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COMMUTATORS OF RIESZ TRANSFORMS WITH LIPSCHITZ FUNCTIONS RELATED TO MAGNETIC SCHRÖDINGER OPERATORS

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Abstract. Let $A := -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$ be a magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$, $n \geqslant 2$, where $\vec{a} := (a_1, \cdots, a_n) \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In this paper, the author shows that the commutators of the Riesz transforms $L_k A^{-1/2}$, $k \in \{1, \cdots, n\}$, with functions in Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ for $\alpha \in (0,1)$, are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/p - 1/q = \alpha/n$ and L_k is the closure of $\frac{\partial}{\partial x_k} - ia_k$ in $L^2(\mathbb{R}^n)$. Let ρ be an admissible function modeled on the known auxiliary function determined by the Schrödinger operator $-\triangle + V$. The author also characterizes a localized Lipschitz space $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ in terms of the localized Riesz transforms $\{\widetilde{R}_j\}_{j=1}^n$ and their adjoint operators.

1. Introduction

Let $\vec{a} := (a_1, \cdots, a_n)$, $n \geqslant 2$, with $a_k \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ real-valued for each $k \in \{1, \cdots, n\}$, $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and L_k be the *closure in* $L^2(\mathbb{R}^n)$ *of* $\frac{\partial}{\partial x_k} - ia_k$ *with domain* $C_c^\infty(\mathbb{R}^n)$, the *set of* $C^\infty(\mathbb{R}^n)$ *functions with compact supports*. We use the same notation as in [4] and define the *sesquilinear form* Q by

$$Q(f,g) := \sum_{k=1}^n \int_{\mathbb{R}^n} L_k f(x) \overline{L_k g(x)}\, dx + \int_{\mathbb{R}^n} V(x) f(x) \overline{g(x)}\, dx$$

with domain

$$\mathscr{D}(Q) := \left\{ f \in L^2(\mathbb{R}^n) : L_k f \in L^2(\mathbb{R}^n), k \in \{1, \cdots, n\}, \sqrt{V} f \in L^2(\mathbb{R}^n) \right\}.$$

It is known that Q is symmetric and closed. Set

$$\mathscr{D}(A) := \Big\{ f \in \mathscr{D}(Q) : \text{ there exists } g \in L^2(\mathbb{R}^n) \text{ such that for all } \varphi \in \mathscr{D}(Q), \quad Q(f,\varphi) = \int_{\mathbb{R}^n} g(x) \overline{\varphi(x)}\, dx \Big\} \tag{1.1}$$

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Paper MIA-19-13

and $Af := g$ for all $f \in \mathscr{D}(A)$ and $g \in L^2(\mathbb{R}^n)$ as in (1.1). The *magnetic Schrödinger operator* A is a self-adjoint operator since Q is symmetric; see [11]. Formally, we write

$$Af = \sum_{k=1}^{n} L_k^* L_k f + Vf \tag{1.2}$$

or $A = -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$, where L_k^* is the adjoint operator of L_k.

The purposes of this paper are two fold. The first one is to study the boundedness of commutators generated by the Riesz transforms $L_k A^{-\frac{1}{2}}$, $k \in \{1, \cdots, n\}$, with functions in the Lipschitz spaces $\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $\alpha \in (0,1)$. It was proved by Duong et al. in [3] that $L_k A^{-\frac{1}{2}}$ for each k is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2]$. Later, when $b \in \mathrm{BMO}(\mathbb{R}^n)$, the *space of functions with bounded mean oscillation*, Duong and Yan [4] further showed that the commutators $[b, L_k A^{-\frac{1}{2}}]$, $k \in \{1, \cdots, n\}$, are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2)$. Recently, when $\vec{a} := 0$ and V belongs to the so-called reverse Hölder class $\mathscr{B}_q(\mathbb{R}^n)$ for some $q > n/2$ (see Remark 1 below), Liu and Sheng [9] showed that the commutator $[b, \nabla(-\triangle + V)^{-1/2}]$ with $b \in \Lambda_\alpha^\theta(\mathbb{R}^n)$, a subspace of $\mathrm{Lip}_\alpha(\mathbb{R}^n)$, is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for p, q with $1/p - 1/q = \alpha/n$. Recall that the *Lipschitz space* $\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $\alpha \in (0,1)$, consists of the functions f such that

$$\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} := \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

In this paper, we obtain the boundedness of the commutators $[b, L_k A^{-1/2}]$ for general A and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ as follows.

