Mathematical
Inequalities
& Applications
Volume 19, Number 1 (2016), 117–125 doi:10.7153/mia-19-09

ON BOUNDS FOR THE SMALLEST AND THE LARGEST EIGENVALUES OF GCD AND LCM MATRICES

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(*Communicated by G. P. H. Styan*)

Abstract. In this paper, improving a famous result of Wolkowicz and Styan for the GCD matrix (S_n) and the LCM matrix $[S_n]$ defined on $S_n = \{1,2,\dots,n\}$, we present new upper and lower bounds for the smallest and the largest eigenvalues of (S_n) and $[S_n]$ in terms of particular arithmetical functions.

1. Introduction and preliminaries

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. Let (x_i, x_j) and $[x_i, x_j]$ denote the greatest common divisor and the least common multiply of x_i and x_j, respectively. The $n\times n$ matrices $(S) = ((x_i,x_j))$ and $[S] = ([x_i,x_j])$ are respectively called the GCD matrix and the LCM matrix on S. Initially, Smith [32] proved that if S is factor closed then $\det(S) = \prod_{k=1}^{n}\varphi(x_k)$, where φ is Euler's totient. Since Smith's paper many generalizations of Smith's result have been published in the literature. For general accounts see e.g. [2, 10, 20, 30].

The eigenstructure is a rich topic in the study of GCD and LCM matrices. Let $\lambda_n^{(1)} \leqslant \lambda_n^{(2)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ be the eigenvalues of the $n\times n$ matrix $(M_n^{\varepsilon}) = \left(\frac{(i,j)^{2\varepsilon}}{i^{\varepsilon} j^{\varepsilon}}\right)$, where ε is a real number. In 1944, Wintner [34] proved that $\limsup_{n\to\infty}\lambda_n^{(n)} < \infty$ if and only if $\varepsilon > 1$. In 1998, Lindqvist and Seip [21] showed that if $\varepsilon > 1$ then $\frac{\zeta(2\varepsilon)}{\zeta(\varepsilon)^2} \leqslant \lambda_n^{(1)} \leqslant \lambda_n^{(n)} \leqslant \frac{\zeta(\varepsilon)^2}{\zeta(2\varepsilon)}$, and if $\frac{1}{2} < \varepsilon \leqslant 1$ then $\liminf_{n\to\infty}\lambda_n^{(1)} = 0$ and $\limsup_{n\to\infty}\lambda_n^{(n)} = \infty$. In 1989, Beslin and Ligh [5] proved that (S) is positive definite for any set S of distinct positive integers but $[S]$ is not positive definite. Therefore, all eigenvalues of (S) are positive reals but all eigenvalues of $[S]$ need not be positive (see [8]). In 2004, Hong and Loewy [13] investigated the asymptotic behavior of the eigenvalues of $n\times n$ matrix $((x_i,x_j)^{\varepsilon})$ on $S = \{x_1,x_2,\dots,x_n\}$, where ε is a real number. Since the paper of Hong and Loewy many results on the asymptotic behavior of the eigenvalues of the GCD and related matrices have been published in the literature, see [3, 4, 11, 12, 14]. In 2008 Ilmonen, Haukkanen and Merikoski [17] examined the eigenvalues of certain abstract generalizations of the GCD matrix and the LCM matrix on posets. Then Mattila and

Mathematics subject classification (2010): 05C88, 05C89.
Keywords and phrases: gcd matrix, lcm matrix, Euler's phi-function, eigenvalue inequalities, trace, Cauchy's interlacing theorem, Rayleigh-Ritz theorem.

Haukkanen [25] and Mattila [22] gave new bounds for the eigenvalues of such abstract generalizations. In addition to above papers, in 2012 Mattila and Haukkanen [23, 24] investigated the eigenvalues of the $n\times n$ matrix $A_n^{\alpha,\beta}=((i,j)^{\alpha}[i,j]^{\beta})$, where $\alpha,\beta\in\mathbb{R}$. They proved that $\lambda_n^{(1)}\geqslant t_n\cdot\min_{1\leqslant i\leqslant n}J_{\alpha-\beta}(i)\cdot\min\{1,n^{2\beta}\}$, where $\lambda_n^{(1)}$ is the smallest eigenvalue of $A_n^{\alpha,\beta}$ and $\alpha>\beta$, and obtained a real interval which provides a broad bounds for the eigenvalues of the matrix $A_n^{\alpha,\beta}$. Here t_n is the smallest eigenvalue of the $n\times n$ matrix E^TE, where E is the $n\times n$ matrix whose the $ij-$entry is 1 if $j|i$ and 0 otherwise.

