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GENERALIZATION OF THE JENSEN–MERCER INEQUALITY BY TAYLOR'S POLYNOMIAL

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Abstract. We present generalizations of the Jensen-Mercer inequality for the class of n-convex functions, obtained by using Taylor's polynomial and Green function. By applying those inequalities we obtain some results related to Čebyšev functionals.

1. Introduction

In paper [1] the following integral version of the Jensen-Mercer inequality for convex functions was proved.

THEOREM A. *Let $g:[a,b]\to\mathbb{R}$ be a continuous and monotonic function and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let function $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation satisfying*

$$\lambda(a)\leqslant\lambda(t)\leqslant\lambda(b)\ \text{ for all }\ t\in[\alpha,\beta],\quad \lambda(b)-\lambda(a)>0. \tag{1}$$

Then for every continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$ the inequality

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\leqslant\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \tag{2}$$

holds.

REMARK 1. Inequality (2) is also valid when the condition (1) is replaced with the more strict condition that λ is a nondecreasing function such that $\lambda(a)\neq\lambda(b)$.

Let us recall the definition of n-convex functions (see [7, pp. 14–15]).

DEFINITION 1. A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex on $[a,b]$, $n\geqslant 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the n-th divided difference of f satisfies $f[x_0,\dots,x_n]\geqslant 0$. If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

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The *divided difference* of order n of the function $f:[a,b]\to\mathbb{R}$ at distinct points $x_0,\dots,x_n\in[a,b]$ is defined recursively by

$$f[x_i]=f(x_i),\qquad (i=0,\dots,n)$$

and

$$f[x_0,\dots,x_n]=\frac{f[x_1,\dots,x_n]-f[x_0,\dots,x_{n-1}]}{x_n-x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.

Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions and 2-convex functions are simply convex functions.

Consider the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \text{for } \alpha\leqslant s\leqslant t,\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} & \text{for } t\leqslant s\leqslant \beta.\end{cases}\tag{3}$$

The Green function is continuous and convex in s and, since it is symmetric, also in t.

It can be easily shown by integrating by parts that every function $\varphi:[\alpha,\beta]\to\mathbb{R}$, $\varphi\in C^2([\alpha,\beta])$ can be represented in the form

$$\varphi(x)=\frac{\beta-x}{\beta-\alpha}\varphi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\varphi(\beta)+\int_\alpha^\beta G(x,s)\varphi''(s)\,\mathrm{d}s.\tag{4}$$

LEMMA 1. *Let $g:[a,b]\to\mathbb{R}$ be a continuous and monotonic function, and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let function $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation satisfying (1), and G the Green function defined by (3). Then for every function $\varphi\in C^2([\alpha,\beta])$ the identity*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\int_\alpha^\beta\left[G\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)+\frac{\int_a^b G(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\varphi''(s)\,\mathrm{d}s\tag{5}$$

holds.

Proof. Using (4) we obtain

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\left(\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-\alpha\right)\frac{\varphi(\alpha)}{\beta-\alpha}+\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\frac{\varphi(\beta)}{\beta-\alpha}$$
$$+\int_\alpha^\beta G\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)\varphi''(s)\,\mathrm{d}s-(\varphi(\alpha)+\varphi(\beta))$$
$$+\frac{\int_a^b\left[\frac{\beta-g(x)}{\beta-\alpha}\varphi(\alpha)+\frac{g(x)-\alpha}{\beta-\alpha}\varphi(\beta)+\int_\alpha^\beta G(g(x),s)\varphi''(s)\,\mathrm{d}s\right]\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}.$$

Since

$$\left(\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-\alpha\right)\frac{\varphi(\alpha)}{\beta-\alpha}+\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\frac{\varphi(\beta)}{\beta-\alpha}-(\varphi(\alpha)+\varphi(\beta))$$
$$+\frac{\int_a^b\left[\frac{\beta-g(x)}{\beta-\alpha}\varphi(\alpha)+\frac{g(x)-\alpha}{\beta-\alpha}\varphi(\beta)\right]\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$
$$=\frac{1}{\beta-\alpha}\left[\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi(\alpha)-\alpha\varphi(\alpha)+\beta\varphi(\beta)-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi(\beta)\right.$$
$$-\beta\varphi(\alpha)-\beta\varphi(\beta)+\alpha\varphi(\alpha)+\alpha\varphi(\beta)+\beta\varphi(\alpha)-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi(\alpha)$$
$$\left.+\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi(\beta)-\alpha\varphi(\beta)\right]=0,$$

(5) immediately follows. □

In the rest of the paper, for the sake of simplicity, let us denote

$$\mathscr{G}(s)=G\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)+\frac{\int_a^b G(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}. \tag{6}$$

2. Generalizations by Taylor's polynomial

In this section we generalize inequality (2) for n-convex functions using the following Taylor's formula with the integral remainder.