THEOREM 1. *Let $\alpha \in (0,1)$, $p, q \in (1,2]$ with $1/p - 1/q = \alpha/n$. Assume that $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. Then the commutators $[b, L_k A^{-1/2}]$, $k \in \{1, \cdots, n\}$, are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

When T is a standard Calderón-Zygmund singular integral operator, that is, T is bounded on $L^2(\mathbb{R}^n)$ and the associated kernel of T satisfies the Hölder continuity condition, it is known that T is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$. Moreover, Coifman, Rochberg and Weiss [2] proved that $b \in \mathrm{BMO}(\mathbb{R}^n)$ is sufficient to guarantee the commutator $[b, T]$ to be bounded on $L^p(\mathbb{R}^n)$ with all $p \in (1, \infty)$, and they also established a partial converse that if $[b, R_k]$ are bounded on $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$, where R_k for $k \in \{1, \cdots, n\}$ are the Riesz transforms. The full converse of this result was obtained by Janson [8], in which Janson also showed that $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p - 1/q = \alpha/n$, where T is a Calderón-Zygmund singular integral operator with kernel being homogeneous of degree $-n$ and satisfying some smoothness condition.

Naturally, one may ask if the space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ can be characterized by the commutators of the Riesz transforms $L_k A^{-1/2}$. That is, if the converse of Theorem 1 holds. It is known that the Riesz transforms $L_k A^{-\frac{1}{2}}$ defined above do not fall in the scope of standard Calderón-Zygmund operators because their associated kernels may not satisfy the Hölder continuity, or even the weaker Hörmander condition. When $\vec{a} := 0$ and $V := 1 \in \cap_{q>1}\mathscr{B}_q(\mathbb{R}^n)$, Guo et al. [7] gave a function f which is not in $\mathrm{BMO}(\mathbb{R}^n)$, while its commutators $[f, \nabla(-\triangle+1)^{-1/2}]$ is bounded on $L^2(\mathbb{R}^n)$. This example implies that $\mathrm{BMO}(\mathbb{R}^n)$ can not be characterized by commutators of Riesz transforms $\nabla(-\triangle+V)^{-1/2}$. Let ρ be an admissible function introduced in [16] (see Definition 1 below), which is modeled on the known auxiliary function determined by V (see (1.3) below). Instead, in [13], a localized BMO-type space $\mathrm{BMO}_\rho(\mathbb{R}^n)$, a subspace of $\mathrm{BMO}(\mathbb{R}^n)$ introduced in [16], was characterized by the commutators of localized Riesz transforms $\{\widetilde{R}_j\}_{j=1}^n$ and their adjoint operators $\{\widetilde{R}_j^*\}_{j=1}^n$ introduced in [14]. Since when ρ is the known auxiliary function determined by V in (1.3), the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ is just the localized space $\mathrm{BMO}_{-\Delta+V}(\mathbb{R}^n)$ introduced in [5]. Recall that $\mathrm{BMO}_{-\Delta+V}(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$ and thus can not be characterized by $\nabla(-\triangle+V)^{-1/2}$ by the example given in [7]. In this sense, the localized Riesz transforms $\{\widetilde{R}_j\}_{j=1}^n$ and $\{\widetilde{R}_j^*\}_{j=1}^n$ are more suitable than $L_kA^{-1/2}$ or $\nabla(-\triangle+V)^{-1/2}$ to study the characterization of localized function spaces. Motivated by [13], instead of characterizing $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ by $L_kA^{-\frac{1}{2}}$, the second aim of this paper is to characterize the localized Lipschitz space $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$, a subspace of $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ introduced in [15], in terms of $\{\widetilde{R}_j\}_{j=1}^n$ and their adjoint operators $\{\widetilde{R}_j^*\}_{j=1}^n$.

To state our second result, we begin with the notion of admissible functions introduced by Yang and Zhou in [16].

DEFINITION 1. A positive function ρ on $\mathbb{R}^n$ is said to be admissible if there exist positive constants $\widetilde{C}$ and k_0 such that for all $x,\, y \in \mathbb{R}^n$,

$$\rho(y) \leqslant \widetilde{C}[\rho(x)]^{1/(1+k_0)}[\rho(x)+|x-y|]^{k_0/(1+k_0)}.$$

REMARK 1. Obviously, constant functions are admissible. Moreover, another class of admissible functions is given by the well-known reverse Hölder class $\mathscr{B}_q(\mathbb{R}^n)$. Recall that a nonnegative potential V is said to be in $\mathscr{B}_q(\mathbb{R}^n)$ with $q \in (1,\infty]$ if there exists a positive constant C such that for all open balls B of $\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B [V(y)]^q\,dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)\,dy$$

with the usual modification made when $q=\infty$. For any $V \in \mathscr{B}_q(\mathbb{R}^n)$ with certain $q \in (1,\infty]$ and all $x \in \mathbb{R}^n$, set

$$\rho(x) := [m(x,V)]^{-1} := \sup\left\{r>0 : \frac{r^2}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy \leqslant 1\right\}; \tag{1.3}$$

see, for example, [12]. It was proved in [12] that ρ in (1.3) is an admissible function if $n \geqslant 3$ and $V \in \mathscr{B}_{n/2}(\mathbb{R}^n)$. Moreover, Yang and Zhou [16] pointed out that ρ is admissible if $n \geqslant 1$, $q > \max\{1, n/2\}$ and $V \in \mathscr{B}_q(\mathbb{R}^n)$.

