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SMALL HANKEL OPERATORS ON DIRICHLET–TYPE SPACES AND APPLICATIONS

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Abstract. In this paper, we characterize the boundedness and compactness of small Hankel operators on Dirichlet-type spaces D_ρ.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. As usual, $H(\mathbb{D})$ denotes the class of functions analytic on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the set of bounded analytic functions on $\mathbb{D}$.

Let $\rho : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The L^2_ρ space is the set of functions such that

$$\|f\|^2_{L^2_\rho} = \int_{\mathbb{D}} |f(z)|^2 \rho(1-|z|^2)\,dA(z) < \infty.$$

When $\rho(t) = 1$, L^2_ρ is denoted by L^2. The weighted Bergman space A^2_ρ is a subset of L^2_ρ consisting of $f \in H(\mathbb{D})$. When $\rho(t) = t^\alpha$, $\alpha > -1$, A^2_ρ gives the classical Bergman space.

The Dirichlet-type spaces D_ρ consist of those functions $f' \in A^2_\rho$ with

$$\|f\|^2_{D_\rho} = |f(0)|^2 + \|f'\|^2_{L^2_\rho}.$$

When $\rho(t) = t^\alpha$, $0 < \alpha < 1$, it gives the weighted Dirichlet spaces D_α. For more information on D_ρ, we refer to [1], [2], [3], [10] and [14].

Under some conditions of ρ (see Lemma 4), for functions $f \in A^2_\rho$ and $\alpha \geqslant 0$, we can define a small Hankel-type operator $h_{\alpha,f}$ on the set of all polynomials on $\mathbb{D}$ (denoted by $\mathscr{P}$), by

$$h_{\alpha,f}(g) = \overline{P_\alpha(f\overline{g})}, \quad g \in \mathscr{P},$$

where

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\alpha}} (1-|w|^2)^\alpha\, dA(w).$$

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For more information on small Hankel operators, see for example, [4], [8], [16] and [20].

In this paper, we always assume that ρ is a right-continuous and nondecreasing function. We say that ρ is so called upper (resp. lower) type $\gamma \in (0,\infty)$ (see [9]), if

$$\rho(xy) \leqslant Cx^{\gamma}\rho(y), \quad x \geqslant 1 \ \ (\text{resp. } x \leqslant 1) \ \text{ and } \ 0 < y < \infty.$$

REMARK 1. If ρ is a right-continuous and nondecreasing function with upper type γ for some $\gamma > 0$, it is easy to see that $\rho(2y) \leqslant C\rho(y)$, where C is a positive constant independent of y.

Recently there has been a huge interest in studying the boundedness and compactness of various operators (composition, integral, product-type etc.) on spaces of analytic functions on various domains (see, for example, [4, 5, 8, 11, 17, 18] and related references therein). In [16], Rochberg and Wu studied the boundedness of $h_{\alpha,f}$ on weighted Dirichlet spaces D_α. Motivated by their work, we are going to study the boundedness of $h_{\alpha,f}$ on more general Dirichlet-type spaces D_ρ.

Here and afterwards, for two functions f and g, by $f \asymp g$ we mean that $g \lesssim f \lesssim g$, where $f \lesssim g$ means that there exists a positive constant C depending only on ρ and parameters $\alpha, \gamma, \ldots$, such that $f \leqslant Cg$.

2. Auxiliary results

To prove the main theorems, we need the following lemmas.

LEMMA 1. *Suppose that ρ is of upper type γ with $0 < \gamma < \infty$. If $s < 1$ and $\tau + s > 2 + \gamma$. Then*

$$\int_{\mathbb{D}} \frac{\rho\left(1-|w|^2\right)}{(1-|w|^2)^s|1-\overline{w}z|^\tau} dA(w) \lesssim \frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^{s+\tau-2}}$$

for all $z \in \mathbb{D}$.

Proof. Since ρ is of upper type γ, applying Lemma 3.10 of [20] gives

$$\int_{\mathbb{D}} \frac{\rho\left(1-|w|^2\right)}{(1-|w|^2)^s|1-\overline{w}z|^\tau} dA(w) \lesssim \int_{\mathbb{D}} \frac{\rho\left(|1-\overline{w}z|\right)}{(1-|w|^2)^s|1-\overline{w}z|^\tau} dA(w)$$
$$\lesssim \frac{\rho\left(1-|z|^2\right)}{(1-|z|^2)^{s+\tau-2}}. \quad \square$$

Using Lemma 1, following the proof of Theorem 3.3 in [12], it is easy to get the lemma below.

