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BOAS–TYPE INEQUALITY FOR 3–CONVEX FUNCTIONS AT A POINT

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Abstract. Starting from a very general form of Boas-type inequality from [5] we get Boas-type inequality for 3-convex functions at a point. For special λ-balanced sets, weight functions and measures we derive various examples.

1. Introduction

In [2], R. P. Boas proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\frac{dx}{x} \tag{1}$$

holds for all continuous convex functions $\Phi\colon [0,\infty) \to \mathbb{R}$, measurable non–negative functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and non–decreasing bounded functions $m\colon [0,\infty) \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1) is the Lebesgue–Stieltjes integral with respect to m. This inequality represent one direction of generalization of the famous Hardy inequality. After its author, relation (1) was named the Boas inequality. In the case of a concave function Φ, (1) holds with the sign of inequality reversed.

S. Kaijser et al. [6] (see also the paper [7] of N. Levinson) established the so-called general Hardy-Knopp-type inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\frac{dx}{x}, \tag{2}$$

for positive measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and a real convex function Φ on $\mathbb{R}_+$. Later on, A. Čižmešija et al. [4] generalized relation (2) to the so-called strengthened Hardy-Knopp-type inequality by inserting a weight function and integrating over intervals of non-negative real numbers. Further, in [3] A. Čižmešija et al. considered a general Borel measure λ on $\mathbb{R}_+$, such that $L = \lambda(\mathbb{R}_+) = \int_0^\infty d\lambda(t) < \infty$, and proved

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that for a convex function Φ on an interval $I \subseteq \mathbb{R}$ and a weight function u on $\mathbb{R}_+$ the inequality
$$\int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \leqslant \frac{1}{L}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}$$
holds for all measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, where $Af(x) = \frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)$ and $w(x) = \int_0^\infty u\left(\frac{x}{t}\right)d\lambda(t) < \infty$, $x \in \mathbb{R}_+$.

Observe that a non-decreasing and bounded function $m\colon [0,\infty) \to \mathbb{R}$ such that $M = m(\infty) - m(0) > 0$ induces a finite Borel measure λ on $\mathbb{R}_+$ and vice versa. For such a function and measure, related Lebesgue and Lebesgue-Stieltjes integrals are equivalent. Thus, all the above results can be stated for $Af(x)$ defined by

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t), \quad x \in \mathbb{R}_+,$$

so they refine and generalize inequality (1).

Another generalization of (1) was given by D. Luor [8] in a setting with σ-finite Borel measures μ and ν on a topological space X and a Borel probability measure λ on $\mathbb{R}_+$. The weighted version of that Luor's result is obtained in [5] in a setting with a topological space and σ-finite Borel measures as following.

Let λ be a finite Borel measure on $\mathbb{R}_+$. By supp λ we denote its support, that is, the set of all $t \in \mathbb{R}_+$ such that $\lambda(N_t) > 0$ holds for all open neighbourhoods N_t of t. Hence,
$$L = \int_{\text{supp }\lambda} d\lambda(t) = \int_0^\infty d\lambda(t) = \lambda(\mathbb{R}_+) < \infty. \tag{3}$$
On the other hand, let X be a topological space equipped with a continuous scalar multiplication $(a,\mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$, $\mathbf{x} \in X$, such that

$$1\,\mathbf{x} = \mathbf{x}, \;\; a(b\mathbf{x}) = (ab)\mathbf{x}, \;\; \mathbf{x} \in X, \;\; a,b \in \mathbb{R}_+.$$

Further, let the Borel set $\Omega \subseteq X$ be λ-balanced, that is, $t\Omega = \{t\mathbf{x}\colon \mathbf{x} \in \Omega\} \subseteq \Omega$, for all $t \in$ supp λ. For a Borel measurable function $f\colon \Omega \to \mathbb{R}$, we define its Hardy–Littlewood average Af as

$$Af(\mathbf{x}) = \frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t), \quad \mathbf{x} \in \Omega. \tag{4}$$

Finally, suppose that μ and ν are σ–finite Borel measures on X. For $t > 0$ and a Borel set $S \subseteq X$ we define
$$\mu_t(S) = \mu\left(\frac{1}{t}S\right). \tag{5}$$
Obviously, μ_t is a σ-finite Borel measure on X for each $t \in \mathbb{R}_+$. Throughout this paper, we suppose that the measures μ_t are absolutely continuous with respect to the measure ν, that is, $\mu_t \ll \nu$ for each $t \in$ supp λ. As usual, by $\frac{d\mu_t}{d\nu}$ we denote the related Radon–Nikodym derivative.