We next recall the notion of the space $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ in [15] associated to a given admissible function ρ. Throughout this paper, $\mathscr{D}$ *denotes the set of all open balls* $B(x,r)$ *such that* $r \geqslant \rho(x)$.

DEFINITION 2. Let ρ be an admissible function and $\alpha \in (0,1)$. A function f on $\mathbb{R}^n$ is said to be in the *localized Lipschitz space* $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ if there exists a nonnegative constant C such that for all $x,y \in \mathbb{R}^n$ and balls B containing x and y with $B \notin \mathscr{D}$,

$$|f(x)-f(y)| \leqslant C|B|^{\alpha/n},$$

and that for all balls $B \in \mathscr{D}$, $\|f\|_{L^\infty(B)} \leqslant C|B|^{\alpha/n}$. The minimal nonnegative constant C as above is called the *norm* of f in $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ and denoted by $\|f\|_{\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)}$.

Let ρ be an admissible function as in Definition 1. For all $j \in \{1,\cdots,n\}$, $f \in \cup_{p=1}^{\infty} L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\widetilde{R}_j(f)(x) := \mathrm{p.v.}\, c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \eta\left(\frac{|x-y|}{\rho(x)}\right) f(y)\,dy,$$

where and in what follows, $c_n := \Gamma((n+1)/2)/[\pi^{(n+1)/2}]$, $\eta \in C^1(\mathbb{R})$ supported in $(-1,1)$ and $\eta(t)=1$ if $|t| \leqslant 1/2$. The adjoint operator of $\widetilde{R}_j$, $j \in \{1,\cdots,n\}$, has the form

$$\widetilde{R}_j^*(f)(x) := -\mathrm{p.v.}\, c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \eta\left(\frac{|x-y|}{\rho(y)}\right) f(y)\,dy.$$

THEOREM 2. *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, ρ be an admissible function, $\alpha \in (0,1)$ and $j \in \{1,\cdots,n\}$. Then the following assertions are equivalent:*

(i) $b \in \mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$.

(ii) b satisfies that for all open balls $B \in \mathscr{D}$,

$$\frac{1}{|B|^{1+\alpha/n}} \int_B |b(y)|\,dy \leqslant C, \tag{1.4}$$

and $[b,\widetilde{R}_j^*]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p,q \in (1,\infty)$ such that $1/p - 1/q = \alpha/n$.

(iii) b satisfies (1.4) and $[b,\widetilde{R}_j^*]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $p,q \in (1,\infty)$ such that $1/p - 1/q = \alpha/n$.

(iv) b satisfies (1.4) and $[b,\widetilde{R}_j]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p,q \in (1,\infty)$ such that $1/p - 1/q = \alpha/n$.

(v) b satisfies (1.4) and $[b,\widetilde{R}_j]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $p,q \in (1,\infty)$ such that $1/p - 1/q = \alpha/n$.

We mention that the condition (1.4) is necessary. In fact, from the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 2 below, it follows that if $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, the commutator $[b,\widetilde{R}_j^*]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p,q \in (1,\infty)$ such that $1/p - 1/q = \alpha/n$. Now let $b(x) := |x|^\alpha, x \in \mathbb{R}^n$. Then it is easy to see that $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. However, (1.4) does not hold for b.

The organization of the paper is as follows. Section 2 is devoted to the proof of Theorem 1. In Section 3, we present the proof of Theorem 2.

Finally, we now make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. If $f \leqslant Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $x \in \mathbb{R}^n$ and $\lambda, r > 0$, $B := B(x,r)$ denotes the ball centered at x with radius r and $\lambda B := B(x, \lambda r)$.

2. Proof of Theorem 1

In this section, we present the proof of Theorem 1. We begin with an analogy of the classical Fefferman-Stein inequality for the following sharp maximal function $\mathscr{M}_A^\sharp$ established in [10]; see also [4]. For any $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, the *sharp maximal function* $\mathscr{M}_A^\sharp$ associated with the semigroup $\{e^{-tA}\}_{t>0}$ is given by

$$\mathscr{M}_A^\sharp(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B \left| f(y) - e^{-t_B A} f(y) \right| dy,$$

where r_B is the *radius of the ball* B and $t_B := r_B^2$.

LEMMA 1. *Let* $p \in (1, \infty)$. *There exists a positive constant* C_p *such that for all* $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} \leqslant C_p \left\| \mathscr{M}_A^\sharp(f) \right\|_{L^p(\mathbb{R}^n)}.$$

The following lemma on the kernel of $(I - e^{-tA})(L_k A^{-\frac{1}{2}})^*$ was established by Duong and Yan in [4].