LEMMA 2. *Suppose that ρ is of upper type γ with $0 < \gamma < 1$ and $\tau > -1$, $\sigma > -1$. Then $f \in D_\rho$ if and only if*

$$I =: \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^{4+\tau+\sigma}} (1-|z|^2)^\tau (1-|w|^2)^\sigma \rho(1-|w|^2) dA(z) A(w) < \infty.$$

Moreover, we have $|f(0)|^2 + I \asymp \|f\|_{D_\rho}^2$.

LEMMA 3. *Suppose that ρ is of upper type γ with $0<\gamma<1$ and $n\in\mathbb{N}^+$. Let*

$$F_a(z)=\frac{(1-|a|^2)^{1+n}z^{n+1}}{\sqrt{\rho(1-|a|^2)}(1-\overline{a}z)^{n+1}}$$

and

$$G_a(z)=\frac{(1-|a|^2)\sqrt{\rho(1-|a|^2)}(1-|z|^2)^{\alpha}}{\rho(1-|z|^2)(1-\overline{a}z)^{2+\alpha}},\quad \alpha>0,\quad a\in\mathbb{D}.$$

Then $F_a\in D_\rho$ and $G_a\in L^2_\rho$.

Proof. By Lemma 1, we can deduce easily that

$$\int_{\mathbb{D}}|F_a'(z)|^2\rho(1-|z|^2)dA(z)\lesssim\frac{(1-|a|^2)^{2+2n}}{\rho(1-|a|^2)}\int_{\mathbb{D}}\frac{\rho(1-|z|^2)}{|1-\overline{a}z|^{2n+4}}dA(z)\lesssim 1.$$

That is, $F_a\in D_\rho$.

Since ρ is of upper type γ, applying Lemma 3.10 of [20] yields

$$\begin{aligned}
&\int_{\mathbb{D}}|G_a(z)|^2\rho(1-|z|^2)dA(z)\\
&\qquad\lesssim(1-|a|^2)^2\rho(1-|a|^2)\int_{\mathbb{D}}\frac{(1-|z|^2)^{2\alpha}}{\rho(1-|z|^2)|1-\overline{a}z|^{2\alpha+4}}dA(z)\\
&\qquad\lesssim(1-|a|^2)^2\int_{\mathbb{D}}\frac{\rho(|1-\overline{a}z|)(1-|z|^2)^{2\alpha}}{\rho(1-|z|^2)|1-\overline{a}z|^{2\alpha+4}}dA(z)\\
&\qquad\lesssim(1-|a|^2)^2\int_{\mathbb{D}}\frac{(1-|z|^2)^{2\alpha-\gamma}}{|1-\overline{a}z|^{2\alpha+4-\gamma}}dA(z)<\infty.
\end{aligned}$$

Thus, $G_a\in D_\rho$. □

LEMMA 4. *Suppose that ρ is of upper type γ with $0<\gamma<1$ and $\alpha\geqslant 0$. Then P_α is bounded on L^2_ρ.*

Proof. Let

$$M(z,w)=\frac{\rho(1-|z|)^{1/2}}{\rho(1-|w|)^{1/2}|1-\overline{w}z|^2}$$

and

$$h(z)=\frac{\rho(1-|z|)^{1/2}}{(1-|z|)^t},\quad \gamma<t<1.$$

Define

$$T_Mg(z)=:\int_{\mathbb{D}}M(z,w)g(w)dA(w),\quad g\in L^2.$$

Using Lemma 1 and [20, Lemma 4.2.2] implies

$$\int_{\mathbb{D}}M(z,w)h(w)dA(w)\lesssim h(z)$$

and

$$\int_{\mathbb{D}} M(z,w)h(z)dA(z) \lesssim h(w).$$

Hence, by Schur' theorem ([20, Theorem 3.6]), we obtain

$$\int_{\mathbb{D}} |T_M g(w)|^2 dA(w) \lesssim \int_{\mathbb{D}} |g(w)|^2 dA(w)$$

for all $g \in L^2$. For $f \in L^2_\rho$, $F(w) = |f(w)|\rho(1-|w|)^{1/2} \in L^2$. We have