Except $A_n^{\alpha,\beta}$, all above matrices of which eigenvalues were investigated are positive definite. Since the LCM matrix $[S]$ defined on any set $S=\{x_1,x_2,\dots,x_n\}$ is not positive definite for $n\geqslant 2$, to investigate the eigenvalues of the LCM matrix $[S]$ is not easy. Except Mattila and Haukkanen [23, 24], the eigenvalues of the LCM matrix have not hitherto been studied and any upper (lower) bounds for the smallest (largest) eigenvalues of the GCD matrix and the LCM matrix have not been presented earlier in the literature.

In a wider point of view, it is a hard task to calculate the eigenvalues of an $n\times n$ matrix so it is very useful to know the approximate location of the eigenvalues in general. Wolkowicz and Styan, in their excellent paper [35], obtained bounds for eigenvalues of $n\times n$ matrices by using traces. Their result has not only theoretical value but also many computational applications. Since their paper many generalizations and improvements of their result have been published in the literature. For general accounts see e.g. [16, 26, 27, 28, 29, 33, 36]. Also, their result has many applications, for example in graph theory and signal processing, see e.g. [1, 19]. We now state their result.

THEOREM 1. (Theorem 2.1 in [35]) *Let A be an $n\times n$ complex matrix with real eigenvalues $\lambda_n^{(1)},\lambda_n^{(2)},\dots,\lambda_n^{(n)}$ in non-decreasing order, and let $m=trA/n$, $s^2=tr(A^2)/n-m^2$. Then*

$$m-s(n-1)^{1/2}\leqslant\lambda_n^{(1)}\leqslant m-s/(n-1)^{1/2}, \tag{1.1}$$

$$m+s/(n-1)^{1/2}\leqslant\lambda_n^{(n)}\leqslant m+s(n-1)^{1/2}. \tag{1.2}$$

Equality holds on the left (right) of (1.1) if and only if equality holds on the left (right) of (1.2) if and only if the $n-1$ largest (smallest) eigenvalues are equal.

The main aim of this paper is to improve the lower and upper bounds for the smallest and the largest eigenvalues of $n\times n$ matrix in Theorem 1 for the GCD matrix (S_n) and the LCM matrix $[S_n]$ defined on $S_n=\{1,2,\dots,n\}$. We perform this improvement by using Rayleigh-Ritz Theorem and Cauchy's Interlacing Theorem and hence we obtain new lower and upper bounds for the smallest and largest eigenvalues of (S_n) and $[S_n]$ in terms of particular arithmetical functions.

2. Main results

Consider the GCD matrix (S_n) and the LCM matrix $[S_n]$ defined on $S_n=\{1,2,\dots,n\}$. In Theorem 2 we will improve the upper bound in (2.1) and the lower bound in (2.2) for the smallest and largest eigenvalues of (S_n), respectively. Moreover, we will obtain a lower bound for the smallest eigenvalue of (S_n) and an upper bound for the largest eigenvalue of (S_n) by direct calculations of the left side of (2.1) and the right side of (2.2), respectively. Then in Theorem 3 we will perform the similar improvement and calculations for the LCM matrix $[S_n]$.

THEOREM 2. *Let* $n>3$ *and* $\lambda_n^{(1)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ *be the eigenvalues of the GCD matrix* (S_n) *defined on* $S_n=\{1,2,\dots n\}$*. Then we have*

$$\frac{n(n+1)}{2} - s(n-1)^{1/2} < \lambda_n^{(1)} < \frac{n(n+1)}{2} - \left(\frac{ns^2+2(n-1)}{n^2-n}\right)^{1/2} \tag{2.1}$$

and

$$\frac{n(n+1)}{2} + \left(\frac{ns^2+2(n-1)}{n^2-n}\right)^{1/2} < \lambda_n^{(n)} < \frac{n(n+1)}{2} + s(n-1)^{1/2}, \tag{2.2}$$

where

$$s = \left(\frac{2}{n}\sum_{i=1}^{n}(N^2 * \varphi)(i) - \frac{7n^2+12n+5}{12}\right)^{1/2}. \tag{2.3}$$