LEMMA 2. *For each* $k \in \{1, \cdots, n\}$, *the kernel* $K_{t,k}^*(y,z)$ *of the operator*

$$(L_k A^{-1/2}(I - e^{-tA}))^* = (I - e^{-tA})(L_k A^{-\frac{1}{2}})^*$$

satisfies the following estimate

$$\sum_{m=0}^{\infty} 2^m \left(2^m t^{1/2}\right)^{n/2} \left(\int_{2^m t^{1/2} \leqslant |y-z| < 2^{m+1} t^{1/2}} \left| K_{t,k}^*(y,z) \right|^2 dz \right)^{1/2} \leqslant C,$$

where C *is a constant independent of* t *and* y.

Proof of Theorem 1. For each $k \in \{1, \cdots, n\}$, let $(L_k A^{-1/2})^*$ be the *adjoint operator of* $L_k A^{-1/2}$. By duality, for given $p, q \in (1, 2]$ with $1/p - 1/q = \alpha/n$, it suffices to prove $[b, (L_k A^{-1/2})^*]$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, where for any $r \in [1, \infty]$, $1/r + 1/r' = 1$. To this end, we first show that there exists a positive constant C such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} &\mathscr{M}_A^\sharp \left(\left[b, \left(L_k A^{-1/2} \right)^* \right] f \right)(x) \\ &\quad \leqslant C \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \left[\mathscr{M}_{2,\alpha} \left(\left(L_k A^{-1/2} \right)^* f \right)(x) + \mathscr{M}_{2,\alpha}(f)(x) \right], \end{aligned} \tag{2.1}$$

where for $r \in [2, n/\alpha)$ and any suitable function f,

$$\mathscr{M}_{r,\alpha}(f)(x) := \sup_{x\in B} \frac{1}{|B|^{-\alpha/n}} \left(\frac{1}{|B|} \int_B |f(y)|^r \, dy \right)^{1/r}. \tag{2.2}$$

Indeed, assume that (2.1) holds. For $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ and each $N \in \mathbb{N}$, define $b_N := \min\{N, |b|\} sgn(b)$. Then $b_N \in L^\infty(\mathbb{R}^n)$ and $\|b_N\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$. Furthermore, by the fact that $(L_k A^{-1/2})^*$ is bounded on $L^s(\mathbb{R}^n)$ for $s \in [2, \infty)$, we see that for all $f \in C_c^\infty(\mathbb{R}^n)$, $[b_N, (L_k A^{-1/2})^*](f) \in L^s(\mathbb{R}^n)$ and

$$\|[b_N, (L_k A^{-1/2})^*](f)\|_{L^s(\mathbb{R}^n)} \lesssim N \|f\|_{L^s(\mathbb{R}^n)}.$$

Recall that $\mathscr{M}_{2,\alpha}$ is bounded from $L^s(\mathbb{R}^n)$ to $L^t(\mathbb{R}^n)$ with $s \in (2, n/\alpha)$ and $1/s - 1/t = \alpha/n$; see Chanillo [1]. By this fact together with $1/q' - 1/p' = \alpha/n$, Lemma 1 and (2.1), we have that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\left\| \left[b_N, (L_k A^{-1/2})^* \right](f) \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\quad \lesssim \left\| \mathscr{M}_A^\sharp \left(\left[b_N, \left(L_k A^{-1/2} \right)^* \right](f) \right) \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\quad \lesssim \|b_N\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \left\| \mathscr{M}_{2,\alpha}\left(\left(L_k A^{-1/2} \right)^* f \right) + \mathscr{M}_{2,\alpha}(f) \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\quad \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)}.
\end{aligned}$$

A standard argument together with the Fatou lemma then implies that for all $f \in L^{q'}(\mathbb{R}^n)$, $[b, (L_k A^{-1/2})^*](f) \in L^{p'}(\mathbb{R}^n)$ and

$$\left\| \left[b, (L_k A^{-1/2})^* \right](f) \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

This implies Theorem 1.

To show (2.1), let $T := (L_k A^{-\frac{1}{2}})^*$. For any $f \in L^{q'}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $B := B(x_B, r_B) \ni x$, $f_1 := f\chi_{2B}$ and $f_2 := f - f_1$. One writes

$$[b, T]f = (b - b_B)Tf - T((b - b_B)f_1) - T((b - b_B)f_2)$$

and

$$e^{-t_B A}([b, T]f) = e^{-t_B A}((b - b_B)Tf) - e^{-t_B A} T((b - b_B)f_1) - e^{-t_B A} T((b - b_B)f_2),$$

where for any function f and ball B, $f_B := \frac{1}{|B|}\int_B f(x)\,dx$. Then we see that