$$\begin{aligned}
\|P_\alpha f\|^2_{L^2_\rho} &= \int_{\mathbb{D}} |P_\alpha f(z)|^2 \rho(1-|z|^2)dA(z) \\
&= \int_{\mathbb{D}} \left| \int_{\mathbb{D}} \frac{f(w)(1-|w|^2)^\alpha}{(1-\overline{w}z)^{2+\alpha}} dA(w) \right|^2 \rho(1-|z|^2)dA(z) \\
&\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|f(w)|}{|1-\overline{w}z|^2} dA(w) \right)^2 \rho(1-|z|^2)dA(z) \\
&= \int_{\mathbb{D}} |T_M F(z)|^2 dA(z) \\
&\lesssim \|f\|^2_{L^2_\rho}.
\end{aligned}$$

That is, P_α is bounded on L^2_ρ. □

LEMMA 5. *Suppose that* $\alpha \geqslant 0$ *and* ρ *is of upper type* γ *with* $0 < \gamma < 1$. *If* $f \in A^2_\rho$ *and* $h_{\alpha,f} : D_\rho \to L^2_\rho$ *is bounded, then*

$$\sup_{a\in\mathbb{D}} (1-|a|^2)|f(a)| < \infty.$$

Proof. For $f \in A^2_\rho$, it is well known that

$$\sup_{a\in\mathbb{D}} (1-|a|^2)|f(a)| \asymp |f(0)| + \sum_{k=1}^{n} |f^{(k)}(0)| + \sup_{a\in\mathbb{D}} (1-|a|^2)^{n+2}|f^{(n+1)}(a)|.$$

Thus, it is sufficient to prove

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^{n+2}|f^{(n+1)}(a)| < \infty.$$

Applying the reproducing formula yields

$$\begin{aligned}
&(1-|a|^2)^{n+2} f^{(n+1)}(a) \\
\asymp &(1-|a|^2)^{n+2} \int_{\mathbb{D}} \frac{\overline{z}^{n+1} f(z)(1-|z|^2)^\alpha}{(1-\overline{z}a)^{3+\alpha+n}} dA(z) \\
\asymp &(1-|a|^2)^{n+2} \int_{\mathbb{D}} \frac{\overline{z}^{n+1} f(z)}{(1-\overline{z}a)^{1+n}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha (1-|z|^2)^\alpha}{(1-\overline{z}w)^{2+\alpha}(1-\overline{w}a)^{2+\alpha}} dA(w)dA(z) \\
= &\int_{\mathbb{D}} \overline{h_{\alpha,f}(F_a)(w) G_a(w)} \rho(1-|w|^2)dA(w).
\end{aligned}$$

By Lemma 3, we know that $F_a \in D_\rho$ and $G_a \in L^2_\rho$. Using Hölder's inequality, we get

$$\begin{aligned}(1-|a|^2)^{n+2}|f^{(n+1)}(a)| &\lesssim \|h_{\alpha,f}(F_a)\|_{L^2_\rho}\|G_a\|_{L^2_\rho} \\ &\lesssim \|h_{\alpha,f}\| < \infty.\end{aligned}$$

The desired result is obtained. □

3. Boundedness and compactness of $h_{\alpha,g}$

Let μ be a finite positive Borel measure on $\mathbb{D}$. We say that μ is a D_ρ-Carleson measure if the inclusion map $i: D_\rho \to L^2(\mu)$ is bounded, that is

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leqslant C\|f\|^2_{D_\rho}$$

for all $f \in D_\rho$. The best constant C, denoted by $\|\mu\|_\rho$, is said to be the norm of μ. Suppose μ is D_ρ-Carleson measure, we say that μ is a vanishing D_ρ-Carleson measure if the inclusion map $i: D_\rho \to L^2(d\mu)$ is compact in the following sense:

$$\lim_{n\to\infty} \int_{\mathbb{D}} |f_n(z)|^2 d\mu(z) = 0,$$

whenever $\{f_n\}$ is a bounded sequence in D_ρ that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

