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STRONG CONVERGENCE THEOREM FOR WALSH–MARCINKIEWICZ MEANS

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Abstract. It is known that the maximal operator of Walsh-Marcinkiewicz means is bounded from the dyadic martingale Hardy space H_p to the space L_p for $p>2/3$ and the condition $p>2/3$ is essential. In the case $p=2/3$ the boundedness of the maximal operator does not hold. This means that the investigation of the maximal operator at the endpoint case $p=2/3$ plays an important role.

The main aim of this paper is to prove a strong convergence theorem for the Walsh-Marcinkiewicz means on the Hardy space $H_{2/3}$.

1. Introduction

In 1987, Simon [17] proved that, if $f\in H_1(G)$, then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{\|S_n(f)\|_1}{n}=\|f\|_1.$$

The trigonometric analogue was verified earlier by Smith [19]. In 1993, the analogous result with respect to Vilenkin systems was proved by Gát [4], with respect to Vilenkin-like systems by Blahota [1], later. In 2000, Simon [18] proved that there exists a constant C_p such that the inequality

$$\sum_{k=1}^{\infty}\frac{\|S_k(f)\|_p^p}{k^{2-p}}\leqslant C_p\|f\|_{H_p}^p \tag{1}$$

holds for any function $f\in H_p(G)$, where $0<p<1$. In [21] it was proved that sequence $\left\{1/k^{2-p}:k\geqslant 1\right\}$ in (1) is sharp in a special sense.

We mention that the endpoint for the boundedness of the maximal operator of Walsh-Fejér means is $p=1/2$ (see [23, 16, 6, 14]). For $0<p\leqslant 1/2$ the second author [20] proved that there exists an absolute constant c_p, depending only on p, such that

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$$\frac{1}{\log^{[1/2+p]} n}\sum_{k=1}^{n}\frac{\|\sigma_k(f)\|_{H_p}^p}{k^{2-2p}} \leqslant c_p\|f\|_{H_p}^p \quad (f\in H_p(G)) \tag{2}$$

holds for the Fejér means (we are interested only in the endpoint case $p=1/2$). Recently, it is proved for Vilenkin system [2].

For the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [10] proved that the means

$$\mathcal{M}_n(f) := \frac{1}{n}\sum_{j=1}^{n} S_{j,j}(f)$$

of a function $f\in L\log L([0,2\pi]^2)$ converges a.e to f as $n\to\infty$. Zhizhiashvili improved this result for $f\in L_1([0,2\pi]^2)$ [26]. Dyachenko [3] showed this result for dimension greater than 2.

For the two-dimensional Walsh-Fourier series Weisz [25] proved that the maximal operator $\mathcal{M}^*(f)$ is bounded from the dyadic martingale Hardy space $H_p\left(G^2\right)$ to the space $L_p\left(G^2\right)$ for $p>2/3$. In the case $p=2/3$ Goginava [6] proved that $\mathcal{M}^*$ is not bounded from the Hardy space $H_{2/3}\left(G^2\right)$ to the space $L_{2/3}\left(G^2\right)$. By interpolation it follows that $\mathcal{M}^*$ is not bounded from the Hardy space $H_p\left(G^2\right)$ to the space $weak-L_p\left(G^2\right)$ for $0<p<2/3$. That is, the endpoint of the boundedness of the maximal operator $\mathcal{M}^*$ is $p=2/3$. This means that it is interesting to discuss what does happen here. Goginava [7] also proved that $\mathcal{M}^*$ is bounded from the Hardy space $H_{2/3}\left(G^2\right)$ to the space $weak-L_{2/3}\left(G^2\right)$.