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left| \left(I - e^{-t_B A}\right)[b, T]f(y) \right| dy \\
&\quad \leqslant \frac{1}{|B|}\int_B |(b - b_B)Tf(y)| \, dy + \frac{1}{|B|}\int_B |T((b - b_B)f_1)(y)| \, dy
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|B|}\int_B \left|e^{-t_BA}((b-b_B)Tf)(y)\right| dy \\
&+\frac{1}{|B|}\int_B \left|e^{-t_BA}T((b-b_B)f_1)(y)\right| dy \\
&+\frac{1}{|B|}\int_B \left|\left(I-e^{-t_BA}\right)T((b-b_B)f_2)(y)\right| dy =: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

By the Hölder inequality, we see that

$$
\mathrm{I} \leqslant \frac{1}{|B|^{1-\frac{\alpha}{n}}}\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\int_B |Tf(y)|\,dy \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(Tf)(x).
$$

For II, from the Hölder inequality and the $L^2(\mathbb{R}^n)$-boundedness of T (see [3, Theorem 1.1]), it follows that

$$
\begin{aligned}
\mathrm{II} &\leqslant \left(\frac{1}{|B|}\int_B |T((b-b_B)f_1)(y)|^2\,dy\right)^{1/2} \\
&\lesssim \left(\frac{1}{|B|}\int_{2B} |b(y)-b_B||f(y)|^2\,dy\right)^{1/2} \\
&\lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\frac{1}{|B|^{-\frac{\alpha}{n}}}\left(\frac{1}{|B|}\int_{2B} |f(y)|^2\,dy\right)^{1/2} \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(f)(x).
\end{aligned}
$$

To estimate III, recall that the kernel $p_t(y,z)$ of e^{-tA} satisfies that for all $t>0$ and almost all $y,z\in\mathbb{R}^n$,

$$
|p_t(y,z)| \leqslant (4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{|y-z|^2}{4t}\right); \tag{2.3}
$$

see [4]. Let $g:=(b-b_B)Tf$. By (2.3), for any $y\in B$,

$$
\begin{aligned}
\left|e^{-t_BA}g(y)\right| &\leqslant \int_{\mathbb{R}^n} |p_{t_B}(y,z)g(z)|\,dz \\
&\lesssim \int_{\mathbb{R}^n} t_B^{-\frac{n}{2}}\exp\left(-\frac{|y-z|^2}{4t_B}\right)|g(z)|\,dz \\
&\sim \int_{|y-z|<2t_B^{1/2}} t_B^{-\frac{n}{2}}\exp\left(-\frac{|y-z|^2}{4t_B}\right)|g(z)|\,dz + \sum_{k=0}^{\infty}\int_{2^kt_B^{1/2}\leqslant|y-z|<2^{k+1}t_B^{1/2}}\cdots \\
&\lesssim \frac{1}{|B|}\int_B |g(z)|\,dz + \sum_{k=0}^{\infty}\frac{t_B^N}{(2^kt_B^{1/2})^{2N}}\frac{1}{t_B^{\frac{n}{2}}}\int_{|y-z|<2^{k+1}t_B^{1/2}} |g(z)|\,dz \\
&\lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\left[\mathscr{M}_{2,\alpha}(Tf)(x)+\sum_{k=0}^{\infty}\frac{1}{2^{k(2N-n)}}\mathscr{M}_{2,\alpha}(Tf)(x)\right] \\
&\lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(Tf)(x),
\end{aligned}
$$

where $N>n/2$. This implies that $\mathrm{III}\lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(Tf)(x)$.

For IV, let $\mathscr{M}(f)$ be the *Hardy-Littlewood maximal function*, defined by setting for all locally integrable functions f and $x\in\mathbb{R}^n$,

$$\mathscr{M}(f)(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\,dy.$$

By (2.3) and the $L^2(\mathbb{R}^n)$-boundedness of $\mathscr{M}$ and T, we conclude that

$$\begin{aligned}
\mathrm{IV} &\lesssim \frac{1}{|B|}\int_B \mathscr{M}[T((b-b_B)f_1)](y)\,dy\\
&\lesssim \left[\frac{1}{|B|}\int_B \{\mathscr{M}[T((b-b_B)f_1)](y)\}^2\,dy\right]^{1/2}\\
&\lesssim \left[\frac{1}{|B|}\int_B |T((b-b_B)f_1)(y)|^2\,dy\right]^{1/2}\\
&\lesssim \left[\frac{1}{|B|}\int_{2B} |(b(y)-b_B)f(y)|^2\,dy\right]^{1/2} \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(f)(x).
\end{aligned}$$