Thus, the following weighted general Boas-type inequality is given in [5].

THEOREM 1. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (3). *Let μ and ν be σ-finite Borel measures on a topological space X, μ_t be defined by* (5) *and such that $\mu_t \ll \nu$ for all $t \in \operatorname{supp} \lambda$. Further, let $\Omega \subseteq X$ be a λ-balanced set and u be a non-negative function on X, such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right) \frac{d\mu_t}{d\nu}(\mathbf{x})\, d\lambda(t) < \infty, \quad \mathbf{x} \in \Omega. \tag{6}$$

Suppose $\Phi\colon I \to \mathbb{R}$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon \Omega \to \mathbb{R}$ is a Borel measurable function, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, and Af is defined by (4), *then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality*

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\mu(\mathbf{x}) \leqslant \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\, d\nu(\mathbf{x}) \tag{7}$$

holds. For a non-positive concave function Φ, the sign of inequality in (7) *is reversed.*

Notice that the condition on non-negativity of the convex function Φ in Theorem 1 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

COROLLARY 1. *If in Theorem 1 we have $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$, then* (7) *holds for all convex functions Φ on an interval $I \subseteq \mathbb{R}$. In that case, for all concave functions Φ relation* (7) *holds with the sign of inequality reversed.*

We will make further generalization based on the inequality (7). Instead of convex functions we will introduce a different class of functions, following the idea of I. A. Baloch et al. [1] and J. Pečarić et al. [10].

This paper is organized in following way: after the Introduction, in Section 2 we define class of functions $\mathscr{K}_1^c(I)$ and prove the Boas inequality of Levinson type for such functions. We point out the dual class of functions and corresponding dual inequality. We discuss 3-convexity at the point and give several one-dimensional results. In Section 3 we obtain multidimensional results and examples of Levinson type concerning balls in $\mathbb{R}^n$ centred at the origin and their dual sets.

CONVENTIONS. An interval I in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while by $\operatorname{Int} I$ we denote its interior. By $\mathbb{R}_+$ we denote the set of all positive real numbers i.e. $\mathbb{R}_+ = (0,\infty)$. A kth order divided difference of a function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0,\dots,x_k \in I$ is defined recursively by

$$[x_i]f = f(x_i), \qquad \text{for} \quad i = 0,\dots,k$$

and

$$[x_0,\dots,x_k]f = \frac{[x_1,\dots,x_k]f - [x_0,\dots,x_{k-1}]f}{x_k - x_0}.$$

A function $f : I \to \mathbb{R}$ is called k-convex if $[x_0,\dots,x_k]f \geqslant 0$ for all choices of $k+1$ distinct points $x_0,\dots,x_k \in I$. If the kth derivative $f^{(k)}$ of a k-convex function exists, then $f^{(k)} \geqslant 0$, but $f^{(k)}$ may not exist (for properties of divided differences and k-convex functions see [11]). For $R > 0$ we denote by $B(R)$ a ball in $\mathbb{R}^n$ centred at the origin and of radius R, that is, $B(R) = \{\mathbf{x} \in \mathbb{R}^n \colon |\mathbf{x}| \leqslant R\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. By its complementary set we mean the set $\mathbb{R}^n \setminus B(R) = \{\mathbf{x} \in \mathbb{R}^n \colon |\mathbf{x}| > R\}$.

2. Main results

In order to make generalization of the inequality (7) in Levinson's sense we will replace convex functions with the following class of functions.

DEFINITION 1. Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a function and $c \in \operatorname{Int} I$. We say that $f \in \mathscr{K}_1^c(I)$ (resp. $f \in \mathscr{K}_2^c(I)$) if there exists a constant α such that the function $F(x) = f(x) - \frac{\alpha}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$.