The first author [11] proved that the modified maximal operator

$$\widetilde{\mathcal{M}}^* := \sup_{n\in\mathbb{N}} |\mathcal{M}_n| / \log^{3/2}(n+1) \tag{3}$$

is bounded from the Hardy space $H_{2/3}\left(G^2\right)$ to the space $L_{2/3}\left(G^2\right)$. He also proved that the sequence $\left\{\log^{3/2}(n+1)\right\}_{n=1}^{\infty}$ is important for the maximal operator $\widetilde{\mathcal{M}}^*$ (for details see [11]). That is, the order of deviant behaviour of the n-th Marcinkiewicz means was given exactly.

A necessary and sufficient condition for the convergence of Walsh-Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_{2/3}\left(G^2\right)$ can be found in [12].

For the case $0<p<2/3$ The authors [13] proved that there exists an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty}\frac{\|\mathcal{M}_k(f)\|_{H_p(G^2)}^p}{k^{3-3p}} \leqslant c_p\|f\|_{H_p(G^2)}^p \quad (f\in H_p(G^2)) \tag{4}$$

holds.

In the present paper we turn our attention back to the endpoint case $p=2/3$. We establish a strong convergence theorem for the Walsh-Marcinkiewicz means in the

Hardy space $H_{2/3}$. We show that there exists a constant c, such that

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\|\mathscr{M}_m(f)\|_{H_{2/3}}^{2/3}}{m}\leqslant c\,\|f\|_{H_{2/3}}^{2/3}, \tag{5}$$

for all $f\in H_{2/3}\left(G^2\right)$. That is we get the two-dimensional version of the above mentioned inequality (2) of the second author in the endpoint case $p=2/3$. We mention that the method presented in [13] to reach inequality (4) is not enough effective in the endpoint case. So we have to improve a new method to prove inequality (5).

2. Definitions and Notations

Now, we give a brief introduction to the theory of dyadic analysis [15, 22]. Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x=(x_0,x_1,...,x_k,...)$ with coordinates $x_k\in\{0,1\}\,(k\in\mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) is the product measure and the topology is the product topology. The compact Abelian group G is called the Walsh group. A base for the neighbourhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G,\\ I_n(x) &:= I_n\left(x_0,...,x_{n-1}\right)\\ &:= \left\{y\in G:\ y=\left(x_0,...,x_{n-1},y_n,y_{n+1},...\right)\right\}, \end{aligned}$$

$(x\in G,n\in\mathbb{N})$. These sets are called dyadic intervals. Let $0=(0:i\in\mathbb{N})\in G$ denote the null element of G, and $I_n:=I_n(0)\ (n\in\mathbb{N})$. Set

$$e_n:=(0,...,0,1,0,...)\in G,$$

the n-th coordinate of which is 1 and the rest are zeros $(n\in\mathbb{N})$.

For $k\in\mathbb{N}$ and $x\in G$ denote

$$r_k(x):=(-1)^{x_k}$$

the k th Rademacher function. The Walsh-Paley functions are constructed as products of Rademacher functions. For the definition let us take the binary form of $n\in\mathbb{N}$, $n=\sum_{i=0}^{\infty}n_i2^i$, where $n_i\in\{0,1\}\quad(i\in\mathbb{N})$. That is, n is expressed in the number system of base 2. Denote $|n|:=\max\{j\in\mathbb{N}:\ n_j\neq 0\}$, that is, $2^{|n|}\leqslant n<2^{|n|+1}$.

The nth Walsh-Paley function w_n is defined as

$$w_n(x):=\prod_{k=0}^{\infty}\left(r_k(x)\right)^{n_k}=(-1)^{\sum_{k=0}^{|n|}n_kx_k},\quad (x\in G,\ n\in\mathbb{N}).$$

The Dirichlet kernels for the Walsh-Paley system are defined as usually by

$$D_n := \sum_{k=0}^{n-1} w_k\,,\;\; D_0 := 0.$$

One can easily check that (see e.g. [15])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{6}$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $weak-L_p(G)$, $(0<p<\infty)$ are respectively defined by

$$\|f\|_p := \left(\int_G |f|^p \, d\mu\right)^{1/p} < \infty, \quad \|f\|_{weak-L_p} := \sup_{\lambda>0} \lambda\mu\,(f>\lambda)^{1/p} < +\infty.$$