For $f \in H(\mathbb{D})$, define positive measure μ_f by

$$d\mu_f(z) = |f(z)|^2 \rho(1-|z|^2) dA(z).$$

THEOREM 1. *Let $0<\gamma<1$ and $\alpha > \frac{1+\gamma}{2}$. Suppose that ρ is of upper type γ and $f \in H(\mathbb{D})$. Then $h_{\alpha,f}: D_\rho \to L^2_\rho$ is bounded if and only if μ_f is a D_ρ-Carleson measure.*

Proof. Sufficiency. Suppose $g \in D_\rho$, then $\overline{g}f \in L^2_\rho$. Combining this with Lemma 4 and the definition of D_ρ-Carleson measure yield

$$\|h_{\alpha,f}(g)\|_{L^2_\rho} \lesssim \|fg\|_{L^2_\rho} \lesssim C_f\|g\|_{D_\rho}.$$

This leads that $h_{\alpha,f}: D_\rho \to L^2_\rho$ is bounded.

Necessity. Since $1 \in D_\rho$, $\overline{f} = h_{\alpha,f}(1) \in L^2_\rho$, furthermore $f \in A^2_\rho$. Note that $f = P_\alpha f$. For $g \in D_\rho$, we have

$$f(z)\overline{g(z)} - \overline{h_{\alpha,f}(g)(z)} = \int_{\mathbb{D}} \frac{f(w)(\overline{g(z)-g(w)})}{(1-\overline{w}z)^{2+\alpha}}(1-|w|^2)^\alpha dA(w).$$

Since ρ is upper type γ, for any $\varepsilon \in (\gamma,1)$, from Lemma 1, we get

$$\int_{\mathbb{D}} \frac{\rho(1-|w|^2)}{|1-\overline{w}z|^{2+\varepsilon}} dA(w) \lesssim \frac{\rho(1-|z|^2)}{(1-|z|^2)^\varepsilon}.$$

Notice the fact that

$$C_f =: \sup_{w\in\mathbb{D}}(1-|w|^2)^2|f(w)|^2 < \infty$$

(see Lemma 5). Using Hölder's inequality gives

$$\begin{aligned}
&\left|f(z)\overline{g(z)} - \overline{h_{\alpha,f}(g)(z)}\right|^2\\
\leqslant &\int_{\mathbb{D}} \frac{|f(w)|^2(1-|w|^2)^2}{|1-\overline{w}z|^{2+\varepsilon}}\rho(1-|w|^2)dA(w)\\
&\qquad\qquad \times \int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon}} \frac{(1-|w|^2)^{2\alpha-2}}{\rho(1-|w|^2)}dA(w)\\
\lesssim & C_f \frac{\rho(1-|z|^2)}{(1-|z|^2)^{\varepsilon}} \int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon}} \frac{(1-|w|^2)^{2\alpha-2}}{\rho(1-|w|^2)}dA(w)\\
\leqslant & \frac{C_f}{(1-|z|^2)^{\varepsilon}} \int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon}} \frac{\rho(|1-\overline{w}z|)(1-|w|^2)^{2\alpha-2}}{\rho(1-|w|^2)}dA(w)\\
\leqslant & \frac{C_f}{(1-|z|^2)^{\varepsilon}} \int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon}} \left(\frac{|1-\overline{w}z|}{1-|w|^2}\right)^{\gamma}(1-|w|^2)^{2\alpha-2}dA(w)\\
= & \frac{C_f}{(1-|z|^2)^{\varepsilon}} \int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon-\gamma}}(1-|w|^2)^{2\alpha-2-\gamma}dA(w).
\end{aligned}$$

From Lemma 2, we get

$$\begin{aligned}
&\|f\overline{g} - \overline{h_{\alpha,f}(g)}\|^2_{L^2_\rho}\\
\lesssim & C_f \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|g(z)-g(w)|^2}{|1-\overline{w}z|^{2+2\alpha-\varepsilon-\gamma}}(1-|w|^2)^{2\alpha-2-\gamma}(1-|z|^2)^{-\varepsilon}\rho(1-|w|^2)dA(w)dA(z)\\
\lesssim & C_f\|g\|^2_{D_\rho}.
\end{aligned}$$

Therefore,

$$\|f\overline{g}\|^2_{L^2_\rho} \lesssim \|\overline{h_{\alpha,f}(g)}\|^2_{L^2_\rho} + C_f\|g\|^2_{D_\rho} \lesssim C_f\|g\|^2_{D_\rho}.$$