REMARK 1. If $f \in \mathscr{K}_i^c(a,b)$, $i = 1,2$, and $f''(c)$ exists, then $f''(c) = \alpha$. Let $f \in \mathscr{K}_1^c(a,b)$. Due to the concavity and convexity of F for every distinct points $x_j \in (a,c]$ and $y_j \in [c,b)$, $j = 1,2,3$, we have

$$[x_1,x_2,x_3]F = [x_1,x_2,x_3]f - \alpha \leqslant 0 \leqslant [y_1,y_2,y_3]f - \alpha = [y_1,y_2,y_3]F.$$

Therefore, if $f''_-(c)$ and $f''_+(c)$ exist, letting $x_j \nearrow c$ and $y_j \searrow c$, we get

$$f''_-(c) \leqslant \alpha \leqslant f''_+(c).$$

Similary, for $f \in \mathscr{K}_2^c(a,b)$, we have $f''_+(c) \leqslant \alpha \leqslant f''_-(c)$. □

REMARK 2. Function $f : I \to \mathbb{R}$ is 3-convex (resp. 3-concave) if and only if $f \in \mathscr{K}_1^c(I)$ (resp. $f \in \mathscr{K}_2^c(I)$) for every $c \in \operatorname{Int} I$. In other words, a function is 3-convex on an interval if and only if it is 3-convex at every point of its interior, so the property from the definition of $\mathscr{K}_1^c(I)$ can be described as "3-convexity at point c".

For the main result we need another set of measures, sets and functions that also satisfying Theorem1. So, let $\hat{\lambda}$ be a finite Borel measure on $\mathbb{R}_+$ such that

$$\hat{L} = \int_0^\infty d\hat{\lambda}(t) = \int_{\operatorname{supp}\hat{\lambda}} d\hat{\lambda}(t) < \infty. \tag{8}$$

Let $\hat{\Omega} \subseteq X$ be a $\hat{\lambda}$-balanced Borel set, let measures $\hat{\mu}$, $\hat{\mu}_t, t \in \mathbb{R}_+$ and $\hat{\nu}$ be σ-finite Borel measures on X such that $\hat{\mu}_t(S) = \hat{\mu}\left(\frac{1}{t}S\right)$, for $t \in \mathbb{R}_+$ and $S \subseteq X$ Borel set and $\hat{\mu}_t \ll \hat{\nu}$, $t \in \operatorname{supp}\hat{\lambda}$. Finally, let $\hat{u}$ be a non-negative function on X, such that

$$\hat{v}(\mathbf{x}) = \int_0^\infty \hat{u}\left(\frac{1}{t}\mathbf{x}\right)\frac{d\hat{\mu}_t}{d\hat{\nu}}(\mathbf{x})d\hat{\lambda}(t) < \infty, \ \mathbf{x} \in \hat{\Omega}. \tag{9}$$

For a Borel measurable function $g \colon \hat{\Omega} \to \mathbb{R}$ the Hardy-Littlewood average $\hat{A}g$ of g defined by

$$\hat{A}g(\mathbf{x}) = \frac{1}{\hat{L}}\int_0^\infty g(t\mathbf{x})d\hat{\lambda}(t), \ \mathbf{x} \in \hat{\Omega}.$$

THEOREM 2. *Let X, Ω, λ, μ, ν, μ_t, L, u, v be as in Theorem 1. Furthermore, let $\hat{\Omega}$, $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$, $\hat{\mu}_t$, $\hat{L}$, $\hat{u}$, $\hat{v}$ be another set of measures and functions that satisfy Theorem 1. If $f\colon \Omega \to I\cap(-\infty,c]$ and $g\colon \hat{\Omega} \to I\cap[c,\infty)$ are measurable functions satisfying*

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x}) f^2(\mathbf{x}) d\nu(\mathbf{x}) = \int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x}) g^2(\mathbf{x}) d\hat{\nu}(\mathbf{x}) \tag{10}$$

then for every $\Phi \in \mathscr{K}_1^c(I)$ the following inequality holds

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x})\Phi(\hat{A}g(\mathbf{x})) d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})\Phi(g(\mathbf{x})) d\hat{\nu}(\mathbf{x}) \leqslant \int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x})) d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x})) d\nu(\mathbf{x}). \tag{11}$$

If $\Phi \in \mathscr{K}_2^c(I)$ in the above setting, then (11) holds with the sign of inequality reversed.