The partial sums of the Walsh-Fourier series are defined

$$S_m(f;x) := \sum_{i=0}^{m-1} \widehat{f}(i)\, w_i(x).$$

The nth Fejér means and Fejér kernel of the Walsh-Fourier series of the function f are given by

$$\sigma_n(f;x) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(f;x), \quad K_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

The σ-algebra generated by the dyadic 2-dimensional $I_n(x^1)\times I_n(x^2)$ square of measure $2^{-n}\times 2^{-n}$ is denoted by $F_{n,n}\,(n\in\mathbb{N})$. Denote by $f=(f_{n,n}\; n\in\mathbb{N})$ one-parameter martingale with respect to $F_{n,n}\,(n\in\mathbb{N})$. The definitions of the spaces $L_p(G^2)$, weak-$L_p(G^2)$ and $H_p(G^2)$ are given analogous way as in the one dimensional case.

The Kronecker product $(w_{n,m}: n,m\in\mathbb{N})$ of two Walsh system is said to be two-dimensional Walsh system. Thus,

$$w_{n,m}(x^1,x^2) = w_n(x^1)\, w_m(x^2).$$

If $f\in L_1(G^2)$ then the numbers $\widehat{f}(n,m) = \int\limits_{G^2} f w_{n,m} d\mu$, $(w_{n,m}: n,m\in\mathbb{N})$ is said to be the (n,m)th Walsh-Fourier coefficient of f. We can extend this definition to the martingales in the usual way. Let us denote $S_{n,m}$ the (n,m)th rectangular partial sum of Walsh-Fourier series of a martingale f. Namely,

$$S_{n,m}(f;x^1,x^2) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \widehat{f}(i,j)\, w_{i,j}(x^1,x^2).$$

A bounded measurable function a is called a p-atom, if there exists a dyadic 2-dimensional cube I^2 such that

$$\int_{I^2} a\,d\mu = 0, \quad \|a\|_\infty \leqslant \mu(I^2)^{-1/p}, \quad \operatorname{supp} a \subset I^2.$$

The dyadic Hardy martingale spaces $H_p\left(G^2\right)$ for $0 < p \leqslant 1$ have an atomic characterization. Namely the following theorem is true (see [24]):

THEOREM W. (Weisz [24]) *A martingale* $f = (f_{n,n},\ n \in \mathbb{N})$ *is in* $H_p\left(G^2\right)$ $(0 < p \leqslant 1)$ *if and only if there exists a sequence* $(a_k,\ k \in \mathbb{N})$ *of* p*-atoms and a sequence* $(\mu_k,\ k \in \mathbb{N})$ *of real numbers such that for every* $n \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n}(a_k) = f_{n,n}, \qquad a.e. \tag{7}$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$*, where the infimum is taken over all decomposition of* f *of the form (7).*

The nth Marcinkiewicz-Fejér means of a martingale f is defined by

$$\mathscr{M}_n(f;x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}(f;x^1,x^2).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D_{k,l}(x^1,x^2) = D_k(x^1)D_l(x^2), \;\; K_n(x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}(x^1,x^2).$$

It is known that [5] there exists a constant c, such that

$$\sup_n \int_{G^2} |K_n(x^1,x^2)|\,d\mu(x^1,x^2) \leqslant c \qquad \text{for all } n \in \mathbb{N}. \tag{8}$$