Let n be a positive integer, function $\varphi:[\alpha,\beta]\to\mathbb{R}$ such that $\varphi^{(n-1)}$ is absolutely continuous, and $c\in[\alpha,\beta]$. Then for all $x\in[\alpha,\beta]$

$$\varphi(x)=T_{n-1}(\varphi;c,x)+R_{n-1}(\varphi;c,x) \tag{7}$$

holds, where

$$T_{n-1}(\varphi;c,x)=\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(c)}{k!}(x-c)^k \tag{8}$$

is Taylor's polynomial of degree $n-1$, and the remainder is given by

$$R_{n-1}(\varphi;c,x)=\frac{1}{(n-1)!}\int_c^x\varphi^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t. \tag{9}$$

Applying Taylor's formula at the points α and β respectively we get

$$\varphi(x)=\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}(x-\alpha)^k+\frac{1}{(n-1)!}\int_\alpha^\beta\varphi^{(n)}(t)\left((x-t)_+\right)^{n-1}\mathrm{d}t, \tag{10}$$

and

$$\varphi(x)=\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\beta)}{k!}(-1)^k(\beta-x)^k-\frac{1}{(n-1)!}\int_\alpha^\beta(-1)^{n-1}\varphi^{(n)}(t)\left((t-x)_+\right)^{n-1}\mathrm{d}t, \tag{11}$$

where

$$(x-t)_+=\begin{cases}x-t, & t\leqslant x,\\ 0, & t>x\end{cases}. \tag{12}$$

Note that for $n\geqslant 1$ function $\left((x-t)_+\right)^{n-1}$ is convex in x and in t.

Applying Taylor's formula (7) for φ'', we can get the following identities.

LEMMA 2. *Let functions $g:[a,b]\to\mathbb{R}$, $\lambda:[a,b]\to\mathbb{R}$ be as in Lemma 1, and $\mathscr{G}:[\alpha,\beta]\to\mathbb{R}$ defined by (6). Then for every function $\varphi:[\alpha,\beta]\to\mathbb{R}$ such that $\varphi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$, the identities*

$$\begin{aligned}&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b\varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b\mathrm{d}\lambda(x)}\right)\\&=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\mathscr{G}(s)(s-\alpha)^k\,\mathrm{d}s\\&\quad+\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_t^\beta\mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\right)\varphi^{(n)}(t)\,\mathrm{d}t,\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b\varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b\mathrm{d}\lambda(x)}\right)\\&=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k\int_\alpha^\beta\mathscr{G}(s)(\beta-s)^k\mathrm{d}s\\&\quad-\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_\alpha^t\mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\right)\varphi^{(n)}(t)\mathrm{d}t\end{aligned} \tag{14}$$

hold.

Proof. Applying Taylor's formula (7) for φ'', at the points α and β, respectively, and replacing n by $n-2$ $(n\geqslant 3)$ we have

$$\varphi''(s)=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}(s-\alpha)^k+\frac{1}{(n-3)!}\int_\alpha^s\varphi^{(n)}(t)(s-t)^{n-3}\mathrm{d}t, \tag{15}$$

and

$$\varphi''(s)=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k(\beta-s)^k-\frac{1}{(n-3)!}\int_s^\beta\varphi^{(n)}(t)(s-t)^{n-3}\mathrm{d}t. \tag{16}$$

Using (15) in (5) we get

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta \mathscr{G}(s)(s-\alpha)^k\,\mathrm{d}s+\frac{1}{(n-3)!}\int_\alpha^\beta \mathscr{G}(s)\left(\int_\alpha^s \varphi^{(n)}(t)(s-t)^{n-3}\,\mathrm{d}t\right)\mathrm{d}s.$$