Proof. From Definition 1 there exists a constant α such that $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ is concave on $I\cap(-\infty,c]$ so we can apply Theorem 1 on the function F and get

$$\int_\Omega u(\mathbf{x}) F(Af(\mathbf{x})) d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x}) F(f(\mathbf{x})) d\nu(\mathbf{x}) \geqslant 0$$

By the definition of the function F we have

$$\int_\Omega u(\mathbf{x})\left[\Phi(Af(\mathbf{x})) - \frac{\alpha}{2}(Af(\mathbf{x}))^2\right] d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x})\left[\Phi(f(\mathbf{x})) - \frac{\alpha}{2} f^2(\mathbf{x})\right] d\nu(\mathbf{x}) \geqslant 0.$$

Since integral is a linear functional we can write

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x})) d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x})) d\nu(\mathbf{x}) \geqslant \frac{\alpha}{2}\left[\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x}) f^2(\mathbf{x}) d\nu(\mathbf{x})\right]. \tag{12}$$

For the same constant α, the second part of Definition 1 gives us a convex function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ on $I\cap[c,\infty)$. Now, from Theorem 1 we have

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x}) F(\hat{A}g(\mathbf{x})) d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x}) F(g(\mathbf{x})) d\hat{\nu}(\mathbf{x}) \leqslant 0.$$

Similarly as in the first part of the proof, we obtain

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x})\left[\Phi(\hat{A}g(\mathbf{x})) - \frac{\alpha}{2}(\hat{A}g(\mathbf{x}))^2\right] d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})\left[\Phi(g(\mathbf{x})) - \frac{\alpha}{2} g^2(\mathbf{x})\right] d\hat{\nu}(\mathbf{x}) \leqslant 0$$

and also

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x})\Phi(\hat{A}g(\mathbf{x}))d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})\Phi(g(\mathbf{x}))d\hat{v}(\mathbf{x})$$
$$\leqslant \frac{\alpha}{2}\left[\int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\hat{v}(\mathbf{x})\right]. \tag{13}$$

Due to assumption (10) the right hand sides of inequalities (12) and (13) are equal. Hence, we obtain (11). In the case $\Phi \in \mathcal{K}_2^c(I)$, the function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ is convex on $I \cap (-\infty, c]$ and concave on $I \cap [c, \infty)$. Following the idea of the first part of the proof we get our statement. □

Similarly as in [9] we analize α from the Definition 1.

REMARK 3. The assumption of equality (10) in Theorem 2 can be weakened. More concretely, if

(a) $\alpha \geqslant 0$ and

$$\int_{\Omega} u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_{\Omega} v(\mathbf{x})f^2(\mathbf{x})dv(\mathbf{x})$$
$$\geqslant \int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\hat{v}(\mathbf{x}), \tag{14}$$

or

(b) $\alpha \leqslant 0$ and

$$\int_{\Omega} u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_{\Omega} v(\mathbf{x})f^2(\mathbf{x})dv(\mathbf{x})$$
$$\leqslant \int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\hat{v}(\mathbf{x}), \tag{15}$$

then (11) holds. Indeed, if we multiply (14) with $\frac{\alpha}{2} \geqslant 0$ we get

$$\frac{\alpha}{2}\left[\int_{\Omega} u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_{\Omega} v(\mathbf{x})f^2(\mathbf{x})dv(\mathbf{x})\right]$$
$$\geqslant \frac{\alpha}{2}\left[\int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\hat{v}(\mathbf{x})\right] \tag{16}$$

so we can chain inequalities (12) and (13) to get (11). In the case when we multiply (15) with $\frac{\alpha}{2} \leqslant 0$ we again get (16) and the conclusion is the same.

COROLLARY 2. *Let X, Ω, $\hat{\Omega}$, λ, $\hat{\lambda}$, μ, $\hat{\mu}$, v, $\hat{v}$, μ_t, $\hat{\mu}_t$, L, $\hat{L}$, u, $\hat{u}$, v, $\hat{v}$ be as in Theorem 2 and assume that (10) holds. If Φ is 3-convex on the interval I, then (11) holds. If Φ is 3-concave, then (11) holds with the sign reversed.*

Proof. If Φ is 3-convex, then by Remark 2 it also belongs to $\mathscr{K}_1^c(I)$ for every $c \in Int\, I$, so we can again apply Theorem 2. $\square$

For Lebesgue measures and some intervention on the weight functions, from Theorem 2 we obtain the following result.