Finally, from the fact that $|y-z|\geqslant r_B$ for any $y\in B$ and $z\notin 2B$, Lemma 2 and the Hölder inequality, we deduce that

$$\begin{aligned}
\mathrm{V} &\leqslant \frac{1}{|B|}\int_B\int_{\mathbb{R}^n\setminus(2B)} \left|K^*_{t_B,k}(y,z)\right| |b(z)-b_B||f(z)|\,dz\,dy\\
&\leqslant \frac{1}{|B|}\int_B\sum_{m=0}^{\infty}\left(\int_{2^m r_B\leqslant|y-z|<2^{m+1}r_B} \left|K^*_{t_B,k}(y,z)\right| |b(z)-b_B||f(z)|dz\right)dy\\
&\lesssim \frac{1}{|B|}\int_B\sum_{m=0}^{\infty}\left|2^{m+1}B\right|^{\frac{\alpha}{n}}\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\left(\int_{2^m r_B\leqslant|y-z|<2^{m+1}r_B}\left|K^*_{t_B,k}(y,z)\right|^2dz\right)^{1/2}\\
&\quad\times\left(\int_{2^{m+2}B}|f(z)|^2\,dz\right)^{1/2}dy\\
&\lesssim \sup_{m\geqslant 0}2^{-m}\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(f)(x)\lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\mathscr{M}_{2,\alpha}(f)(x).
\end{aligned}$$

Combining the estimates from I through V, we see that (2.1) holds, which completes the proof of Theorem 1.

3. Proof of Theorem 2

Let ρ be a given admissible function. In this section, we establish characterizations of $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ via commutators of localized Riesz transforms and their adjoint operators in [14]. We begin with the following lemma established in [14].

LEMMA 3. *Let ρ be an admissible function, and for all $j\in\{1,\cdots,n\}$, let $\widetilde{R}_j$ and $\widetilde{R}^*_j$ be defined as above. Then $\widetilde{R}_j$ and $\widetilde{R}^*_j$ are bounded on $L^p(\mathbb{R}^n)$ for $p\in(1,\infty)$ and bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.*

The following characterization of $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ was obtained in [15], which is a localized version of Lemma 4 in [8].

LEMMA 4. *Let ρ be an admissible function, $\mathscr{D}$ as in Section 1 and $\alpha\in(0,1)$. Define the localized Morrey-Campanato space $\mathscr{E}^{\alpha}_{\rho}(\mathbb{R}^n)$ by*

$$\mathscr{E}^{\alpha}_{\rho}(\mathbb{R}^n) := \left\{ f\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\ \|f\|_{\mathscr{E}^{\alpha}_{\rho}(\mathbb{R}^n)} := \sup_{B\notin\mathscr{D}} \frac{1}{|B|^{1+\alpha/n}}\int_B |f(y)-f_B|dy \right.$$
$$\left. + \sup_{B\in\mathscr{D}} \frac{1}{|B|^{1+\alpha/n}}\int_B |f(y)|dy<\infty \right\}.$$

Then $\mathscr{E}^{\alpha}_{\rho}(\mathbb{R}^n)=\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)$ with equivalent norms.

Proof of Theorem 2. To show that (i) implies (ii), by $\mathrm{Lip}_{\alpha,\rho}(\mathbb{R}^n)\subset\mathrm{Lip}_{\alpha}(\mathbb{R}^n)$, it suffices to prove that if $b\in\mathrm{Lip}_{\alpha}(\mathbb{R}^n)$, then $[b,\widetilde{R}^*_j]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p\in(1,n/\alpha)$ and $q\in(p,\infty)$ such that $1/p-1/q=\alpha/n$. Moreover, by a standard argument as in Theorem 1, we only need to show that there exists a positive constant C such that for all $f\in C^\infty_c(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\mathscr{M}^\sharp\left(\left[b,\widetilde{R}^*_j\right]f\right)(x)\leqslant C\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\left\{\mathscr{M}_{r,\alpha}\left[\widetilde{R}^*_j(f)\right](x)+\mathscr{M}_{r,\alpha}(f)(x)\right\}, \tag{3.1}$$

where $\mathscr{M}_{r,\alpha}$ is as in (2.3) with $r\in(1,n/\alpha)$ and $\mathscr{M}^\sharp(f)$ is the *classical maximal function* defined by

$$\mathscr{M}^\sharp(f)(x):=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)-f_B|\,dy.$$

To show (3.1), let $x\in\mathbb{R}^n$, $B:=B(x_0,r_0)$ be any ball containing x, $f_1:=f\chi_{2B}$ and $f_2:=f\chi_{\mathbb{R}^n\setminus(2B)}$. We first prove that for all $j\in\{1,\cdots,n\}$ and B as above,

$$\frac{1}{|B|}\int_B\left|\left[b,\widetilde{R}^*_j\right](f)(y)+\widetilde{R}^*_j\left[(b-b_B)f_2\right](x_0)\right|dy$$
$$\lesssim\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\left\{\mathscr{M}_{r,\alpha}\left[\widetilde{R}^*_j(f)\right](x)+\mathscr{M}_{r,\alpha}(f)(x)\right\}. \tag{3.2}$$