Applying Fubini's theorem we obtain (13). Analogously using (16) in (5) and applying Fubini's theorem we obtain (14). □

LEMMA 3. *Let $g:[a,b]\to\mathbb{R}$ and $\lambda:[a,b]\to\mathbb{R}$ be as in Lemma 1. Then for every function $\varphi:[\alpha,\beta]\to\mathbb{R}$ such that $\varphi^{(n-1)}$ is absolutely continuous for some $n\geqslant 1$ the identities*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\sum_{k=1}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}\left[\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^k-(\beta-\alpha)^k+\frac{\int_a^b (g(x)-\alpha)^k\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]$$
$$+\frac{1}{(n-1)!}\int_\alpha^\beta\left[\left(\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-t\right)_+\right)^{n-1}\right.$$
$$\left.-(\beta-t)^{n-1}+\frac{\int_a^b \left((g(x)-t)_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\varphi^{(n)}(t)\,\mathrm{d}t, \tag{17}$$

and

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\sum_{k=1}^{n-1}\frac{\varphi^{(k)}(\beta)}{k!}(-1)^k\left[\left(\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-\alpha\right)^k-(\beta-\alpha)^k+\frac{\int_a^b (\beta-g(x))^k\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]$$
$$-\frac{1}{(n-1)!}\int_\alpha^\beta(-1)^{n-1}\left[\left(\left(t-\alpha-\beta+\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)_+\right)^{n-1}\right.$$
$$\left.-(t-\alpha)^{n-1}+\frac{\int_a^b \left((t-g(x))_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\varphi^{(n)}(t)\,\mathrm{d}t \tag{18}$$

hold.

Proof. Using formula (10) and the facts that $(\alpha-t)_+=0$ and $(\beta-t)_+=\beta-t$

for $t \in [\alpha,\beta]$, we have

$$\begin{aligned}
&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\\
&=\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^k\\
&\quad+\frac{1}{(n-1)!}\int_\alpha^\beta \varphi^{(n)}(t)\left(\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-t\right)_+\right)^{n-1}\mathrm{d}t\\
&\quad-\varphi(\alpha)-\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}(\beta-\alpha)^k-\frac{1}{(n-1)!}\int_\alpha^\beta \varphi^{(n)}(t)(\beta-t)^{n-1}\,\mathrm{d}t\\
&\quad+\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}\frac{\int_a^b (g(x)-\alpha)^k\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\\
&\quad+\frac{1}{(n-1)!}\int_\alpha^\beta \varphi^{(n)}(t)\frac{\int_a^b \left((g(x)-t)_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\mathrm{d}t.
\end{aligned}$$

By regrouping and canceling all the first members in the above sums we obtain (17). Analogously using formula (11) we obtain (18). □

Using Lemmas 2 and 3 we can get the following generalizations of the Jensen-Mercer inequality for n-convex functions.

THEOREM 1. *Let $g:[a,b]\to\mathbb{R}$ be a continuous and monotonic function, and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation satisfying (1), and $\mathscr{G}:[\alpha,\beta]\to\mathbb{R}$ defined by (6). Let $\varphi:[\alpha,\beta]\to\mathbb{R}$ be a n-convex function such that $\varphi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$.*

(i) *Then the inequality*

$$\begin{aligned}
&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\\
&\leqslant\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta \mathscr{G}(s)(s-\alpha)^k\,\mathrm{d}s \qquad (19)
\end{aligned}$$

holds. Moreover, if $\varphi^{(k)}(\alpha)\geqslant 0$ for $k=2,3,\ldots,n-1$, then the right hand side of (19) is negative or equals zero and (2) holds.

(*ii*) *If n is even then the inequality*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$\leqslant \sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k\int_\alpha^\beta \mathscr{G}(s)(\beta-s)^k\,\mathrm{d}s \tag{20}$$

holds. Moreover, if $\varphi^{(k)}(\beta)\geqslant 0$ for $k=2,4,\ldots,n-2$ and $\varphi^{(k)}(\beta)\leqslant 0$ for $k=3,5,\ldots,n-1$, then the right hand side of (20) is negative or equals zero and (2) holds.