COROLLARY 3. *Let λ and $\hat{\lambda}$ be finite Borel measures on $\mathbb{R}_+$ and L and $\hat{L}$ be defined by (3) and (8) respectively. Let $\Omega \subseteq \mathbb{R}_+$ be a λ-balanced set such that $t\Omega = \Omega$ for λ-a.e. $t \in \sup\lambda$ and $\hat{\Omega} \subseteq \mathbb{R}_+$ be a $\hat{\lambda}$-balanced such that $t\hat{\Omega} = \hat{\Omega}$ for $\hat{\lambda}$-a.e. $t \in \sup\hat{\lambda}$. Suppose that u and $\hat{u}$ are non-negative functions on $\mathbb{R}_+$, such that*

$$w(x) = \int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) < \infty, \quad x \in \Omega$$

and

$$\hat{w}(x) = \int_0^\infty \hat{u}\left(\frac{x}{t}\right) d\hat{\lambda}(t) < \infty, \quad x \in \hat{\Omega}.$$

If $f\colon \Omega \to I \cap (-\infty, c]$ and $g\colon \hat{\Omega} \to I \cap [c,\infty)$ are measurable functions satisfying

$$\int_\Omega u(x)(Af(x))^2 \frac{dx}{x} - \frac{1}{L}\int_\Omega w(x) f^2(x)\frac{dx}{x}$$
$$= \int_{\hat{\Omega}} \hat{u}(x)(\hat{A}g(x))^2 \frac{dx}{x} - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{w}(x) g^2(x)\frac{dx}{x},$$

the following inequality

$$\int_{\hat{\Omega}} \hat{u}(x)\Phi(\hat{A}g(x))\frac{dx}{x} - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{w}(x)\Phi(g(x))\frac{dx}{x}$$
$$\leqslant \int_\Omega u(x)\Phi(Af(x))\frac{dx}{x} - \frac{1}{L}\int_\Omega w(x)\Phi(f(x))\frac{dx}{x} \tag{17}$$

holds for $\Phi \in \mathscr{K}_1^c(I)$. If $\Phi \in \mathscr{K}_2^c(I)$ in the above setting, then (17) holds with the sign of inequality reversed.

Proof. It follows directly from Theorem 2 if we set $X = \mathbb{R}_+$, the measures μ, ν, $\hat{\mu}$ and $\hat{\nu}$ to be the Lebesgue measures and replace the weight functions u and $\hat{u}$ with $x \mapsto \frac{u(x)}{x}$, $x \mapsto \frac{\hat{u}(x)}{x}$ respectively. For such measures we get $\frac{d\mu_t}{d\nu}(x) = \frac{d\hat{\mu}_t}{d\hat{\nu}}(x) = \frac{1}{t}$, $t \in \mathbb{R}_+$. In this setting, we have

$$v(x) = \int_0^\infty u\left(\frac{x}{t}\right)\cdot\frac{t}{x}\cdot\frac{1}{t}\,d\lambda(t) = \frac{1}{x}\int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) = \frac{w(x)}{x}, \quad x \in \Omega,$$

and $\hat{v}(x) = \frac{\hat{w}(x)}{x}$, $x \in \hat{\Omega}$ where the function v and $\hat{v}$ are defined by (6) and (9). $\square$

EXAMPLE 1. Consider the Theorem 2 with $X = \Omega = \mathbb{R}_+$ $d\lambda(t) = \chi_{(0,1)}(t)\,dt$. For $0 < b \leqslant \infty$, let $d\mu(x) = \chi_{(0,b)}(x)\,dx$, and $\nu(x) = dx$. Instead of the weight u

we take the function $x \mapsto \frac{u(x)}{x}\chi_{(0,b)}(x)$. Then $\operatorname{supp}\lambda = (0,1]$, $L = 1$, $t\Omega = \Omega$ for $t \in \operatorname{supp}\lambda$, $\frac{d\mu_t}{d\nu}(x) = \frac{1}{t}\chi_{(0,tb)}(x)$,

