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NOTES ON THE HERZ–TYPE HARDY SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

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Abstract. The aim of this paper is twofold. First we give a new norm equivalents of the variable Herz spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. Secondly we use these results to prove the atomic decomposition for Herz-type Hardy spaces of variable smoothness and integrability. Also, we prove the boundedness of a wide class of sublinear operators on these spaces, which includes maximal, potential and Calderón-Zygmund operators.

1. Introduction

It is well-known that function spaces have been a central topic in modern analysis, and are now of increasing applications in areas such as harmonic analysis and partial differential equations. Some examples of these spaces can be mentioned such as: Herz spaces. It is well known that these spaces play an important role in Harmonic Analysis. After they have been introduced in [11], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [10] and in regularity theory for elliptic and parabolic equations in divergence form [20], [21].

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [22], image restoration [4] and PDE with non-standard growth conditions.

Herz spaces $K_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and $q \in (0,\infty]$ were recently studied by Izuki [12, 13]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [2], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [14]. Many authors are interested in Herz spaces with variable exponents, for example, [8], [9], [23], [24] and

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[26]. See also [5] and [27] for further results on Herz type Besov and Triebel-Lizorkin spaces with fixed exponents.

The main purpose of this paper is to consider Herz-type Hardy spaces $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where the exponents α and q are variables.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basic facts on function spaces with variable integrability. In the preliminary section we also give some key technical lemmas needed in the proofs of the main statements. We then define the Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in Section 3 and give several basic properties. We show that for some special parameters, the spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are just the Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q_\infty}(\mathbb{R}^n)$ and

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\{2^{k\alpha(0)} f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

Also we define the Herz-type Hardy spaces $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we present the relation between these function spaces and Herz spaces by using the boundedness results for a wide class of classical operators, where for making the presentation clearer, we give the proofs of the boundedness of these class of operators later in Section 5. The main statements are formulated in Section 4, where we give the atomic decomposition of these function spaces.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$. The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$. The expression $f \lesssim g$ means that $f \leqslant c\,g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By $\operatorname{supp} f$ we denote the support of the function f, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The symbol $\mathscr{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $\mathscr{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

The variable exponents that we consider are always measurable functions on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathscr{P}_0$. The subset of variable exponents with range $[1, \infty)$ is denoted by $\mathscr{P}$. For $p \in \mathscr{P}_0(\mathbb{R}^n)$, we use the notation

$$p^+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x), \quad p^- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x).$$

Everywhere below we shall consider bounded exponents.

Let $p \in \mathscr{P}_0(\mathbb{R}^n)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This is a qausi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\Big\{\mu > 0 : \rho_{p(\cdot)}\Big(\frac{1}{\mu} f\Big) \leqslant 1\Big\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

A useful property is that $\rho_{p(\cdot)}(f) \leqslant 1$ if and only if $\|f\|_{p(\cdot)} \leqslant 1$ (*unit ball property*). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case.

We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* $\log$*-Hölder continuous*, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\ln(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \leqslant \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is $\log$*-Hölder continuous at the origin* (or has a log *decay at the origin*). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is $\log$*-Hölder continuous at infinity* (or has a log *decay at infinity*).

By $\mathscr{P}_0^{\ln}(\mathbb{R}^n)$ and $\mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathscr{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathscr{P}^{\ln}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathscr{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x| \to \infty} p(x)$. Obviously we have $\mathscr{P}^{\ln}(\mathbb{R}^n) \subset \mathscr{P}_0^{\ln}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$. Note that $p \in \mathscr{P}^{\ln}(\mathbb{R}^n)$ if and only if $p' \in \mathscr{P}^{\ln}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for any of these quantities.

Let $p, q \in \mathscr{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\Big\{\lambda_v > 0 : \rho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leqslant 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leqslant 1\Big\}. \tag{2.1}$$

Since $q^+ < \infty$, then we can replace (2.1) by the simpler expression $\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$. Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. It is known, cf. [1] and [15], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geqslant 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geqslant 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leqslant 1$ a.e. on $\mathbb{R}^n$, or if $1 \leqslant q(x) \leqslant p(x) < \infty$ a.e. on $\mathbb{R}^n$.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathscr{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{2.2}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B \tag{2.3}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leqslant 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \tag{2.4}$$

for large balls ($|B| \geqslant 1$), with constants only depending on the log-Hölder constant of p (see, for example, [7, Section 4.5]). Here p' denotes the conjugate exponent of p given by $1/p(\cdot) + 1/p'(\cdot) = 1$.

We refer the reader to the recent monograph [7, Section 4.5] for further details, historical remarks and more references on variable exponent spaces.

The following lemma is from [6, Lemma 2.11], see also [17, Lemma 2.6].

LEMMA 1. *Let $p \in \mathscr{P}^{\log}$. For any cubes (balls) P and Q, such that $P \subset Q$, we have*

$$C \left(\frac{|Q|}{|P|} \right)^{1/p^+} \leqslant \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leqslant c \left(\frac{|Q|}{|P|} \right)^{1/p^-}$$

with $c, C > 0$ are independent of $|Q|$ and $|P|$.

The next lemma is a Hardy-type inequality which is easy to prove.

LEMMA 2. *Let $0 < a < 1$ and $0 < q \leqslant \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\| \{\varepsilon_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leqslant k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geqslant k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\| \{\delta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} + \left\| \{\eta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} \leqslant cI,$$

with $c > 0$ only depending on a and q.

The proof of the following results are given in [2], where the second lemma is a generalization of (2.2), (2.3) and (2.4) to the case of dyadic annuli.

LEMMA 3. *Let $\alpha \in L^{\infty}(\mathbb{R}^n)$ and $r_1 > 0$. If α is* log*-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \text{if } \; 0 < r_2 \leqslant \frac{r_1}{2} \\ 1 & \text{if } \frac{r_1}{2} < r_2 \leqslant 2r_1 \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \text{if } \quad r_2 > 2r_1 \end{cases}$$

for any $x \in B(0,r_1) \setminus B(0,\frac{r_1}{2})$ and $y \in B(0,r_2) \setminus B(0,\frac{r_2}{2})$, with the implicit constant not depending on x, y, r_1 and r_2.