Proof. First we calculate m and s defined in Theorem 1 for the matrix (S_n) in terms of the power function N^2 and the Euler totient φ, see [30, 31]. It is clear that $m=n(n+1)/2$. Then, we have

$$\begin{aligned} s^2 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(i,j)^2 - \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{1}{n}\left[2\sum_{i=1}^{n}\sum_{j=1}^{i}(i,j)^2 - \sum_{i=1}^{n} i^2\right] - \frac{(n+1)^2}{4} \\ &= \frac{2}{n}\sum_{i=1}^{n}(N^2 * \varphi)(i) - \frac{7n^2+12n+5}{12}. \end{aligned}$$

Here $N^2 * \varphi$ is the Dirichlet convolution of the power function N^2 and the Möbius funciton μ, see [30, 31] and the last equality follows from Lemma 1 in [7] which was proved by Cesáro [9] for the first time. From Theorem 1 we obtain the left-hand side of

(2.1) and the right-hand side of (2.2). On the other hand, we have

$$
\begin{aligned}
n^2(m-\lambda_n^{(1)})^2 &= n^2\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_n^{(i)}-\lambda_n^{(1)}\right)^2 \\
&= \left(\sum_{i=1}^{n}\lambda_n^{(i)}-n\lambda_n^{(1)}\right)^2 \\
&= \sum_{i=1}^{n}(\lambda_n^{(i)}-\lambda_n^{(1)})^2+\sum_{j\neq k}(\lambda_n^{(j)}-\lambda_n^{(1)})(\lambda_n^{(k)}-\lambda_n^{(1)}).
\end{aligned}
$$

Here it should be noted that in the proof of Lemma 2.2 in [35] the second sum is underestimated by 0 but here we shall find a better lower bound for it and thus we are able to improve the upper bound for $\lambda_n^{(1)}$ and the lower bound for $\lambda_n^{(n)}$ of (S_n) in Theorem 1. From Rayleigh-Ritz Theorem (see Theorem 4.2.2 in [15]) it is clear that $\lambda_{min}(A) \leqslant a_{ii} \leqslant \lambda_{max}(A)$ for an $n\times n$ Hermitian matrix $A=(a_{ij})$. Thus we have $\lambda_n^{(n)}-\lambda_n^{(1)} > n-1$ for all $n \geqslant 2$. From Cauchy's Interlacing Theorem (see Theorem 4.3.8 in [15]) we have that $\lambda_n^{(n-1)}-\lambda_n^{(1)} > \lambda_3^{(2)}-\lambda_3^{(1)}$ for all $n>3$. By a simple calculation in Maple $\lambda_3^{(2)} \cong 1{,}460$ and $\lambda_3^{(1)} \cong 0{,}324$. Thus we have

$$
\sum_{j\neq k}(\lambda_n^{(j)}-\lambda_n^{(1)})(\lambda_n^{(k)}-\lambda_n^{(1)}) > 2(n-1). \tag{2.4}
$$

In addition, we have

$$
\begin{aligned}
\sum_{i=1}^{n}(\lambda_n^{(i)}-\lambda_n^{(1)})^2 &= \sum_{i=1}^{n}(\lambda_n^{(i)}-m+m-\lambda_n^{(1)})^2 \\
&= \sum_{i=1}^{n}[(\lambda_n^{(i)}-m)(\lambda_n^{(i)}+m-2\lambda_n^{(1)})]+n(m-\lambda_n^{(1)})^2 \\
&= \sum_{i=1}^{n}[(\lambda_n^{(i)})^2-2\lambda_n^{(i)}\lambda_n^{(1)}-m^2+2m\lambda_n^{(1)})]+n(m-\lambda_n^{(1)})^2 \\
&= \sum_{i=1}^{n}(\lambda_n^{(i)})^2-nm^2+n(m-\lambda_n^{(1)})^2 \\
&= ns^2+n(m-\lambda_n^{(1)})^2.
\end{aligned}
$$

Thus, it is clear that

$$
n^2(m-\lambda_n^{(1)})^2 > ns^2+n(m-\lambda_n^{(1)})^2+2(n-1).
$$

Finally we have

$$
\lambda_n^{(1)} < \frac{n(n+1)}{2}-\left(\frac{ns^2+2(n-1)}{n^2-n}\right)^{1/2}.
$$

Similarly we expand $n^2(\lambda_n^{(n)} - m)^2$ to obtain the left-hand inequality in (2.2).

$$n^2(\lambda_n^{(n)} - m)^2 = n(\lambda_n^{(n)} - m)^2 + ns^2 + \sum_{j \neq k} (\lambda_n^{(n)} - \lambda_n^{(j)})(\lambda_n^{(n)} - \lambda_n^{(k)}).$$