(*iii*) *If n is odd then the reversed inequality (20) holds. Moreover, if $\varphi^{(k)}(\beta)\leqslant 0$ for $k=2,4,\ldots,n-1$ and $\varphi^{(k)}(\beta)\geqslant 0$ for $k=3,5,\ldots,n-2$, then the right hand side of the reversed inequality (20) is nonnegative and reverse inequality in (2) holds.*

Proof. Since the Green function G is convex and $G(\alpha,s)=G(\beta,s)=0$, from Theorem A follows

$$\mathscr{G}(s)=G\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)+\frac{\int_a^b G(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\leqslant 0.$$

Hence, if n is even then

$$\int_t^\beta \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\leqslant 0,\quad \text{for } t\leqslant s\leqslant\beta, \tag{21}$$

and

$$\int_\alpha^t \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\geqslant 0,\quad \text{for } \alpha\leqslant s\leqslant t. \tag{22}$$

Also, if n is odd then the inequality in (21) remains the same while the inequality in (22) becomes reversed.

Since the function φ is n-convex, without loss of generality we can assume that φ is n-times differentiable and $\varphi^{(n)}\geqslant 0$ (see [7, p. 16 and p. 293]). Therefore we have

$$\int_\alpha^\beta\left(\int_t^\beta \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\right)\varphi^{(n)}(t)\,\mathrm{d}t\leqslant 0. \tag{23}$$

Analogously, for even n we have

$$\int_\alpha^\beta\left(\int_\alpha^t \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s\right)\varphi^{(n)}(t)\,\mathrm{d}t\geqslant 0, \tag{24}$$

while for odd n we have reversed inequality in (24). Now, applying Lemma 2 we conclude (*i*), (*ii*) and (*iii*). □

REMARK 2. The right hand side of (19) can be written in the form

$$\int_{\alpha}^{\beta} \mathscr{G}(s)\left(\sum_{k=0}^{n-3} \frac{\varphi^{(k+2)}(\alpha)}{k!}(s-\alpha)^{k}\right) \mathrm{d}s.$$

Hence, in case $T_{n-3}(\varphi;\alpha,s)=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}(s-\alpha)^{k}\geqslant 0$ it is negative or equals zero and inequality (2) holds. Similarly, the right hand side of (20) can be written in the form

$$\int_{\alpha}^{\beta} \mathscr{G}(s)\left(\sum_{k=0}^{n-3} \frac{\varphi^{(k+2)}(\beta)}{k!}(s-\beta)^{k}\right) \mathrm{d}s.$$

Therefore, in case n is even and $T_{n-3}(\varphi;\beta,s)\geqslant 0$ inequality (2) holds, while in case n is odd and $T_{n-3}(\varphi;\beta,s)\leqslant 0$ reverse inequality in (2) holds.

THEOREM 2. *Let $g:[a,b]\to\mathbb{R}$ be a continuous and monotonic function, and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation satisfying (1), and $\varphi:[\alpha,\beta]\to\mathbb{R}$ a n-convex function such that $\varphi^{(n-1)}$ is absolutely continuous for some $n\geqslant 1$.*

(*i*) *Then the inequality*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$\leqslant \sum_{k=1}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}\left[\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^{k}-(\beta-\alpha)^{k}+\frac{\int_a^b (g(x)-\alpha)^{k}\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right] \tag{25}$$

holds. Moreover, if $\varphi^{(k)}(\alpha)\geqslant 0$ for $k=2,3,\ldots,n-1$, then the right hand side of (25) is negative or equals zero and (2) holds.

(*ii*) *If n is even then the inequality*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$\leqslant \sum_{k=1}^{n-1}\frac{\varphi^{(k)}(\beta)}{k!}(-1)^{k}\left[\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-\alpha\right)^{k}-(\beta-\alpha)^{k}+\frac{\int_a^b(\beta-g(x))^{k}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right] \tag{26}$$

holds. Moreover, if $\varphi^{(k)}(\beta)\geqslant 0$ for $k=2,4,\ldots,n-2$ and $\varphi^{(k)}(\beta)\leqslant 0$ for $k=3,5,\ldots,n-1$, then the right hand side of (26) is negative or equals zero and (2) holds.