LEMMA 4. *Let $p \in \mathscr{P}_{\infty}^{\ln}(\mathbb{R}^n)$ and let $R = B(0,r) \setminus B(0,\frac{r}{2})$. If $|R| \geqslant 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.
The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathscr{P}_0^{\ln}(\mathbb{R}^n) \cap \mathscr{P}_{\infty}^{\ln}(\mathbb{R}^n)$.

3. Variable Herz-type Hardy spaces

For convenience, we set

$$B_k := B(0,2^k), \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

DEFINITION 1. Let $p,q \in \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^{\infty}(\mathbb{R}^n)$. The *inhomogeneous Herz space* $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\chi_{B_0}\|_{p(\cdot)} + \left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\geqslant 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{3.1}$$

Similarly, the *homogeneous Herz space* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{3.2}$$

If α and p,q are constant, then $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} := \left(\sum_{k=0}^{\infty} \|g_k\|^q_{p(\cdot)}\right)^{1/q} \quad \text{and} \quad \|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} := \left(\sum_{k=-\infty}^{-1} \|g_k\|^q_{p(\cdot)}\right)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification if $q = \infty$).

Now we present the main result of this section.

PROPOSITION 1. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p,q \in \mathscr{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\{2^{k\alpha(0)} f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}. \tag{3.3}$$

Proof. Step 1. We will prove that $K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n) \hookrightarrow K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which is equivalent to $\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{K^{\alpha_\infty}_{p(\cdot),q_\infty}}$ for any $f \in K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K^{\alpha_\infty}_{p(\cdot),q_\infty}} = 1$ and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha(\cdot)} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1 \tag{3.4}$$

for some constant $c>0$. Since α has logarithmic decay at infinity, then for $k \geqslant 1$ and $x \in R_k$ we have

$$k|\alpha(x)-\alpha_\infty| \lesssim \frac{k}{\ln(e+|x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$ with constants independent of k and x, and hence

$$\sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha(\cdot)} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \approx \sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha_\infty} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Our estimate (3.4), clearly follows from the inequality

$$\left\| \left| c\, 2^{k\alpha_\infty} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \left\| 2^{k\alpha_\infty} f\chi_k \right\|^{q_\infty}_{p(\cdot)} + 2^{-k} = \delta. \tag{3.5}$$

This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| c\, 2^{k\alpha_\infty} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\left\| c\, \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\chi_k \right\|_{p(\cdot)} \leqslant 1.$$

We have for any $x \in R_k$

$$\delta^{-\frac{1}{q(x)}} = (2^k\delta)^{\frac{1}{q_\infty}-\frac{1}{q(x)}} 2^{k(\frac{1}{q(x)}-\frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}}.$$

Since q has logarithmic decay at infinity, then for $k \geqslant 1$ and $x \in R_k$ we have

$$\frac{k|q(x)-q_\infty|}{q_\infty q(x)} \leqslant \frac{k|q(x)-q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\ln(e+|x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty}-\frac{1}{q(x)})} \approx 1$ with constants independent of k and x. Also, since $1 < 2^k\delta < 2^{k+1}$,

$$(2^k\delta)^{\frac{1}{q_\infty}-\frac{1}{q(x)}} \leqslant (2^{k+1})^{|\frac{1}{q_\infty}-\frac{1}{q(x)}|} \lesssim 1.$$

Hence, with an appropriate choice of $c > 0$

$$\left\| c\,\delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\chi_k \right\|_{p(\cdot)} \leqslant \left\| \delta^{-\frac{1}{q_\infty}} 2^{k\alpha_\infty} f\chi_k \right\|_{p(\cdot)} \leqslant 1,$$

because of $\|2^{k\alpha_\infty} f\chi_k\|_{p(\cdot)} \leqslant \delta^{\frac{1}{q_\infty}}$.

Step 2. We will prove that $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n)$, which is equivalent to $\|f\|_{K^{\alpha_\infty}_{p(\cdot),q_\infty}} \lesssim \|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ for any $f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\sum_{k=1}^{\infty} \left\| c\,2^{k\alpha_\infty} f\chi_k \right\|_{p(\cdot)}^{q_\infty} \lesssim 1 \tag{3.6}$$

for some constant $c > 0$. As before, we have for $k \geqslant 1$

$$\left\|2^{k\alpha_\infty} f\chi_k\right\|_{p(\cdot)}^{q_\infty} \lesssim \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}^{q_\infty}.$$

Now, our estimate (3.6), clearly follows from the inequality

$$\left\| c\,2^{k\alpha(\cdot)} f\chi_k \right\|_{p(\cdot)}^{q_\infty} \leqslant \left\| \left|2^{k\alpha(\cdot)} f\chi_k\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{3.7}$$

This claim can be reformulated as showing that

$$\left\| c\,\delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\chi_k \right\|_{p(\cdot)} \leqslant 1.$$

From above, $\delta^{-\frac{1}{q_\infty}} \lesssim \delta^{-\frac{1}{q(x)}}$, then with an appropriate choice of $c > 0$

$$\left\| c\,\delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\chi_k \right\|_{p(\cdot)} \leqslant \left\| \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\chi_k \right\|_{p(\cdot)}.$$

The left-hand side is less than or equal 1 if and only if

$$\left\| \left| \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1.$$

We see that the right-hand side can be rewritten us

$$\delta^{-1}\left\|\left|2^{k\alpha(\cdot)}f\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\leqslant 1$$

which follows immediately from the definition of δ.

Step 3. Let us prove that

$$\|\{2^{k\alpha(0)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}+\|\{2^{k\alpha_\infty}f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\lesssim\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We suppose that $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\leqslant 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)}\approx 2^{k\alpha(0)}$ for $k<0$ and $x\in R_k$. Thus

$$\|\{2^{k\alpha(0)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}\approx\|\{2^{k\alpha(\cdot)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}.$$

As in Step 2 we can prove that

$$\left\|c\,2^{k\alpha(\cdot)}f\chi_k\right\|_{p(\cdot)}^{q(0)}\leqslant\left\|\left|2^{k\alpha(\cdot)}f\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}+2^k$$

for any $k<0$ and for some constant $c>0$. Then $\|\{2^{k\alpha(0)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}\lesssim 1$. Using the estimate (3.7) we obtain $\|\{2^{k\alpha_\infty}f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\lesssim 1$. Therefore,

$$\|\{2^{k\alpha(0)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}+\|\{2^{k\alpha_\infty}f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\lesssim 1.$$

The desired estimate can be obtained by the scaling argument.