From the linearity of $\widetilde{R}^*_j$, we deduce that for all $y\in\mathbb{R}^n$,

$$\left[b,\widetilde{R}^*_j\right](f)(y)=(b(y)-b_B)\widetilde{R}^*_j(f)(y)-\widetilde{R}^*_j\left[(b-b_B)f_1\right](y)-\widetilde{R}^*_j\left[(b-b_B)f_2\right](y)$$
$$=:\mathrm{I}_1(y)+\mathrm{I}_2(y)+\mathrm{I}_3(y).$$

Similar to the estimates of I and II in Theorem 1, by the Hölder inequality and Lemma 3, we have that

$$\sum_{j=1}^2\frac{1}{|B|}\int_B\left|\mathrm{I}_j(y)\right|dy\lesssim\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\left\{\mathscr{M}_{r,\alpha}\left[\widetilde{R}^*_j(f)\right](x)+\mathscr{M}_{r,\alpha}(f)(x)\right\}.$$

By Definition 1, for any given positive constant a, there exists a positive constant $\widetilde{C}_a \in [1,\infty)$ such that for all $y \in \mathbb{R}^n$ and $x \in B(y, a\rho(y))$,

$$\frac{1}{\widetilde{C}_a}\rho(y) \leqslant \rho(x) \leqslant \widetilde{C}_a\rho(y). \tag{3.3}$$

We denote the kernel of $\widetilde{R}_j^*$ still by $\widetilde{R}_j^*$. Let $y \in B$. From (3.3), we deduce that

$$\operatorname{supp}(\widetilde{R}_j^*(y,\cdot)) \subset B(y, \widetilde{C}_1\rho(y)).$$

Because x_0, $y \in B$, another application of (3.3) yields that there exists positive constant C_1 such that

$$\operatorname{supp}(\widetilde{R}_j^*(y,\cdot)) \subset B(x_0, C_1\rho(x_0))$$

if $B \notin \mathscr{D}$, and

$$\operatorname{supp}(\widetilde{R}_j^*(y,\cdot)) \subset B(x_0, C_1 r_0)$$

if $B \in \mathscr{D}$, which further implies that

$$\operatorname{supp}(\widetilde{R}_j^*(y,\cdot)) \subset [C_1(\rho(x_0)/r_0 + 1)B].$$

From this and the Hölder inequality, it follows that

$$\begin{aligned}
&|\mathrm{I}_3(y) - \mathrm{I}_3(x_0)| \\
&\quad \leqslant \int_{[C_1(\rho(x_0)/r_0+1)B]\setminus(2B)} \left|\widetilde{R}_j^*(y,z) - \widetilde{R}_j^*(x_0,z)\right| |b(z) - b_B||f(z)|\,dz \\
&\quad \lesssim \int_{[C_1(\rho(x_0)/r_0+1)B]\setminus(2B)} \left\{\left|\left(\frac{y_j - z_j}{|y-z|^{n+1}} - \frac{x_{0\,j} - z_j}{|x_0 - z|^{n+1}}\right)\eta\left(\frac{|y-z|}{\rho(z)}\right)\right|\right. \\
&\qquad \left. + \left|\frac{x_{0\,j} - z_j}{|x_0 - z|^{n+1}}\left[\eta\left(\frac{|y-z|}{\rho(z)}\right) - \eta\left(\frac{|x_0-z|}{\rho(z)}\right)\right]\right|\right\} |b(z) - b_B||f(z)|\,dz \\
&\quad \lesssim \int_{[C_1(\rho(x_0)/r_0+1)B]\setminus(2B)} \frac{|y - x_0||b(z) - b_B||f(z)|}{|x_0 - z|^{n+1}}\,dz \\
&\quad \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{r_0}{(2^k r_0)^{n+1}} \left|2^{k+2}B\right|^{\alpha/n} \int_{2^{k+1}B} |f(z)|\,dz \\
&\quad \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \mathscr{M}_{r,\alpha}(f)(x).
\end{aligned}$$

This together with estimates of I_1 and I_2 yields (3.2), which implies (3.1), and hence, completes the proof of the implication (i) $\Longrightarrow$ (ii).

Since the implications from (ii) to (iii) and from (iv) to (v) are obvious, and the implications from (ii) to (iv) and from (iii) to (v) follow from a standard duality argument, Theorem 2 is reduced to showing that (v) implies (i). We borrow some idea from [8]. Let $\{R_j\}_{j=1}^n$ be the kernels of the classical Riesz transforms. Observe that for all $j \in \{1,\cdots,n\}$, $1/R_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, there exist $z_0 \in \mathbb{R}^n \setminus \{0\}$ and $\delta \in (0,\infty)$ such that $1/R_j(z)$ is expressed as an absolutely convergent Fourier series in $B(z_0,\delta)$ (see, for example, [6, Theorem 3.2.16]). That is, there exist $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and numbers