(*iii*) *If n is odd then the reversed inequality* (26) *holds. Moreover, if* $\varphi^{(k)}(\beta) \leqslant 0$ *for* $k = 2,4,\ldots,n-1$ *and* $\varphi^{(k)}(\beta) \geqslant 0$ *for* $k = 3,5,\ldots,n-2$*, then the right hand side of the reversed inequality* (26) *is nonnegative and reverse inequality in* (2) *holds.*

Proof. Since $\left((x-t)_+\right)^{n-1}$ is convex function and $(\alpha - t)_+ = 0$ and $(\beta - t)_+ = \beta - t$ for $t \in [\alpha, \beta]$, from Theorem A follows

$$\left(\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-t\right)_+\right)^{n-1} - (\beta-t)^{n-1} + \frac{\int_a^b \left((g(x)-t)_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leqslant 0.$$

Since the function φ is n-convex, without loss of generality we can assume that φ is n-times differentiable and $\varphi^{(n)} \geqslant 0$. Hence, applying Lemma 3 we conclude (i). Analogously, we conclude (ii) and (iii). $\square$

REMARK 3. In case $T_{n-1}(\varphi;\alpha,x)$ is convex function the right hand side of (25) is negative or equals zero and inequality (2) holds. In case $T_{n-1}(\varphi;\beta,x)$ is convex function and n is even the right hand side of (26) is negative or equals zero and (2) holds. In case $T_{n-1}(\varphi;\beta,x)$ is convex function and n is odd the right hand side of the reversed inequality (26) is nonnegative and reverse inequality in (2) holds.

3. Related results

For two Lebesgue integrable functions $f,h:[\alpha,\beta]\to\mathbb{R}$ the Čebyšev functional is given as

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)\,\mathrm{d}t - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)\,\mathrm{d}t\right)\cdot\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,\mathrm{d}t\right). \quad (27)$$

In paper [3] the following theorems were proved.

THEOREM B. *Let* $f:[\alpha,\beta]\to\mathbb{R}$ *be a Lebesgue integrable and* $h:[\alpha,\beta]\to\mathbb{R}$ *be an absolutely continuous function with* $(\cdot - a)(b - \cdot)[h']^2 \in L[\alpha,\beta]$*. Then the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (t-\alpha)(\beta-t)\left[h'(t)\right]^2\mathrm{d}t\right)^{\frac{1}{2}} \quad (28)$$

holds, where the constant $\frac{1}{\sqrt{2}}$ *is the best possible.*

THEOREM C. *Assume that* $h:[\alpha,\beta]\to\mathbb{R}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $f:[\alpha,\beta]\to\mathbb{R}$ *is absolutely continuous with* $f'\in L_\infty[\alpha,\beta]$*. Then the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{2(\beta-\alpha)}\left\|f'\right\|_\infty\left(\int_\alpha^\beta (t-\alpha)(\beta-t)\,\mathrm{d}h(t)\right)^{\frac{1}{2}} \quad (29)$$

holds, where the constant $\frac{1}{2}$ *is the best possible.*

Proofs of the following theorems utilize the main ideas from the proofs of the similar theorems in [2] and [6], so we omit them here.

For a continuous and monotonic function $g:[a,b]\to\mathbb{R}$, an interval $[\alpha,\beta]$ such that the image of g is subset of $[\alpha,\beta]$, a function $\lambda:[a,b]\to\mathbb{R}$ either continuous or of bounded variation satisfying (1), and function $\mathscr{G}:[\alpha,\beta]\to\mathbb{R}$ defined by (6), let us denote

$$\mathscr{R}(t)=\int_t^\beta \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s, \tag{30}$$

$$\widetilde{\mathscr{R}}(t)=\int_\alpha^t \mathscr{G}(s)(s-t)^{n-3}\,\mathrm{d}s, \tag{31}$$

$$\mathscr{B}(t)=\left(\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}-t\right)_+\right)^{n-1}-(\beta-t)^{n-1}+\frac{\int_a^b\left((g(x)-t)_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, \tag{32}$$

and

$$\widetilde{\mathscr{B}}(t)=\left(\left(t-\alpha-\beta+\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)_+\right)^{n-1}-(t-\alpha)^{n-1}+\frac{\int_a^b\left((t-g(x))_+\right)^{n-1}\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}. \tag{33}$$

Considering the function $\mathscr{R}$ we have the following identity in which the remainder $\mathscr{K}_n$ is estimated by using Theorem B.