Now let $\|\{2^{k\alpha(0)}f\chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}\leqslant 1$ and $\|\{2^{k\alpha_\infty}f\chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\leqslant 1$. As in Step 1 we have for any $k<0$ and for some constant $c>0$

$$\left\|\left|c\,2^{k\alpha(\cdot)}f\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\leqslant\left\|2^{k\alpha(0)}f\chi_k\right\|_{p(\cdot)}^{q(0)}+2^k$$

and using (3.5) we obtain

$$\sum_{k=-\infty}^{\infty}\left\|\left|2^{k\alpha(\cdot)}f\chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\lesssim 1.$$

Therefore, $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim 1$ and hence the result follows by the scaling argument. $\square$

Let G_Nf be the grand maximal function of f defined by

$$G_Nf(x)=\sup_{\varphi\in\mathscr{A}_N}|\varphi^*_N(f)(x)|,$$

where $\mathscr{A}_N=\{\varphi\in\mathscr{S}(\mathbb{R}^n):\sup_{|\alpha|\leqslant N,|\beta|\leqslant N}|x^\alpha\partial^\beta\varphi(x)|\leqslant 1\}$ and $\varphi^*_N(f)(x)=\sup_{t>0}|\varphi_t*f(x)|$, with $\varphi_t=t^{-n}\varphi(\frac{\cdot}{t})$.

DEFINITION 2. Let $p,q \in \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The *inhomogeneous Herz-type Hardy space* $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $G_N f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define $\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

Similarly, the *homogeneous Herz-type Hardy space* $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define $\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

We consider sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \operatorname{supp} f \tag{3.8}$$

for integrable and compactly supported functions f. Condition (3.8) is satisfied by several classical operators in Harmonic Analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [18] and [25]).

Using the same of arguments as in [2], we obtain the following results.

THEOREM 1. *Let $q \in \mathscr{P}_0(\mathbb{R}^n)$, $p \in \mathscr{P}^{\ln}_\infty(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, and let α and q are* log*-Hölder continuous at infinity, with $\alpha \in L^\infty(\mathbb{R}^n)$ and*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}.$$

Suppose that T is a sublinear operator satisfying estimate (3.8). If T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then T is bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

For homogeneous spaces we have the following statement:

THEOREM 2. *Let $q \in \mathscr{P}_0(\mathbb{R}^n)$, $p \in \mathscr{P}^{\ln}_0(\mathbb{R}^n) \cap \mathscr{P}^{\ln}_\infty(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, and let α and q are* log*-Hölder continuous, both at the origin and at infinity, such that $\alpha \in L^\infty(\mathbb{R}^n)$ and*

$$-\frac{n}{p^+} < \alpha^- \leqslant \alpha^+ < n\left(1 - \frac{1}{p^-}\right).$$

Then every sublinear operator T satisfying (3.8) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The proof of Theorems 1 and 2 is postponed to the Appendix.

We have G_N satisfies the size condition (3.8). Let α, q, p as in Theorem 1, with $p \in \mathscr{P}^{\ln}(\mathbb{R}^n)$. Then

$$HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) = K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Let α, q, p as in Theorem 2, with $p \in \mathscr{P}^{\ln}(\mathbb{R}^n)$. Then

$$H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) = \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

4. Atomic decomposition

In recent years, it turned out that atomic decomposition of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of some operators, such as Calderón-Zygmund operators, the commutator of Calderón-Zygmund operator with a BMO function and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. The main goal of this section is to prove an atomic decomposition result for $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. First we introduce the basic notation.

DEFINITION 3. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathscr{P}(\mathbb{R}^n)$, $q \in \mathscr{P}_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot),p(\cdot))$-atom, if

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leqslant r\}$, $r > 0$.

(ii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha(0)/n}$, $0 < r < 1$.

(iii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha_\infty/n}$, $r \geqslant 1$.

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$, $|\beta| \leqslant s$.

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot),p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\operatorname{supp} a \subset B(0,r)$, $r \geqslant 1$.

Now we come to the atomic decomposition theorems.

THEOREM 3. *Let α and q are log-Hölder continuous at infinity and $p \in \mathscr{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{4.1}$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom of restricted type with $\operatorname{supp} a \subset B_k$ and

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Conversely, if $\alpha_\infty \geqslant n(1-\frac{1}{p_\infty})$ and $s \geqslant [\alpha_\infty + n(\frac{1}{p_\infty}-1)]$, and if (4.1) holds, then $f \in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \inf\left\{\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

THEOREM 4. *Let α and q are be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathscr{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{4.2}$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha(\cdot) \geqslant n(1 - \frac{1}{p^-})$ and $s \geqslant [\alpha^+ + n(\frac{1}{p^-} - 1)]$, and if (4.2) holds, then $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

REMARK 1. Corresponding statements to Theorems 3 and 4 were proved by Liu and Wang [19], with α and q constants, under the assumption that the maximal operator $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).

Proof. By similarity, we only consider $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. The proof follows the ideas in [18], see also [19].