$\{a_k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^{\infty}|a_k|<\infty$ such that for all $z\in B(z_0,\delta)$, $1/R_j(z)=\sum_{k=1}^{\infty}a_k e^{i\nu_k\cdot z}$. Let $z_1:=\delta^{-1}z_0$. If $|z-z_1|<1$, then we have that $|\delta z-z_0|<\delta$ and

$$\frac{1}{R_j(z)}=\frac{\delta^{-n}}{R_j(\delta z)}=\delta^{-n}\sum_{k=1}^{\infty}a_k e^{i\nu_k\cdot(\delta z)}. \tag{3.4}$$

Let $B:=B(\widetilde{x},r)\notin\mathscr{D}$ being any ball and $C_2:=[4(1+|z_1|)\widetilde{C}_1]$, where $\widetilde{C}_1$ is as in (3.3) with $a=1$. Then we have that $r<\rho(\widetilde{x})$. To show (i), we first consider the case that $r<\rho(\widetilde{x})/C_2$. Let $\widetilde{y}:=\widetilde{x}-2rz_1$ and $\widetilde{B}:=B(\widetilde{y},r)$. Then we obtain that for all $x\in B$ and $y\in\widetilde{B}$,

$$\left|\frac{x-y}{2r}-z_1\right|\leqslant\frac{|x-\widetilde{x}|}{2r}+\frac{|y-\widetilde{y}|}{2r}<1 \tag{3.5}$$

and $|\widetilde{x}-x|<r<\rho(\widetilde{x})$, which together with (3.3) implies that $|x-y|<2r(1+|z_1|)$ and $\rho(\widetilde{x})\leqslant\widetilde{C}_1\rho(x)$. Therefore, we see that

$$|x-y|<2r(1+|z_1|)<\frac{\rho(\widetilde{x})}{2\widetilde{C}_1}\leqslant\frac{\rho(x)}{2}. \tag{3.6}$$

Note that $\widetilde{R}_j(x,y)=R_j(x-y)$ for all $x,\,y\in\mathbb{R}^n$ with $|x-y|<\frac{\rho(x)}{2}$ and $j\in\{1,\cdots,n\}$. From this, (3.4), (3.5), the Hölder inequality and the boundedness of $[b,\widetilde{R}_j]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we then deduce that

$$\begin{aligned}
&\int_B\left|b(x)-b_{\widetilde{B}}\right|dx\\
&=\int_{\mathbb{R}^n}\left[b(x)-b_{\widetilde{B}}\right]sgn\left(b-b_{\widetilde{B}}\right)\chi_B(x)\,dx\\
&=\frac{1}{|\widetilde{B}|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[b(x)-b(y)]sgn(b(x)-b_{\widetilde{B}})\chi_B(x)\chi_{\widetilde{B}}(y)\frac{(2r)^n\widetilde{R}_j(x,y)}{R_j(\frac{x-y}{2r})}\,dy\,dx\\
&\lesssim\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[b(x)-b(y)]sgn(b(x)-b_{\widetilde{B}})\chi_B(x)\chi_{\widetilde{B}}(y)\widetilde{R}_j(x,y)\sum_{k=1}^{\infty}a_k e^{i\frac{\delta\nu_k}{2r}\cdot(x-y)}\,dy\,dx\\
&\lesssim\sum_{k=1}^{\infty}|a_k|\int_{\mathbb{R}^n}\left|\left[b,\widetilde{R}_j\right]\left(\chi_{\widetilde{B}}e^{-i\frac{\delta\nu_k}{2r}}\right)(x)\right|\chi_B(x)\,dx\\
&\lesssim\sum_{k=1}^{\infty}|a_k|\left\|\left[b,\widetilde{R}_j\right]\left(\chi_{\widetilde{B}}e^{-i\frac{\delta\nu_k}{2r}}\right)\right\|_{L^q(\mathbb{R}^n)}\|\chi_B\|_{L^{q'}(\mathbb{R}^n)}\lesssim|B|^{1+\frac{\alpha}{n}},
\end{aligned}$$

which implies that

$$\int_B|b(x)-b_B|\,dx\lesssim|B|^{1+\frac{\alpha}{n}}. \tag{3.7}$$

Now consider the case that $r\in[\rho(\widetilde{x})/C_2,\rho(\widetilde{x}))$. In this case, $C_2B\in\mathscr{D}$. From this fact together with $C_2>1$ and (1.4), we deduce that

$$\frac{1}{|B|}\int_B|b(y)-b_B|\,dy\lesssim\frac{1}{|B|}\int_B|b(y)|\,dy\lesssim\frac{1}{|C_2B|}\int_{C_2B}|b(y)|\,dy\lesssim|B|^{1+\frac{\alpha}{n}}.$$

This and the estimate for the case that $r < \rho(\widetilde{x})/C_2$ imply that (3.7) holds for all balls $B \notin \mathscr{D}$, which together with (1.4) and Lemma 4 further yields (i), and hence, completes the proof of Theorem 2.

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