$$Af(x) = \int_0^1 f(tx)\,dt = Hf(x),$$

and

$$v(x) = \int_0^1 \frac{u\left(\frac{1}{t}x\right)}{\frac{1}{t}x}\cdot\frac{1}{t}\chi_{(0,tb)}(x)\,dt = \frac{1}{x}\int_{\frac{x}{b}}^1 u\left(\frac{x}{t}\right)\,dt = \int_x^b u(y)\,\frac{dy}{y^2} = \frac{w(x)}{x},$$

for $x \in (0,b)$.

For measurable functions $f\colon \mathbb{R}_+ \to I \cap (-\infty, c]$ and $g\colon \hat{\Omega} \to I \cap [c,\infty)$ the condition (10) becomes

$$\int_0^b u(x)(Hf(x))^2\,\frac{dx}{x} - \int_0^b w(x)f^2(x)\,\frac{dx}{x}$$
$$= \int_{\hat{\Omega}} \hat{u}(x)(\hat{A}g(x))^2\,d\hat{\mu}(x) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(x)g^2(x)\,d\hat{\nu}(x),$$

and for a function Φ from $\mathscr{K}_1^c(I)$ the following inequality

$$\int_{\hat{\Omega}} \hat{u}(x)\Phi(\hat{A}g(x))\,d\hat{\mu}(x) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(x)\Phi(g(x))\,d\hat{\nu}(x)$$
$$\leqslant \int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x} - \int_0^b w(x)\Phi(f(x))\,\frac{dx}{x} \tag{18}$$

holds. If Φ is from $\mathscr{K}_2^c(I)$, then the sign of inequality (18) is reversed.

On the other hand, we have dual example.

EXAMPLE 2. Let $X = \hat{\Omega} = \mathbb{R}_+$ and $d\hat{\lambda}(t) = \chi_{[1,\infty)}(t)\frac{dt}{t^2}$ in the Theorem 2. For $0 \leqslant b < \infty$, let $\hat{u}\colon (b,\infty) \to \mathbb{R}$ be a non-negative locally integrable function on its domain. Let $d\hat{\mu}(x) = \chi_{(b,+\infty)}(x)dx$ and $d\hat{\nu}(x) = dx$. Then we get a dual result to (18) (see also [3, 4, 6]). So $\operatorname{supp}\hat{\lambda} = [1,\infty)$, $\hat{L} = 1$, $\hat{w}(x) = \frac{1}{x}\int_b^x \hat{u}(t)\,dt$ and

$$\hat{A}g(x) = \int_1^\infty g(tx)\frac{dt}{t^2} = x\int_x^\infty g(t)\,\frac{dt}{t^2} = \hat{H}g(x), \quad x \in (b,\infty).$$

For measurable functions $f\colon \Omega \to I \cap (-\infty, c]$ and $g\colon \hat{\Omega} \to I \cap [c,\infty)$ the condition (10) becomes

$$\int_\Omega u(x)(Af(x))^2\,d\mu(x) - \frac{1}{L}\int_\Omega v(x)f^2(x)d\nu(x)$$
$$= \int_0^\infty \hat{u}(x)(\hat{H}g(x))^2\,\frac{dx}{x} - \int_0^\infty \hat{w}(x)g^2(x)\,\frac{dx}{x},$$

and for $\Phi \in \mathscr{K}_1^c(I)$ the following inequality

$$\int_0^\infty \hat{u}(x)\Phi(\hat{H}g(x))\,\frac{dx}{x} - \int_0^\infty \hat{w}(x)\Phi(g(x))\,\frac{dx}{x} \leqslant \int_\Omega u(x)\Phi(Af(x))\,d\mu(x) - \frac{1}{L}\int_\Omega v(x)\Phi(f(x))\,d\nu(x) \tag{19}$$

holds. If $\Phi \in \mathscr{K}_2^c(I)$, then the sign of inequality (19) is reversed.

3. Multidimensional examples

In Corollary 1 the condition $t\Omega = \Omega$, λ-a.e. $t \in \operatorname{supp}\lambda$ is emphasized, so the logical choice of the multidimensional examples is setting with balls in $\mathbb{R}^n$ centred at the origin.