Step 1. We prove the necessity part of the theorem. Let $f \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{D}(\mathbb{R}^n)$ such that $\varphi \geqslant 0$, $\int \varphi(x)dx = 1$ and $\operatorname{supp}\varphi \subset \overline{B_0}$. Set $\varphi_j = 2^{jn}\varphi(2^j\cdot)$ and $f_j = f * \varphi_j$, $j \in \mathbb{N}_0$. It is well know that $f_j \in C^\infty(\mathbb{R}^n)$ and $\lim_{j\to\infty} f_j = f$. Let ψ be a radial smooth function such that $\operatorname{supp}\psi \subset \{x : \frac{1}{2} - \varepsilon \leqslant |x| \leqslant 1 + \varepsilon\}$, $0 < \varepsilon < \frac{1}{4}$, and $\psi(x) = 1$, if $\frac{1}{2} \leqslant |x| \leqslant 1$. Let $\widetilde{R_{k,\varepsilon}} = \{x : 2^{k-1} - 2^k\varepsilon \leqslant |x| \leqslant 2^k + 2^k\varepsilon\}$ and $\psi_k = \psi(2^{-k}\cdot)$. It is easy to see that $\operatorname{supp}\psi_k \subset \widetilde{R_{k,\varepsilon}}$, $\psi_k(x) = 1$ if $x \in R_k$ and $1 \leqslant \sum_{k=-\infty}^{\infty} \psi_k(x) \leqslant 2$, $x \neq 0$. Set $\Phi_k(x) = \begin{cases} \frac{\psi_k(x)}{\sum_{j=-\infty}^{\infty}\psi_j(x)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$, $x \neq 0$. We denote by $\mathscr{P}_m$ the class of all the real polynomials with the degree less than m. Let $P_k^{(j)}(x) = P_{\widetilde{R_{k,\varepsilon}}}(f_j\Phi_k)(x)\chi_{\widetilde{R_{k,\varepsilon}}}(x) \in \mathscr{P}_m$ be the unique polynomial satisfying

$$\int_{\widetilde{R_{k,\varepsilon}}} \left(f_j(x)\Phi_k(x) - P_k^{(j)}(x)\right) x^\beta dx = 0, \quad |\beta| \leqslant \left[\alpha^+ + n\left(\frac{1}{p^-} - 1\right)\right] = m.$$

Observe that

$$\begin{aligned} f_j(x) &= \sum_{k=-\infty}^{\infty} \left(f_j(x)\Phi_k(x) - P_k^{(j)}(x) \right) + \sum_{k=-\infty}^{\infty} P_k^{(j)}(x) \\ &= I_j + II_j. \end{aligned}$$

For I_j, let $g_k^{(j)}(x) = f_j(x)\Phi_k(x) - P_k^{(j)}(x)$ and $a_k^{(j)} = \frac{g_k^{(j)}}{\lambda_k}$, where

$$\lambda_k = b \sum_{l=k-1}^{k+1} \left\| |B_{k+1}|^{\alpha(\cdot)/n} (G_N f) \chi_l \right\|_{p(\cdot)}$$

and b is a constant which will be chosen later. It is easy to see that $\operatorname{supp} g_k^{(j)} \subset \widetilde{R_{k,\varepsilon}} \subset B_{k+1}$ and $I_j = \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(j)}$.

Let us prove that $a_k^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom. Let $\{\varphi_d^k : |d| \leqslant m\}$ be the orthogonal polynomials restricted to $\widetilde{R_{k,\varepsilon}}$ with respect to the weight $\frac{1}{|\widetilde{R_{k,\varepsilon}}|}$, which are obtained from $\{x^\beta : |\beta| \leqslant m\}$ by Gram-Schmidt's method, that is

$$\langle \varphi_d^k, \varphi_v^k \rangle = \frac{1}{|\widetilde{R_{k,\varepsilon}}|} \int_{\widetilde{R_{k,\varepsilon}}} \varphi_d^k(x) \varphi_v^k(x) dx = \delta_{dv}.$$

Therefore, $P_k^{(j)}(x) = \sum_{|d| \leqslant m} \langle f_j \Phi_k, \varphi_d^k \rangle \varphi_d^k(x)$ for $x \in \widetilde{R_{k,\varepsilon}}$. Observe that $\varphi_v^k(x) = \varphi_v^1(2^{k-1}x)$ a.e for $x \in \widetilde{R_{k,\varepsilon}}$, by the Hölder inequality

$$\begin{aligned} |B_{k+1}|^{\alpha(x)/n} |P_k^{(j)}(x)| &\leqslant \frac{c}{|\widetilde{R_{k,\varepsilon}}|} \int_{\widetilde{R_{k,\varepsilon}}} |B_{k+1}|^{\alpha(x)/n} |f_j(y)| |\Phi_k(y)| dy \\ &\leqslant \frac{c \left\| |B_{k+1}|^{\alpha(x)/n} f_j \Phi_k \right\|_{p(\cdot)} \left\| \chi_{\widetilde{R_{k,\varepsilon}}} \right\|_{p'(\cdot)}}{|\widetilde{R_{k,\varepsilon}}|}. \end{aligned}$$

Using Lemma 3 to obtain $|B_{k+1}|^{\alpha(x)/n} \approx 2^{k\alpha(x)} \approx 2^{k\alpha(y)}$ for any $x, y \in \widetilde{R_{k,\varepsilon}}$. Hence

$$\begin{aligned} &\left\| |B_{k+1}|^{\alpha(\cdot)/n} g_k^{(j)} \right\|_{p(\cdot)} \\ &\leqslant c \left\| |B_{k+1}|^{\alpha(\cdot)/n} f_j \Phi_k \right\|_{p(\cdot)} + \frac{c}{|\widetilde{R_{k,\varepsilon}}|} \left\| |B_{k+1}|^{\alpha(\cdot)/n} f_j \Phi_k \right\|_{p(\cdot)} \left\| \chi_{\widetilde{R_{k,\varepsilon}}} \right\|_{p'(\cdot)} \left\| \chi_{\widetilde{R_{k,\varepsilon}}} \right\|_{p(\cdot)} \\ &\leqslant c \left\| |B_{k+1}|^{\alpha(\cdot)/n} f_j \Phi_k \right\|_{p(\cdot)} \leqslant C \sum_{l=k-1}^{k+1} \left\| |B_{k+1}|^{\alpha(\cdot)/n} G_N(f) \chi_l \right\|_{p(\cdot)}, \end{aligned}$$

where we used the fact that $\left\| \chi_{\widetilde{R_{k,\varepsilon}}} \right\|_{p'(\cdot)} \left\| \chi_{\widetilde{R_{k,\varepsilon}}} \right\|_{p(\cdot)} \approx |\widetilde{R_{k,\varepsilon}}|$, see Lemma 4. Choose $b = C$, we obtain

$$\left\| |B_{k+1}|^{\alpha(\cdot)/n} a_k^{(j)} \right\|_{p(\cdot)} \leqslant c.$$