Let us define the maximal operator $\mathscr{M}^*$ by

$$\mathscr{M}^*(f) = \sup_{n \geqslant 1} |\mathscr{M}_n(f)|.$$

For the martingale

$$f = \sum_{n=0}^{\infty} (f_n - f_{n-1})$$

the conjugate transforms are defined as

$$\widetilde{f^{(t)}} = \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in G$ is fixed. Note that $\widetilde{f^{(0)}} = f$. It is well-known (see [22]) that

$$\left\|\widetilde{f^{(t)}}\right\|_{H_p(G^2)} = \|f\|_{H_p(G^2)}, \quad \|f\|_{H_p(G^2)}^p \sim \int_G \left\|\widetilde{f^{(t)}}\right\|_p^p dt, \tag{9}$$
$$\left(\widetilde{\mathscr{M}_m(f)}\right)^{(t)} = \mathscr{M}_m\left(\widetilde{(f)^{(t)}}\right).$$

To prove our main theorem we need the following lemmas of Goginava [7, Lemma 7, Lemma 9]:

LEMMA 1. (Goginava [7]) *Let* $(x^1,x^2) \in (I_{l^1} \backslash I_{l^1+1}) \times (I_{m^2} \backslash I_{m^2+1})$ *and* $0 \leqslant l^1 \leqslant m^2 < N$. *Then*

$$\int_{I_N \times I_N} |K_n(x^1+t^1, x^2+t^2)| d\mu(t^1,t^2)$$
$$\leqslant \frac{c}{n2^{2N}} \left\{ 2^{l^1-m^2} \sum_{r^1=l^1+1}^{m^2+1} 2^{r^1} D_{2^{m^2+1}}(x^1+e_{l^1}+e_{r^1}) \sum_{s=m^2+1}^{N} D_{2^s}(x^2+e_{m^2}+x^1_{m^2+1,s-1}) \right.$$
$$\left. +2^{l^1+m^2} \sum_{s=l^1}^{m^2} \sum_{r^1=l^1+1}^{s} D_{2^s}(x^1+e_{l^1}+e_{r^1}) \right\}, \text{ for } n \geqslant 2^N,$$

with the notation $x_{i,j} := \sum_{s=i}^{j} x_s e_s$ $(x_{i,i-1} = 0)$.

LEMMA 2. (Goginava [7]) *Let* $(x^1,x^2) \in I_N \times (I_{m^2} \backslash I_{m^2+1})$ *and* $0 \leqslant m^2 < N$. *Then*

$$\int_{I_N \times I_N} |K_n(x^1+t^1, x^2+t^2)| d\mu(t^1,t^2) \leqslant c \frac{2^{m^2}}{n2^N} \sum_{s=m^2}^{N-1} D_{2^s}(x^2+e_{m^2}), \text{ for } n > 2^N.$$

3. Strong convergence theorems

THEOREM 1. *There exists an absolute constant* c, *such that*

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\mathscr{M}_m(f)\|_{H_{2/3}}^{2/3}}{m} \leqslant c \|f\|_{H_{2/3}}^{2/3},$$

for all $f \in H_{2/3}(G^2)$.

Proof. In the next step we follow the method of the authors in [13], but we have to make the necessary changes, because the method presented in [13] is not enough effective in the endpoint case.

Suppose that

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\mathscr{M}_m(f)\|_{2/3}^{2/3}}{m} \leqslant c \|f\|_{H_{2/3}}^{2/3}. \tag{10}$$

Combining (9) and (10) we have that

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\|\mathscr{M}_m(f)\|_{H_{2/3}}^{2/3}}{m}\sim\frac{1}{\log n}\sum_{m=1}^{n}\int_G\frac{\left\|\left(\widetilde{\mathscr{M}_m(f)}\right)^{(t)}\right\|_{2/3}^{2/3}}{m}dt \tag{11}$$

$$=\int_G\frac{1}{\log n}\sum_{m=1}^{n}\frac{\left\|\left(\widetilde{\mathscr{M}_m(f)}\right)^{(t)}\right\|_{2/3}^{2/3}}{m}dt=\int_G\frac{1}{\log n}\sum_{m=1}^{n}\frac{\left\|\mathscr{M}_m\left(\widetilde{(f)^{(t)}}\right)\right\|_{2/3}^{2/3}}{m}dt$$

$$\leqslant\int_G\left\|\left(\widetilde{(f)^{(t)}}\right)\right\|_{H_{2/3}}^{2/3}dt\sim\int_G\|f\|_{H_{2/3}}^{2/3}dt\sim\|f\|_{H_{2/3}}^{2/3}.$$

