

Stevo Stević
Mathematical Institute of the Serbian Academy of Sciences
Knez Mihailova 36/III, 11000 Beograd, Serbia
and
Operator Theory and Applications Research Group
Department of Mathematics, King Abdulaziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: sstevic@ptt.rs