COROLLARY 4. *Suppose that $0 < b \leqslant \infty$ and that a positive function ψ on $[0,1]$ and a non-negative function u on $\mathbb{R}^n$ are such that*

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{b}}^1 u\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in B(b) \tag{20}$$

and

$$P_1 = \int_0^1 \psi(t)\,dt < \infty. \tag{21}$$

If $f\colon B(b) \to I \cap (-\infty, c]$ and $g\colon \hat{\Omega} \to I \cap [c,\infty)$ are measurable functions satisfying

$$\frac{1}{P_1^2}\int_{B(b)} u(\mathbf{x})\left(\int_0^1 \psi(t)f(t\mathbf{x})dt\right)^2 d\mathbf{x} - \frac{1}{P_1}\int_{B(b)} v(\mathbf{x})f^2(\mathbf{x})d\mathbf{x} = \int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\hat{\nu}(\mathbf{x}), \tag{22}$$

then for $\Phi \in \mathscr{K}_1^c(I)$ the inequality

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x})\Phi(\hat{A}g(\mathbf{x}))d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x})\Phi(g(\mathbf{x}))d\hat{\nu}(\mathbf{x}) \leqslant \int_{B(b)} u(\mathbf{x})\Phi\left(\frac{1}{P_1}\int_0^1 \psi(t)f(t\mathbf{x})dt\right)d\mathbf{x} - \frac{1}{P_1}\int_{B(b)} v(\mathbf{x})\Phi(f(\mathbf{x}))d\mathbf{x} \tag{23}$$

holds.

Proof. Follows from Theorem 2 rewritten with $X = \mathbb{R}^n$, $\Omega = B(b)$, $d\lambda(t) = \psi(t)\chi_{(0,1)}(t)\,dt$, $d\mu(\mathbf{x}) = \chi_{B(b)}(\mathbf{x})\,d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$. Here we have $\operatorname{supp}\lambda = (0,1]$, $\frac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{B(tb)}(\mathbf{x})$, and $Af(\mathbf{x}) = \frac{1}{P_1}\int_0^1 \psi(t)f(t\mathbf{x})\,dt$. It is easy to see that in this setting (20) reduces to (6), and (10) and (11) becomes (22) and (23). □

Applying Corollary 4 to some particular u and Φ we get the following result.

EXAMPLE 3. Apply Corollary 4 for $u(\mathbf{x}) \equiv 1$ and the 3-convex function $\Phi(x) = x^p$, $p > 2$ or $p \in (0,1)$. In this setting, if $f\colon B(b) \to I \cap (-\infty, c]$ and $g\colon \hat{\Omega} \to I \cap [c, \infty)$ are measurable functions satisfying

$$\frac{1}{P_1^2}\int_{B(b)}\left(\int_0^1 \psi(t) f(t\mathbf{x})dt\right)^2 d\mathbf{x} - \frac{1}{P_1}\int_{B(b)} v(\mathbf{x}) f^2(\mathbf{x})d\mathbf{x}$$
$$= \int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^2 d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x}) g^2(\mathbf{x}) d\hat{\nu}(\mathbf{x}),$$

then the following inequality

$$\int_{\hat{\Omega}} \hat{u}(\mathbf{x})(\hat{A}g(\mathbf{x}))^p d\hat{\mu}(\mathbf{x}) - \frac{1}{\hat{L}}\int_{\hat{\Omega}} \hat{v}(\mathbf{x}) g^p(\mathbf{x}) d\hat{\nu}(\mathbf{x})$$
$$\leqslant \frac{1}{P_1^p}\int_{B(b)}\left(\int_0^1 \psi(t) f(t\mathbf{x})dt\right)^p d\mathbf{x} - \frac{1}{P_1}\int_{B(b)} v(\mathbf{x}) f^p(\mathbf{x})d\mathbf{x}$$

holds, where P_1 is defined by (21). Notice that $\Phi(x) = x^p$, $p \in (1,2)$ or $p < 0$ is a 3-concave function.

Similarly, we get the dual result by using the set $\mathbb{R}^n \setminus B(b)$.