This relation is equivalent to the inequalities (ii) and (iii) in Definition 3 and hence each $a_k^{(j)}$ is a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_{k+1}. Furthermore, since $|B_{k+1}|^{\alpha(x)/n} \approx 2^{k\alpha(x)} \approx 2^{l\alpha(x)} \approx 2^{l\alpha(0)}$ for any $x \in R_l$ with $k \leqslant -1$ and $k-1 \leqslant l \leqslant k+1$,

$$\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \leqslant C \sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{k+1} \left\| |B_{k+1}|^{\alpha(\cdot)/n} (G_N f)\, \chi_l \right\|_{p(\cdot)} \right)^{q(0)}$$

$$\leqslant C \sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{k+1} \left\| 2^{l\alpha(0)} (G_N f)\, \chi_l \right\|_{p(\cdot)} \right)^{q(0)} \leqslant c \|G_N f\|^{q(0)}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

which is the desired estimate. Similarly, there exists a constant $c > 0$ such that

$$\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \leqslant c \|G_N f\|^{q_\infty}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Here we use the fact that $|B_{k+1}|^{\alpha(x)/n} \approx 2^{k\alpha(x)} \approx 2^{l\alpha(x)} \approx 2^{l\alpha_\infty}$ for any $x \in R_l$ with $k \geqslant 0$ and $k-1 \leqslant l \leqslant k+1$.

It remains to estimate II_j. Let $\{\psi_d^k : |d| \leqslant m\}$ be the dual basis of $\{x^\beta : |\beta| \leqslant m\}$ with respect to the weight $\frac{1}{|\widetilde{R_{k,\varepsilon}}|}$, on $\widetilde{R_{k,\varepsilon}}$, that is

$$\langle \psi_d^k, x^\beta \rangle = \frac{1}{|\widetilde{R_{k,\varepsilon}}|} \int_{\widetilde{R_{k,\varepsilon}}} \psi_d^k(x) x^\beta \, dx = \delta_{\beta d}.$$

Let

$$h_{k,d}^{(j)}(x) = \sum_{l=-\infty}^{k} \left(\frac{\psi_d^k(x)\chi_{\widetilde{R_{k,\varepsilon}}}(x)}{|\widetilde{R_{k,\varepsilon}}|} - \frac{\psi_d^{k+1}(x)\chi_{\widetilde{R_{k+1,\varepsilon}}}(x)}{|\widetilde{R_{k+1,\varepsilon}}|} \right) \int_{\mathbb{R}^n} f_j(x)\Phi_l(x) x^d \, dx.$$

Therefore,

$$\begin{aligned} II_j &= \sum_{k=-\infty}^{\infty} \sum_{|d|\leqslant m} \langle f_j \Phi_k, x^d \rangle \varphi_d^k(x) \chi_{\widetilde{R_{k,\varepsilon}}}(x) \\ &= \sum_{|d|\leqslant m} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{k} \langle f_j \Phi_l, x^d \rangle \left(\frac{\psi_d^k(x)\chi_{\widetilde{R_{k,\varepsilon}}}(x)}{|\widetilde{R_{k,\varepsilon}}|} - \frac{\psi_d^{k+1}(x)\chi_{\widetilde{R_{k+1,\varepsilon}}}(x)}{|\widetilde{R_{k+1,\varepsilon}}|} \right) \\ &= \sum_{|d|\leqslant m} \sum_{k=-\infty}^{\infty} h_{k,d}^{(j)}(x) = \sum_{|d|\leqslant m} \sum_{k=-\infty}^{\infty} \sigma_{k,d} a_{k,d}^{(j)}(x), \end{aligned}$$

where

$$a_{k,d}^{(j)}(x) = \frac{h_{k,d}^{(j)}(x)}{\sigma_{k,d}}$$

and

$$\sigma_{k,d} = \tilde{b} \sum_{l=k-1}^{k+2} \left\| |B_{k+2}|^{\alpha(\cdot)/n} (G_N f)\, \chi_l \right\|_{p(\cdot)},$$

where $\tilde{b} > 0$ is a constant which will be chosen later. Observe that

$$\left| \sum_{l=-\infty}^{k} \int_{\mathbb{R}^n} f_j(y)\Phi_l(y)y^d dy \right| \leqslant c\, 2^{k(n+|d|)} (G_N f)(x), \quad x \in B_{k+2}$$

and

$$\left| \frac{\psi_d^k(x)\chi_{\widetilde{R_{k,\varepsilon}}}(x)}{|\widetilde{R_{k,\varepsilon}}|} - \frac{\psi_d^{k+1}(x)\chi_{\widetilde{R_{k+1,\varepsilon}}}(x)}{|\widetilde{R_{k+1,\varepsilon}}|} \right| \leqslant c\, 2^{-k(n+|d|)} \sum_{l=k-1}^{k+1} \chi_l(x),$$

then

$$\left\| |B_{k+2}|^{\alpha(\cdot)/n} h_{k,d}^{(j)} \right\|_{p(\cdot)} \leqslant C \sum_{l=k-1}^{k+2} \left\| |B_{k+2}|^{\alpha(\cdot)/n} (G_N f)\chi_l \right\|_{p(\cdot)}.$$

Thus if we take $\tilde{b} = C$, then $a_{k,d}^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in $\widetilde{R_{k,\varepsilon}} \cup \widetilde{R_{k+1,\varepsilon}} \subset B_{k+2}$. Furthermore, since $|B_{k+1}|^{\alpha(x)/n} \approx 2^{k\alpha(x)} \approx 2^{l\alpha(x)} \approx 2^{l\alpha(0)}$ for any $x \in R_l$ with $k \leqslant -1$ and $k-1 \leqslant l \leqslant k+2$,

$$\sum_{k=-\infty}^{-1} |\sigma_{k,d}|^{q(0)} \leqslant C \sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{k+1} \left\| 2^{l\alpha(0)} (G_N f)\chi_l \right\|_{p(\cdot)} \right)^{q(0)} \leqslant c \|G_N f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}$$

and

$$\sum_{k=0}^{\infty} |\sigma_{k,d}|^{q_\infty} \leqslant c \|G_N f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q_\infty},$$

by Proposition 1. Hence f_j can be written as

$$f_j = \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(j)}$$

where $a_k^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in $\widetilde{R_{k,\varepsilon}} \cup \widetilde{R_{k+1,\varepsilon}} \subset B_{k+2}$, λ_k is independent of j and

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \leqslant c \|G_N f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},$$

where c is independent of j and f.