That is, μ_f is a D_ρ-Carleson measure. The proof is completed. □

THEOREM 2. *Let $0<\gamma<1$ and $\alpha > \frac{1+\gamma}{2}$. Suppose that ρ is of upper type γ and $f \in H(\mathbb{D})$. Then $h_{\alpha,f} : D_\rho \to L^2_\rho$ is compact if and only if μ_f is a vanishing D_ρ-Carleson measure.*

Proof. Let μ_f be a vanishing D_ρ-Carleson measure. Suppose that $g_n \in D_\rho$ and $g_n \to 0$ weakly as $n\to\infty$. Since D_ρ is a Hilbert space, $g_n \to 0$ weakly if and only if $\{g_n\}$ is a bounded sequence in D_ρ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. From the proof of Theorem 1, we have

$$\|h_{\alpha,f}(g_n)\|_{L^2_\rho} \lesssim \|g_n f\|_{L^2_\rho} \to 0, \ \ as\ n\to\infty.$$

That is, $h_{\alpha,f}$ is compact.

Conversely, if $h_{\alpha,f}: D_\rho \to L^2_\rho$ is compact, let $\{g_n\}$ as above. Following the proof of Theorem 1 gives

$$\|f\overline{g_n} - \overline{h_{\alpha,f}(g_n)}\|_{L^2_\rho} \lesssim C_f \|g_n\|_{D_\rho}.$$

So,

$$\|f\overline{g_n}\|_{L^2_\rho} \lesssim C_f \|g_n\|_{D_\rho} + \|\overline{h_{\alpha,f}(g_n)}\|_{L^2_\rho} \to 0, \ \ as\ n \to \infty.$$

Hence μ_f is a vanishing D_ρ-Carleson measure. □

4. An application

In this section, as an application of the boundedness of small Hankel operators, we establish a relationship between decompositions of f and μ_f being a D_ρ-Carleson measure for functions f in weighted Bergman spaces A^2_ρ. To prove the result, we need some preliminaries.

Let $a \in \mathbb{D}$ and

$$S(a) = \left\{ z = re^{i\theta} \in \mathbb{D} : |a| \leqslant |z|, \ \left|\frac{\arg(a\bar{z})}{2\pi}\right| \leqslant \frac{1-|a|}{2} \right\}$$

be the Carleson box with vertex at a. When $a = 0$, $S(0) = \mathbb{D}$.

The hyperbolic distance of z and w in $\mathbb{D}$ is denoted by

$$d(z,w) = \log \frac{1 + |\frac{w-z}{1-\overline{w}z}|}{1 - |\frac{w-z}{1-\overline{w}z}|}.$$

A sequence $\{a_k\} \subseteq \mathbb{D}$ is called a d-lattice ($d > 0$), if every point of $\mathbb{D}$ is within hyperbolic distance $5d$ of some a_k and no two points of this sequence are within hyperbolic distance $d/5$ of each other.

We also need a few lemmas, the following one is a special case of Theorem 4.1 in [13].

LEMMA 6. *Suppose that ρ satisfies*

$$\int_{S(a)} \rho(1-|z|^2)\,dA(z) \int_{S(a)} \frac{(1-|z|^2)^{2\alpha}}{\rho(1-|z|^2)}\,dA(z) \lesssim (1-|a|^2)^{4+2\alpha}, \tag{4.1}$$

where $\alpha \geqslant -1/2$. Then there exists a sequence $\{a_k\}_{k=1}^\infty$ which is d-lattice for some $d > 0$, such that any $f \in A^2_\rho$ has the form

$$f(z) = \sum_{k=1}^\infty c_k \frac{(1-|a_k|^2)^{2+\alpha}}{(1-\overline{a_k}z)^{2+\alpha}} \left(\int_{E(a_k,r)} \rho(1-|w|^2)\,dA(w) \right)^{-1/2}$$

for some $\{c_k\}_{k=1}^\infty \in l^2$, where $0 < r < 1$ and

$$E(a_k,r) = \{w \in \mathbb{D} : \left|\frac{a_k - w}{1-\overline{a_k}w}\right| < r\}$$

is the pseudo-hyperbolic disk. Moreover,

$$\|f\|_{A^2_\rho} \lesssim \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}. \tag{4.2}$$

The a_k are not uniquely determined by f but the series sets up an isomorphism with a quotient space of l^2. Convergence in series is pointwise and in norm.