COROLLARY 5. *Suppose that $0 \leqslant b < \infty$ and that the positive function ψ on $[1,\infty)$ and the non-negative function u on $\mathbb{R}^n$ are such that*

$$\hat{v}(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{b}} \hat{u}\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(b) \tag{24}$$

and

$$P_\infty = \int_1^\infty \psi(t)\,dt < \infty. \tag{25}$$

If $f\colon \Omega \to I \cap (-\infty, c]$ and $g\colon \mathbb{R}^n \setminus B(b) \to I \cap [c, \infty)$ are measurable functions satisfying

$$\int_{\Omega} u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_{\Omega} v(\mathbf{x}) f^2(\mathbf{x}) d\nu(\mathbf{x})$$
$$= \frac{1}{P_\infty^2}\int_{\mathbb{R}^n\setminus B(b)} \hat{u}(\mathbf{x})\left(\int_1^\infty \psi(t) g(t\mathbf{x})dt\right)^2 d\mathbf{x} - \frac{1}{P_\infty}\int_{\mathbb{R}^n\setminus B(b)} \hat{v}(\mathbf{x}) g^2(\mathbf{x})d\mathbf{x} \tag{26}$$

then for $\Phi \in \mathscr{K}_1^c(I)$ the following inequality

$$\int_{\mathbb{R}^n\setminus B(b)} \hat{u}(\mathbf{x})\Phi\left(\frac{1}{P_\infty}\int_1^\infty \psi(t) g(t\mathbf{x})dt\right) d\mathbf{x} - \frac{1}{P_\infty}\int_{\mathbb{R}^n\setminus B(b)} \hat{v}(\mathbf{x})\Phi(g(\mathbf{x}))d\mathbf{x}$$
$$\leqslant \int_{\Omega} u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}) - \frac{1}{L}\int_{\Omega} v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}) \tag{27}$$

holds.

Proof. The proof follows from Theorem 2 if we set $d\hat{\lambda}(t) = \psi(t)\chi_{(1,\infty)}(t)\,dt$, $X = \mathbb{R}^n$, $\hat{\Omega} = \mathbb{R}^n \setminus B(b)$, $d\hat{\mu}(\mathbf{x}) = \chi_{\mathbb{R}^n\setminus B(b)}(\mathbf{x})\,d\mathbf{x}$ and $d\hat{\nu}(\mathbf{x}) = d\mathbf{x}$. Then we get supp $\hat{\lambda} = [1,\infty)$, $\frac{d\hat{\mu}_t}{d\hat{\nu}}(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n\setminus B(tb)}(\mathbf{x})$ and $\hat{A}g(\mathbf{x}) = \frac{1}{P_\infty}\int_1^\infty \psi(t)g(t\mathbf{x})\,dt$. So (6), (10) and (11) become (24), (26) and (27), respectively. □

EXAMPLE 4. Apply Corollary 5 for $\hat{u}(\mathbf{x}) \equiv 1$ and the 3-convex function $\Phi(x) = x^p$, $p > 2$ or $p \in (0,1)$. If $f\colon \Omega \to I \cap (-\infty, c]$ and $g\colon \mathbb{R}^n \setminus B(b) \to I \cap [c,\infty)$ are measurable functions satisfying

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2 d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x}) f^2(\mathbf{x}) d\nu(\mathbf{x})$$
$$= \frac{1}{P_\infty^2}\int_{\mathbb{R}^n\setminus B(b)} \left(\int_1^\infty \psi(t)g(t\mathbf{x})dt\right)^2 d\mathbf{x} - \frac{1}{P_\infty}\int_{\mathbb{R}^n\setminus B(b)} \hat{v}(\mathbf{x})g^2(\mathbf{x})d\mathbf{x},$$

then the following inequality

$$\int_{\mathbb{R}^n\setminus B(b)} \left(\frac{1}{P_\infty}\int_1^\infty \psi(t)g(t\mathbf{x})dt\right)^p d\mathbf{x} - \frac{1}{P_\infty}\int_{\mathbb{R}^n\setminus B(b)} \hat{v}(\mathbf{x})g^p(\mathbf{x})d\mathbf{x}$$
$$\leqslant \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x}) f^p(\mathbf{x}) d\nu(\mathbf{x})$$

holds, where P_∞ is defined by (25).

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