Using the Banach-Alaoglu theorem and the usual diagonal method, we can fined a subsequence $\{j_v\}$ of $\mathbb{N}$ such that for each $k \in \mathbb{Z}$, $\lim_{v\to\infty} a_k^{(j_v)} = a_k$ in $\mathscr{S}'(\mathbb{R}^n)$, which is a central $(\alpha(\cdot), p(\cdot))$-atom supported on B_{k+2}. Now it remain to prove that

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \tag{4.3}$$

in $\mathscr{S}'(\mathbb{R}^n)$. For each $\varphi \in \mathscr{S}(\mathbb{R}^n)$, note that $\operatorname{supp} a_k^{(j_v)} \subset \cup_{i=k-1}^{k+2} R_i = \tilde{R}_k$ and

$$\langle f, \varphi \rangle = \lim_{v\to\infty} \sum_{k=-\infty}^{\infty} \lambda_k \int a_k^{(j_v)}(x)\varphi(x)dx.$$

It will know that, see [19]

$$\left|\int a_k^{(j_\nu)}(x)\varphi(x)dx\right| \leqslant c\, 2^{k(m+1)}\left\|a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_1.$$

This term is bounded by

$$\begin{aligned}
&c\, 2^{k(m+1-\alpha(0))}\left\|2^{k\alpha(\cdot)}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_1\\
&\leqslant c\, 2^{k(m+1-\alpha(0))}\left\|2^{k\alpha(\cdot)}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_k}\right\|_{p'(\cdot)}\\
&\leqslant c\, 2^{k(m+1-\alpha(0)+n(1-\frac{1}{p^-})}\left\|2^{k\alpha(\cdot)}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}, \quad k\leqslant 0\\
&\leqslant c\, 2^{k(m+1-\alpha(0)+n(1-\frac{1}{p^-}))}
\end{aligned}$$

where we have used successively $2^{k\alpha(\cdot)}\approx 2^{k\alpha(0)}$, $k<0$, Hölder's inequality and Lemma 4. If $k>0$, let $k_0\in\mathbb{N}$ such that $k_0+\alpha_\infty-n+\frac{n}{p_\infty}>0$, then by the fact that $2^{k\alpha(\cdot)}\approx 2^{k\alpha_\infty}$, $k>0$, Hölder inequality and Lemma 4 we have

$$\begin{aligned}
\left|\int a_k^{(j_\nu)}(x)\varphi(x)dx\right| &\leqslant c\int_{\tilde{R}_k}|a_k^{(j_\nu)}(x)||x|^{-k_0}dx\\
&\leqslant c\, 2^{-k(k_0+\alpha_\infty)}\left\|2^{k\alpha(\cdot)}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_k}\right\|_{p'(\cdot)}\\
&\leqslant c\, 2^{-k(k_0+\alpha_\infty-n+\frac{n}{p_\infty})},
\end{aligned}$$

where $c>0$ is independent of k. If we set

$$b_k=\begin{cases}\lambda_k 2^{k(m+1-\alpha(0)+n(1-\frac{1}{p^-}))} & \text{if } k\leqslant 0\\ \lambda_k 2^{k(n-k_0-\alpha_\infty-\frac{n}{p_\infty})} & \text{if } k>0,\end{cases}$$

then

$$\sum_{k=-\infty}^{\infty}|b_k|\leqslant c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}+\left(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\right)^{1/q_\infty}\leqslant c\|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}<\infty.$$

Therefore,

$$\langle f,\varphi\rangle=\sum_{k=-\infty}^{\infty}\lambda_k\int a_k(x)\varphi(x)dx.$$

This means that (4.3) holds in the sense of distribution.

Step 2. We prove the sufficiency part of the theorem. In view of Proposition 1, it suffices to estimate

$$\|\{2^{k\alpha(0)}(G_N f)\chi_k\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})} \quad\text{and}\quad \|\{2^{k\alpha_\infty}(G_N f)\chi_k\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}.$$

For any $k<0$,

$$2^{k\alpha(0)}\|(G_N f)\chi_k\|_{p(\cdot)} \leqslant \sum_{l=-\infty}^{\infty}|\lambda_l|2^{k\alpha(0)}\|(G_N a_l)\chi_k\|_{p(\cdot)} = \sum_{l=-\infty}^{k-2}\cdots+\sum_{l=k-1}^{-1}\cdots+\sum_{l=0}^{\infty}\cdots. \tag{4.4}$$

By the $L^{p(\cdot)}$ boundedness of the grand maximal operator G_N, the third term is bounded by

$$\begin{aligned} c\,2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\|a_l\|_{p(\cdot)} &\lesssim \sum_{l=0}^{\infty}|\lambda_l|2^{k\alpha(0)-l\alpha^-}\|2^{l\alpha(\cdot)}a_l\|_{p(\cdot)} \\ &\lesssim 2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|2^{-l\alpha^-} \\ &\lesssim 2^{k\alpha(0)}\left(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty}\right)^{1/q_\infty}. \end{aligned}$$

The $\ell_<^{q(0)}$ of this expression is bounded by $(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty})^{1/q_\infty}$. Now the second term is bounded

$$c\sum_{l=k-1}^{-1}|\lambda_l|2^{(k-l)\alpha(0)}\|2^{l\alpha(\cdot)}a_l\|_{p(\cdot)} \lesssim \sum_{l=k-1}^{-1}|\lambda_l|2^{(k-l)\alpha(0)}.$$

Here we use that fact that $2^{l\alpha(0)}\approx 2^{l\alpha(y)}$, $y\in B_l$ and $l\leqslant 0$, since

$$-l|\alpha(x)-\alpha(0)|\leqslant \frac{-cl}{\log(e+\frac{1}{|x|})}\leqslant c, \qquad x\in B_l, l\leqslant 0.$$