Since $\mathscr{M}_n$ is bounded (see inequality 8) from the space L_∞ to the space L_∞, by Lemma 2 it is enough to prove that

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\|\mathscr{M}_m(a)\|_{2/3}^{2/3}}{m}<c<\infty$$

for every arbitrary $2/3$-atom a.

Let a be an arbitrary $2/3$-atom with support I^2 and $\mu(I^2)=2^{-2N}$. Without loss of generality, we may assume that $I^2:=I_N\times I_N$. It is easy to see that $\mathscr{M}_n(a)=0$ if $n\leqslant 2^N$. Therefore, we set $n>2^N$.

According to inequality (11) if we invoke the theorem of Levi, we can write

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\|\mathscr{M}_m(a)\|_{2/3}^{2/3}}{m}\leqslant\frac{1}{\log n}\sum_{m=2^N}^{n}\frac{\|\mathscr{M}_m(a)\|_{2/3}^{2/3}}{m}$$

$$\leqslant\frac{1}{\log n}\sum_{m=2^N}^{n}\int_{I_N\times I_N}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu+\frac{1}{\log n}\sum_{m=2^N}^{n}\int_{I_N\times\overline{I_N}}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu$$

$$+\frac{1}{\log n}\sum_{m=2^N}^{n}\int_{\overline{I_N}\times I_N}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu+\frac{1}{\log n}\sum_{m=2^N}^{n}\int_{\overline{I_N}\times\overline{I_N}}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu$$

$$=:I_1+I_2+I_3+I_4.$$

Inequality (8) implies

$$I_1\leqslant\frac{1}{\log n}\sum_{m=2^N}^{\infty}\int_{I_N\times I_N}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu$$

$$\leqslant\frac{1}{\log n}\sum_{m=2^N}^{\infty}\frac{1}{m}\|a\|_\infty^{2/3}/2^{2N}\leqslant\frac{1}{\log n}\sum_{m=2^N}^{n}\frac{1}{m}<c<\infty$$

Now, we estimate the expression I_2. We introduce the notation $J_t:=I_t\backslash I_{t+1}$ $(t\in\mathbb{N})$. We decompose $\overline{I_N}$ and J_{m^2} as the following disjoint union:

$$\overline{I_N}=\bigcup_{m^2=0}^{N-1}J_{m^2},\quad J_{m^2}=\bigcup_{q^2=m^2+1}^{N}I_N^{m^2,q^2}, \tag{12}$$

where

$$I_N^{m^2,q^2} := \begin{cases} I_{q^2+1}(0,\dots,0,x_{m^2}=1,0,\dots,0,x_{q^2}=1), & \text{for } m^2<q^2<N,\\ I_N(0,\dots,0,x_{m^2}=1,0,\dots,0), & \text{for } q^2=N. \end{cases}$$

Let us set $(x^1,x^2)\in I_N\times I_N^{m^2,q^2}$. Lemma 2 immediately gives

$$\begin{aligned} |\mathscr{M}_n(a;x^1,x^2)| &\leqslant \|a\|_\infty \int_{I_N\times I_N} |K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)\\ &\leqslant c2^{3N}\frac{2^{m^2}}{n2^N}\sum_{s=m^2}^{q^2} D_{2^s}(x^2+e_{m^2})\\ &\leqslant \frac{c2^{2N+m^2}}{n}\sum_{s=m^2}^{q^2}2^s \leqslant \frac{c2^{2N+m^2+q^2}}{n}. \end{aligned}$$