By the inequality (2.4) we have

$$n^2(\lambda_n^{(n)} - m)^2 > n(\lambda_n^{(n)} - m)^2 + ns^2 + 2(n-1).$$

This completes the proof. □

THEOREM 3. *Let $n > 4$ and $\mu_n^{(1)} \leqslant \cdots \leqslant \mu_n^{(n)}$ be the eigenvalues of the LCM matrix $[S_n]$ on $S_n = \{1, 2, \ldots n\}$. Then, we have*

$$\frac{n(n+1)}{2} - s(n-1)^{1/2} < \mu_n^{(1)} < \frac{n(n+1)}{2} - \left(\frac{s^2 + 32(n-1)}{n-1}\right)^{1/2} \tag{2.5}$$

and

$$\frac{n(n+1)}{2} + \left(\frac{s^2 + 32(n-1)}{n-1}\right)^{1/2} < \mu_n^{(n)} < \frac{n(n+1)}{2} + s(n-1)^{1/2}, \tag{2.6}$$

where

$$s = \left[\frac{1}{3n} \sum_{i=1}^{n} i^2 \Big(N\big(N\mu * (N+1)(2N+1)\big) * \zeta\Big)(i) - \frac{7n^2 + 12n + 5}{12}\right]^{1/2}. \tag{2.7}$$

Proof. First we calculate m and s defined in Theorem 1 for the matrix $[S_n]$ in terms of arithmetical functions N, μ and ζ by a similar method as in [6]. Again $m = n(n+1)/2$. Then, we have

$$\begin{aligned} s^2 &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} [i,j]^2 - \frac{1}{n^2} \left(\sum_{i=1}^{n} i\right)^2 \\ &= \frac{1}{n} \left[2 \sum_{i=1}^{n} \sum_{j=1}^{i} [i,j]^2 - \sum_{i=1}^{n} i^2\right] - \frac{(n+1)^2}{4} \\ &= \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} [i,j]^2 - \frac{7n^2 + 12n + 5}{12}. \end{aligned}$$

Now we calculate the sum of squares of lcms.

$$\begin{aligned} \sum_{j=1}^{i} [i,j]^2 &= \sum_{j=1}^{i} i^2 \frac{j^2}{(i,j)^2} \\ &= i^2 \sum_{d \mid i} \frac{1}{d^2} \sum_{\substack{j=1 \\ (j,i)=d}}^{i} j^2 \\ &= i^2 \sum_{d \mid i} \sum_{\substack{k \leqslant i/d \\ (k,i/d)=1}} k^2. \end{aligned}$$

Here we have for every positive integer t,

$$\begin{aligned}\sum_{\substack{k\leqslant t\\(k,t)=1}} k^2 &= \sum_{k\leqslant t} k^2\delta((k,t))\\ &= \sum_{k\leqslant t} k^2 \sum_{d|(k,t)} \mu(d)\\ &= \sum_{d|t}\mu(d)\sum_{md\leqslant t} m^2d^2\\ &= \sum_{d|t}\mu(d)d^2\sum_{m\leqslant t/d} m^2\\ &= \sum_{d|t} d^2\mu(d)\frac{t/d(t/d+1)(2t/d+1)}{6}\\ &= \frac{t}{6}\sum_{d|t} d\mu(d)(t/d+1)(2t/d+1)\\ &= \left[\frac{N}{6}\big((N\mu)*((N+1)(2N+1))\big)\right](t).\end{aligned}$$

Thus, we obtain (2.7).

Again, by using the same method in the proof of Lemma 2.2 in [35], we expand $n^2(m-\mu_n^{(1)})^2$. Then, we have

$$n^2(m-\mu_n^{(1)})^2 = ns^2 + n(m-\mu_n^{(1)})^2 + \sum_{j\neq k}(\mu_n^{(j)}-\mu_n^{(1)})(\mu_n^{(k)}-\mu_n^{(1)}).$$