Again, the $\ell_<^{q(0)}$ of this expression is bounded by $\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}$ (by Lemma 2). Using the similar arguments used in [19] and, again, the fact that $2^{l\alpha(0)}\approx 2^{l\alpha(y)}$ ($y\in B_l$ and $l\leqslant 0$), we obtain

$$\begin{aligned} 2^{k\alpha(0)}G_N a_l(x) &\leqslant c\,2^{l(m+1)+k\alpha(0)}|x|^{-(n+m+1)}\|a_l\|_1, \quad x\in R_k \\ &\leqslant c\,2^{(k-l)\alpha^+}2^{l(m+1)-k(n+m+1)}\|2^{l\alpha(\cdot)}a_l\|_1, \quad k\geqslant l+2. \end{aligned}$$

Applying Hölder's inequality and the fact that a_l is a central $(\alpha(\cdot),p(\cdot))$-atom, we obtain

$$\begin{aligned} \left\|2^{k\alpha(0)}(G_N a_l)\chi_k\right\|_{p(\cdot)} &\leqslant c\,2^{(k-l)\alpha^+}2^{l(m+1)-k(n+m+1)}\|\chi_{B_l}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} \\ &\leqslant c\,2^{(k-l)\alpha^+}2^{l(m+1)-k(n+m+1)}\|\chi_{B_l}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} \\ &\leqslant c\,2^{(l-k)(m+1+n-\alpha^+-\frac{n}{p^-})}, \end{aligned}$$

where in the last estimate we have used Lemma 1. Hence the first sum in (4.4) is bounded by

$$c\sum_{l=-\infty}^{k-2}|\lambda_l|2^{(l-k)(m+1+n-\alpha^+-\frac{n}{p^-})}.$$

Therefore, the $\ell_<^{q(0)}$ of this expression is bounded by $\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}$.

Again, for any $k \geqslant 0$, we have

$$\begin{aligned}2^{k\alpha_\infty}\|(G_N f)\chi_k\|_{p(\cdot)} &\leqslant \sum_{l=-\infty}^{\infty}|\lambda_l|2^{k\alpha_\infty}\|(G_N a_l)\chi_k\|_{p(\cdot)}\\ &= \sum_{l=-\infty}^{-1}\cdots+\sum_{l=0}^{k-2}\cdots+\sum_{l=k-1}^{\infty}\cdots.\end{aligned}$$

As before, we have

$$2^{k\alpha_\infty}G_N a_l(x) \lesssim 2^{(l-k)(m+1+n-\alpha^+-\frac{n}{p^-})}, \quad k\geqslant 0>l.$$

Hence the first sum is bounded by

$$\begin{aligned}&c\,2^{-k(m+1+n-\alpha^+-\frac{n}{p^-})}\sum_{l=-\infty}^{-1}|\lambda_l|2^{l(m+1+n-\alpha^+-\frac{n}{p^-})}\\ &\lesssim 2^{-k(m+1+n-\alpha^+-\frac{n}{p^-})}\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}.\end{aligned}$$

Since again, $m+1+n-\alpha^+-\frac{n}{p^-}>0$, the $\ell_>^{q_\infty}$-norm of this expression is bounded by

$$c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}.$$

The remaining terms they are essentially similar to the estimate of the first term and the second term in (4.4). Hence the theorem is proved. □

REMARK 2. In the necessity part of the theorem, the atoms in the decompositions (4.1) and (4.2) can be taken to be supported in dyadic annuli. Also the assumption $p\in\mathscr{P}^{\ln}(\mathbb{R}^n)$ can be replaced by $p\in\mathscr{P}_0^{\ln}(\mathbb{R}^n)\cap\mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$, respectively $p\in\mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$, in the $H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ spaces , respectively $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ spaces.

5. Appendix

Here we present the more technical proofs of the Theorems 1 and 2. We omit the proof of Theorem 1 since is essentially similar to the proof of Theorem 4.2 in [2], then we need only to prove the Theorem 2. Our proofs use partially some decomposition techniques already used in [16] and [2]. In view of Proposition 1 we use the property (3.3). We split the operator into

$$|Tf(x)| \leqslant \left|T\left(f\chi_{B_{k-2}}\right)(x)\right| + \left|T(f\chi_{\widetilde{R}_k})(x)\right| + \left|T\left(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\right)(x)\right|,$$

where $\widetilde{R}_k := \left\{x \in \mathbb{R}^n : 2^{k-2} \leqslant |x| < 2^{k+2}\right\}$ with $k \in \mathbb{Z}$ and $x \in R_k$.

Estimation of $T\left(f\chi_{B_{k-2}}\right)$. We have

$$2^{k\alpha(0)}\left|T\left(f\chi_{B_{k-2}}\right)(x)\right| \lesssim 2^{k\alpha(0)} \int_{B_{k-2}} |x-y|^{-n}|f(y)|\,dy = 2^{k\alpha(0)} \sum_{j=-\infty}^{k-2} \int_{R_j} |x-y|^{-n}|f(y)|\,dy$$

for any $x \in R_k$, $k < 0$. To estimate the last integral we note that $|x-y| \geqslant |x| - |y| > \frac{2^k}{4}$ and $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ if $x \in R_k$. Hence by Lemma 3 we arrive at the inequality

$$2^{k\alpha(0)}\left|T\left(f\chi_{B_{k-2}}\right)(x)\right| \lesssim \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha^+ - kn} \int_{R_j} 2^{j\alpha(y)}|f(y)|\,dy$$

for any $k < 0$. After applying Hölder's inequality to the last integral, we get

$$\left\|2^{k\alpha(0)}T\left(f\chi_{B_{k-2}}\right)\chi_k\right\|_{p(\cdot)} \lesssim \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha^+ - kn}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \|\chi_k\|_{p(\cdot)}.$$

Since $p \in \mathscr{P}_0^{\ln}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$ implies $p' \in \mathscr{P}_0^{\ln}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\ln}(\mathbb{R}^n)$, then Lemma 4 gives

$$\|\chi_j\|_{p'(\cdot)} \approx |R_j|^{\frac{1}{p'(x_j)}}, \quad x_j \in R_j, \quad \text{and} \quad \|\chi_k\|_{p(\cdot)} \approx |R_k|^{\frac{1}{p(x_k)}}, \quad x_k \in R_k.$$

Hence the sum above can be rewritten as

$$\sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n)}\,|R_j|^{-\frac{1}{p(x_j)}}\,|R_k|^{\frac{1}{p(x_k)}}\,\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}.$$