Hence,

$$\begin{aligned} I_2 &\leqslant \frac{c2^{4N/3}}{\log n}\sum_{m=2^N}^{n}\sum_{m^2=0}^{N-1}\sum_{q^2=m^2+1}^{N}\int_{I_N\times I_N^{m^2,q^2}}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu\\ &\leqslant \frac{c2^{4N/3}}{\log n}\sum_{m=2^N}^{n}\sum_{m^2=0}^{N-1}\sum_{q^2=m^2+1}^{N}\int_{I_N\times I_N^{m^2,q^2}}\frac{2^{2(m^2+q^2)/3}}{m^{5/3}}d\mu\\ &\leqslant \frac{c2^{4N/3}}{\log n}\sum_{m=2^N}^{n}\sum_{m^2=0}^{N-1}\sum_{q^2=m^2+1}^{N}\frac{2^{2(m^2+q^2)/3}}{m^{5/3}}2^{-N-q^2}\\ &\leqslant \frac{c2^{N/3}}{\log n}\sum_{m=2^N}^{\infty}\frac{1}{m^{5/3}}\sum_{m^2=0}^{N-1}2^{2m^2/3}\sum_{q^2=m^2+1}^{N}2^{-q^2/3}\\ &\leqslant \frac{c2^{N/3}}{\log n}\sum_{m=2^N}^{\infty}\frac{2^{N/3}}{m^{5/3}} \leqslant \frac{c2^{2N/3}}{\log n}\sum_{m=2^N}^{\infty}\frac{1}{m^{5m/3}} \leqslant \frac{c}{N}. \end{aligned}$$

Analogously we can prove that $I_3\leqslant c<\infty$.

At last, we estimate I_4. By the decomposition (12) we write

$$\begin{aligned} I_4 \leqslant{}& \frac{1}{\log n}\sum_{m=2^N}^{n}\sum_{l^1=0}^{N-1}\sum_{m^2=0}^{l^1-1}\int_{J_{l^1}\times J_{m^2}}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu\\ &+\frac{1}{\log n}\sum_{m=2^N}^{n}\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}\int_{J_{l^1}\times J_{m^2}}\frac{|\mathscr{M}_m(a)|^{2/3}}{m}d\mu =: I_{4,1}+I_{4,2}. \end{aligned}$$

We discuss $I_{4,2}$ (the estimate of $I_{4,1}$ goes analogously). For a fixed $(x^1,x^2)\in J_{l^1}\times J_{m^2}$

we apply Lemma 1.

$$
\begin{aligned}
|\mathscr{M}_n(a;x^1,x^2)| &\leqslant \|a\|_\infty \int_{I_N\times I_N} |K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)\\
&\leqslant \frac{2^{N+l^1-m^2}}{n}\sum_{r^1=l^1+1}^{m^2+1} 2^{r^1} D_{2^{m^2+1}}(x^1+e_{l^1}+e_{r^1})\\
&\quad\times \sum_{s=m^2+1}^{N} D_{2^s}(x^2+e_{m^2}+x^1_{m^2+1,s-1})\\
&\quad+\frac{2^{N+l^1+m^2}}{n}\sum_{s=l^1}^{m^2}\sum_{r^1=l^1+1}^{s} D_{2^s}(x^1+e_{l^1}+e_{r^1}).
\end{aligned}
$$

It is easy to see that

$$
\int_{J_{l^1}\times J_{m^2}} D^{2/3}_{2^{m^2+1}}(x^1+e_{l^1}+e_{r^1})D^{2/3}_{2^s}(x^2+e_{m^2}+x^1_{m^2+1,s-1})d\mu(x^1,x^2)
$$

$$
\leqslant c2^{-\left(m^2+s\right)/3}
$$

and

$$
\int_{J_{l^1}\times J_{m^2}} D^{2/3}_{2^s}(x^1+e_{l^1}+e_{r^1})d\mu(x^1,x^2)\leqslant c2^{-m^2-s/3}.
$$