Since $\mu_{max}(A)-\mu_{min}(A)\geqslant 2\max_{i\neq j}|a_{ij}|$ for an $n\times n$ Hermitian matrix $A=(a_{ij})$ (see [18]) and $[n,n-1]=n(n-1)$ for every integer $n>1$, we have $\mu_n^{(n)}-\mu_n^{(1)}>2n(n-1)$ for all $n\geqslant 2$. By using Cauchy's Interlacing Theorem (see Theorem 4.3.8 in [15]), we have $\mu_n^{(n-1)}-\mu_n^{(1)}>\mu_4^{(3)}-\mu_4^{(1)}$ for all $n>4$. By a simple calculation in Maple $\mu_4^{(3)}\cong -0,312$ and $\mu_4^{(1)}\cong -8,843$. Thus, we have

$$\sum_{j\neq k}(\mu_n^{(j)}-\mu_n^{(1)})(\mu_n^{(k)}-\mu_n^{(1)}) > 32n(n-1). \tag{2.8}$$

Finally we have

$$\mu_n^{(1)} < m-\left(\frac{s^2+32(n-1)}{n-1}\right)^{1/2}.$$

Similarly expanding $n^2(\mu_n^{(n)}-m)^2$ and using the inequality in (2.8) we obtain the left-hand inequality in (2.6). □

3. Comments and an open problem

In the study of eigenvalues of GCD, LCM and related matrices, some authors [3, 11, 12, 13, 14] investigated asymptotic behavior of the eigenvalues of such matrices defined on $S=\{x_1,x_2,\ldots,x_n\}$ and they gave lower bounds for the smallest eigenvalues and upper bounds for the largest eigenvalues of such matrices. In addition, authors of [17, 22, 23, 24, 25] obtained such bounds on some certain constants and particular arithmetical functions by using matrix theoretic techniques. In this paper, we have obtained not only a lower (an upper) bound but also an upper (a lower) bound for the smallest (largest) eigenvalue of the GCD matrix as well as the LCM matrix defined on S_n by using a different technique from above papers. Indeed, our bounds depend on only particular arithmetical functions and the size n of our matrices.

In this context, Mattila and Haukkanen [23, 24] proved that every eigenvalue of the matrix $A_n^{\alpha,\beta}$ lies in the real interval

$$\left[2\min\{1,n^{\alpha+\beta}\}-T_n\max\{1,n^{2\beta}\}\max_{1\leqslant i\leqslant n}|J_{\alpha-\beta}(i)|, T_n\max\{1,n^{2\beta}\}\max_{1\leqslant i\leqslant n}|J_{\alpha-\beta}(i)|\right].$$

Here t_n and T_n are the smallest and the largest eigenvalues of EE^T, where E is the $n\times n$ matrix (e_{ij}) whose the $ij-$entry is 1 if $j|i$ and 0 otherwise. These bounds provided by above interval are valid for all values of α and β but it is natural that these bounds are not good enough for particular values of α and β. For example, $A_n^{1,0}$ is the GCD matrix (S_n) and above interval is

$$\left[2-T_n\cdot\max_{1\leqslant i\leqslant n}|\varphi(i)|, T_n\cdot\max_{1\leqslant i\leqslant n}|\varphi(i)|\right].$$

For $n=20$ the interval approximately is $[-595.8214, 597.8214]$. On the other hand, from Theorem 2 we have approximately -40.2114 and 7.8123 as a lower and an upper bound for the smallest eigenvalue of (S_{20}) and also we have approximately 13.1876 and 61.2114 as a lower and an upper bound for the largest eigenvalue of (S_{20}).

Moreover, so far the eigenvalues of the usual LCM matrices have not been studied much in the literature. Contrary to the GCD matrix, the LCM matrix $[S]$ defined on any set S is positive definite if and only if $|S|=1$. For example, the eigenvalues of $[S_2]$ are $\lambda_1=\frac{3}{2}-\frac{1}{2}\sqrt{17}$ and $\lambda_2=\frac{3}{2}+\frac{1}{2}\sqrt{17}$. Thus by using Cauchy's Interlacing Theorem and Rayleigh-Ritz Theorem one can easily see that at least one eigenvalue of $[S]=([x_i,x_j])$ is negative and at least one eigenvalue of $[S]=([x_i,x_j])$ is positive for each $n\geqslant 2$. From this observation we conclude our paper with an open problem.

PROBLEM. How many eigenvalues of $[S_n]=([i,j])$ defined on $S_n=\{1,2,\ldots,n\}$ are positive? More generally, how many eigenvalues of $[S]=([x_i,x_j])$ defined on $S=\{x_1,x_2,\ldots,x_n\}$ are positive?

Acknowledgements. The authors would like to thank two anonymous referees for their valuable remarks and suggestions which helped improve the manuscript. The authors also would like to thank Professor Iva Franjić for her support.

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(Received December 11, 2014)

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