Now we can distinguish three cases as follows (here we present the all cases):
$0 \leqslant j \leqslant k-2$: by Lemma 4 we get

$$|R_j|^{-\frac{1}{p(x_j)}}\,|R_k|^{\frac{1}{p(x_k)}} \approx |R_j|^{-\frac{1}{p_\infty}}\,|R_k|^{\frac{1}{p_\infty}} \approx 2^{(k-j)\frac{n}{p_\infty}} \lesssim 2^{(k-j)\frac{n}{p^-}}.$$

$j < 0 \leqslant k-2$: in this case we obtain

$$|R_j|^{-\frac{1}{p(x_j)}}\,|R_k|^{\frac{1}{p(x_k)}} \lesssim |R_j|^{-\frac{1}{p^-}}\,|R_k|^{\frac{1}{p^-}} \lesssim 2^{(k-j)\frac{n}{p^-}}.$$

$j \leqslant k-2 < 0$: here we have

$$|R_j|^{-\frac{1}{p(x_j)}}\,|R_k|^{\frac{1}{p(x_k)}} \approx \left(|R_k||R_j|^{-1}\right)^{\frac{1}{p(x_j)}}\,|R_k|^{\frac{1}{p(x_k)}-\frac{1}{p(x_j)}} \lesssim 2^{(k-j)\frac{n}{p^-}}.$$

Indeed, since $|x_k|<2^k$, $|x_j|<2^j<2^k$ we make use of local log-Hölder continuity of p at the origin and get, for $k\leqslant 0$,

$$\left|\frac{1}{p(x_k)}-\frac{1}{p(x_j)}\right|\log\frac{1}{|R_k|}\lesssim\frac{\log\left(\frac{1}{2^k}\right)}{\log\left(e+\frac{1}{2^k}\right)}\leqslant c$$

with $c>0$ independent of k,j,x_k,x_j.

Therefore, in all cases we have essentially the same bound and hence, combining the estimates above, we arrive at the inequality

$$\left\|2^{k\alpha(0)}T\left(f\chi_{B_{k-2}}\right)\chi_k\right\|_{p(\cdot)}\lesssim\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha^+-n+\frac{n}{p^-})}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}. \tag{5.1}$$

Since $\alpha^+-n+\frac{n}{p^-}<0$, we apply Lemma 2 and get

$$\left(\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}T\left(f\chi_{B_{k-2}}\right)\chi_k\right\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)}\lesssim\left(\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}f\chi_{R_k}\right\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)}\lesssim\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

To estimate $2^{k\alpha_\infty}T\left(f\chi_{B_{k-2}}\right)$ in $\ell^{q_\infty}_{>}$-norm, we have the same estimate (5.1), with $2^{k\alpha_\infty}$ in place of $2^{k\alpha(0)}$. We write

$$\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha^+-n+\frac{n}{p^-})}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}=\sum_{j=-\infty}^{0}\dots+\sum_{j=1}^{k-2}\dots \tag{5.2}$$

for any $k\geqslant 0$ (we put $\sum_{j=1}^{k-2}\dots=0$ if $k=0,1,2$). Since $\alpha^+-n+\frac{n}{p^-}<0$, the first term is bounded by

$$2^{k(\alpha^+-n+\frac{n}{p^-})}\sum_{j=-\infty}^{0}2^{-j(\alpha^+-n+\frac{n}{p^-})}\sup_{j\leqslant 0}\left\|2^{j\alpha(0)}f\chi_j\right\|_{p(\cdot)}\leqslant c\,2^{k(\alpha^+-n+\frac{n}{p^-})}\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

The $\ell^{q_\infty}_{>}$-norm of this expression is bounded by $c\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. Again by Lemma 2, we can estimate the second term in (5.2).

Estimation of $T\left(f\chi_{\widetilde{R}_k}\right)$. Using the boundedness of T

$$\begin{aligned}&\|\{T\left(2^{k\alpha(0)}f\chi_{\widetilde{R}_k}\right)\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}+\|\{T\left(2^{k\alpha_\infty}f\chi_{\widetilde{R}_k}\right)\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\\&\lesssim\|\{2^{k\alpha(0)}f\chi_{\widetilde{R}_k}\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}+\|\{2^{k\alpha_\infty}f\chi_{\widetilde{R}_k}\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\lesssim\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}.\end{aligned}$$

Estimation of $T\left(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\right)$. Using a combination of the arguments used in the proof of Theorem 1 in [2], we arrive at the inequality

$$\left\|2^{k\alpha(0)}T\left(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\right)\chi_k\right\|_{p(\cdot)}\lesssim\sum_{j=k}^{\infty}2^{(k-j)(\alpha^-+\frac{n}{p^+})}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)} \tag{5.3}$$

for any $k<0$. This sum can be rewritten us

$$\sum_{j=k}^{-1} \ldots + \sum_{j=0}^{\infty} \ldots .$$

Observing that $\alpha^- + \frac{n}{p^+} > 0$, an application of Lemma 2 yields that the $\ell_<^{q(0)}$-norm of the first sum is bounded by

$$c\left(\sum_{k=-\infty}^{-1} \left\|2^{k\alpha(0)} f\chi_k\right\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Taking into account that $2^{j\alpha(y)} \approx 2^{j\alpha_\infty}$ for any $y \in R_j$, $j \geqslant 0$, the second sum is bounded by

$$c\,2^{k(\alpha^- + \frac{n}{p^+})} \sum_{j=0}^{\infty} 2^{-j(\alpha^- + \frac{n}{p^+})} \sup_{j \geqslant 0} \left\|2^{j\alpha_\infty} f\chi_j\right\|_{p(\cdot)} \lesssim 2^{k(\alpha^- + \frac{n}{p^+})} \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

The $\ell_<^{q(0)}$-norm of this expression is bounded by $c\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. To estimate the term $2^{k\alpha_\infty} T\left(f\chi_{\mathbb{R}^n \setminus B_{k+2}}\right)$ in $\ell_>^{q_\infty}$-norm, we have the same estimate (5.3), with $2^{k\alpha_\infty}$ in place of $2^{k\alpha(0)}$, an application of Lemma 2 yields the desired inequality and hence the proof of Theorem 2 is complete.

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