Thus, we immediately get

$$
\begin{aligned}
&\int_{J_{l^1}\times J_{m^2}} |\mathscr{M}_m(a)|^{2/3}d\mu\\
&\leqslant \frac{c2^{2\left(N+l^1-m^2\right)/3}}{m^{2/3}}\sum_{r^1=l^1+1}^{m^2+1}\sum_{s=m^2+1}^{N} 2^{2r^1/3}\\
&\quad\times\int_{J_{l^1}\times J_{m^2}} D^{2/3}_{2^{m^2+1}}(x^1+e_{l^1}+e_{r^1})D^{2/3}_{2^s}(x^2+e_{m^2}+x^1_{m^2+1,s-1})d\mu\left(x^1,x^2\right)\\
&\quad+\frac{c2^{2\left(N+l^1+m^2\right)/3}}{m^{2/3}}\sum_{s=l^1}^{m^2}\sum_{r^1=l^1+1}^{s}\int_{J_{l^1}\times J_{m^2}} D^{2/3}_{2^s}(x^1+e_{l^1}+e_{r^1})d\mu(x^1,x^2)\\
&\leqslant \frac{c2^{2(N+l^1-m^2)/3}}{m^{2/3}}\sum_{r^1=l^1+1}^{m^2+1}2^{2r^1/3}\sum_{s=m^2+1}^{N}2^{-\left(m^2+s\right)/3}+\frac{c2^{2(N+l^1+m^2)/3}}{m^{2/3}}\sum_{s=l^1}^{m^2}\sum_{r^1=l^1+1}^{s}2^{-m^2-s/3}\\
&\leqslant \frac{c2^{2(N+l^1-m^2)/3}}{m^{2/3}}2^{-2m^2/3}\sum_{r^1=l^1+1}^{m^2+1}2^{2r^1/3}+\frac{c2^{2\left(N+l^1+m^2\right)/3}}{m^{2/3}}\sum_{s=l_1}^{m^2}(s-l_1-1)\,2^{-m^2-s/3}\\
&\leqslant \frac{c2^{2\left(N+l^1-m^2\right)/3}}{m^{2/3}}+\frac{c2^{\left(2N+l^1-m^2\right)/3}}{m^{2/3}}.
\end{aligned}
$$

and

$$I_{4,2} \leqslant \frac{c}{\log n} \sum_{m=2^N}^{n} \frac{1}{m} \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} \frac{2^{2(N+l^1-m^2)/3} + 2^{\left(2N-m^2+l^1\right)/3}}{m^{2/3}}$$
$$\leqslant \frac{c2^{2N/3}}{\log n} \sum_{m=2^N}^{n} \frac{1}{m^{5/3}} \sum_{l^1=0}^{N-1} 1 \leqslant c. \quad \square$$

COROLLARY 1. *Let* $f \in H_{2/3}\left(G^2\right)$. *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\mathscr{M}_m(f) - f\|_{H_{2/3}}^{2/3}}{m} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\mathscr{M}_m(f)\|_{H_{2/3}}^{2/3}}{m} = \|f\|_{H_{2/3}}^{2/3}.$$

Our theorem and inequality (4) immediately give the two-dimensional version of inequality (2).

COROLLARY 2. *Let* $0 < p \leqslant 2/3$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\frac{1}{\log^{[1/3+p]} n} \sum_{m=1}^{n} \frac{\|\mathscr{M}_m(f)\|_{H_p}^{p}}{m^{3-3p}} \leqslant c_p \|f\|_{H_p}^{p} \quad (f \in H_p).$$

PROBLEM 1. The Walsh system has got three well known rearrangements (for details see e.g. [15]), the original definition belongs to Walsh, the common used ordering given by Paley and the Kaczmarz rearrangement used in telecommunication. The authors think that it is not possible to reach main theorem and corollaries of this paper for Walsh-Kaczmarz system by using lemmas and methods, which have been applied before. So, it is interesting to investigate the validity of these statements for Kaczmarz ordering.

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