THEOREM 3. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous and monotonic function, and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let function* $\lambda:[a,b]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying* (1). *Let function* $\varphi:[\alpha,\beta]\to\mathbb{R}$ *be such that* $\varphi^{(n)}$ *is absolutely continuous for some* $n\geqslant 3$ *with* $(\cdot-a)(b-\cdot)\left[\varphi^{(n+1)}\right]^2\in L[\alpha,\beta]$, *and let the functions* $\mathscr{G}$ *and* $\mathscr{R}$ *be defined by* (6) *and* (30), *respectively. Then the remainder* $\mathscr{K}_n$ *given in the following formula*

$$\begin{aligned}&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\\&=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta \mathscr{G}(s)(s-\alpha)^k\,\mathrm{d}s+\frac{\varphi^{(n-1)}(\beta)-\varphi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta \mathscr{R}(t)\,\mathrm{d}t\\&\quad+\mathscr{K}_n(\varphi;\alpha,\beta)\end{aligned} \tag{34}$$

satisfies the estimation

$$|\mathscr{K}_n(\varphi;\alpha,\beta)|\leqslant\frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left[\Delta(\mathscr{R},\mathscr{R})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)\left[\varphi^{(n+1)}(t)\right]^2\mathrm{d}t\right|^{\frac{1}{2}}. \tag{35}$$

Application of Theorem C gives the following Ostrowsky type inequality.

THEOREM 4. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous and monotonic function, and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let function* $\lambda:[a,b]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying* (1), *and let the functions* $\mathscr{G}$ *and* $\mathscr{R}$ *be defined by* (6) *and* (30), *respectively. Let function* $\varphi:[\alpha,\beta]\to\mathbb{R}$ *be such that* $\varphi^{(n)}$ *is absolutely continuous for some* $n\geqslant 3$ *with* $\varphi^{(n+1)}\geqslant 0$ *on* $[\alpha,\beta]$. *Then the remainder* $\mathscr{K}_n$ *in* (34) *satisfies the estimation*

$$|\mathscr{K}_n(\varphi;\alpha,\beta)|\leqslant\frac{1}{(n-3)!}\left\|\mathscr{R}'\right\|_\infty\left[\frac{\varphi^{(n-1)}(\beta)+\varphi^{(n-1)}(\alpha)}{2}-\frac{\varphi^{(n-2)}(\beta)-\varphi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{36}$$

If the function $\varphi^{(n)}$ belongs to L_p, then we have the following theorem.

THEOREM 5. *Assume* (p,q) *is a pair of conjugate exponents, i.e.* $1\leqslant p,q\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous and monotonic function, and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let function* $\lambda:[a,b]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying* (1), *and let the functions* $\mathscr{G}$ *and* $\mathscr{R}$ *be defined by* (6) *and* (30), *respectively. Let function* $\varphi:[\alpha,\beta]\to\mathbb{R}$ *be such that* $\varphi^{(n-1)}$ *is absolutely continuous and* $\left|\varphi^{(n)}\right|^p$ *is an* R*-integrable for some* $n\geqslant 3$. *Then*

$$\begin{aligned}&\left|\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b\mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b\varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b\mathrm{d}\lambda(x)}\right)\right.\\&\qquad\left.-\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\mathscr{G}(s)(s-\alpha)^k\,\mathrm{d}s\right|\\&\leqslant\frac{1}{(n-3)!}\left(\int_\alpha^\beta|\mathscr{R}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|\varphi^{(n)}\right\|_p,\end{aligned}\tag{37}$$

where the constant on the right-hand side of (37) *is sharp for* $1<p\leqslant\infty$ *and the best possible for* $p=1$.

Analogous identities and Ostrowski type inequalities hold for the other three functions $\widetilde{\mathscr{R}}$, $\mathscr{B}$ and $\widetilde{\mathscr{B}}$.

REMARK 4. We can also obtain other related results, analogous to those in [2] and [6], using the main ideas from [4] and [5]. In particular, we can produce new families of n-exponentially convex and exponentially convex functions, applying functionals, constructed as differences of the right hand side and left hand side of some of the inequalities derived earlier, on some given families with the same property.

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