LEMMA 7. *Suppose that $0<\delta \leqslant \gamma < 1$, ρ is of upper type γ and lower type δ, then ρ satisfies the condition in (4.1) for $\alpha > 0$.*

Proof. An easy computation gives

$$\begin{aligned}
&\int_{S(a)} \rho(1-|z|^2)dA(z)\int_{S(a)} \frac{(1-|z|^2)^{2\alpha}}{\rho(1-|z|^2)}dA(z)\\
\asymp&(1-|a|)^2\int_{|a|}^1 \rho(1-r^2)rdr\int_{|a|}^1 \frac{1}{\rho(1-r^2)}(1-r^2)^{2\alpha}rdr\\
=&(1-|a|)^2\int_{|a|}^1 \frac{\rho(1-r^2)}{\rho(1-|a|^2)}rdr\int_{|a|}^1 \frac{\rho(1-|a|^2)}{\rho(1-r^2)}(1-r^2)^{2\alpha}rdr.
\end{aligned}$$

Since

$$\rho(xy) \lesssim x^\gamma \rho(y), \ \ x \geqslant 1, \ \ 0<y<\infty$$

and

$$\rho(xy) \lesssim x^\delta \rho(y), \ \ x \leqslant 1, \ \ 0<y<\infty.$$

It follows that ρ satisfies the condition in (4.1). □

LEMMA 8. *Suppose that ρ is of upper type γ with $0<\gamma<1$. For $g \in H(\mathbb{D})$, if μ_g is a D_ρ-Carleson measure, then $\mu_{T(g)}$ is also a D_ρ-Carleson measure. Here*

$$T(g)(z) = \int_{\mathbb{D}} \frac{|g(w)|}{|1-\overline{w}z|^{2+\alpha}}(1-|w|^2)^\alpha dA(w), \ \ \alpha \geqslant 0.$$

Proof. Let $0<\gamma<t<1$ and $\eta \in \mathbb{D}$, then $F_\eta(z) = \frac{1}{(1-\overline{\eta}z)^{t/2}} \in D_\rho$. Since μ_g is a D_ρ-Carleson measure,

$$\int_{\mathbb{D}} \frac{|g(z)|^2}{|1-\overline{\eta}z|^t}\rho(1-|z|^2)dA(z) \lesssim \int_{\mathbb{D}} \frac{\rho(1-|z|^2)}{|1-\overline{\eta}z|^{2+t}}dA(z) \lesssim \frac{\rho(1-|\eta|^2)}{(1-|\eta|^2)^t},$$

where we used Lemma 1 in the last inequality. Hence, for $f \in D_\rho$

$$\begin{aligned}
&|(|f(z)|T(g)(z)-T(fg))(z)|^2\\
\leqslant&\left(\int_{\mathbb{D}} \frac{|g(w)||f(z)-f(w)|}{|1-\overline{w}z|^2}dA(w)\right)^2\\
\lesssim&\int_{\mathbb{D}} \frac{|g(w)|^2}{|1-\overline{w}z|^t}\rho(1-|w|^2)dA(w)\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^{4-t}\rho(1-|w|^2)}dA(w)\\
\lesssim&\frac{\rho(1-|z|^2)}{(1-|z|)^t}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{w}z|^{4-t}\rho(1-|w|^2)}dA(w).
\end{aligned}$$

Following the proof of Theorem 1, we obtain

$$\int_{\mathbb{D}} |(|f(z)|T(g)(z) - T(fg))(z)|^2 \rho(1-|z|^2)dA(z) \lesssim \|f\|_{D_\rho}^2.$$

By a similar argument in the proof of Lemma 4, we have

$$\int_{\mathbb{D}} |T(fg)(z)|^2 \rho(1-|z|^2)dA(z) \lesssim \int_{\mathbb{D}} |f(z)g(z)|^2 \rho(1-|z|^2)dA(z) \lesssim \|f\|_{D_\rho}^2.$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)T(g)(z)|^2 \rho(1-|z|^2)dA(z) \\ \lesssim &\int_{\mathbb{D}} |f(z)T(g)(z) - T(fg)(z)|^2 \rho(1-|z|^2)dA(z) \\ &\qquad + \int_{\mathbb{D}} |T(fg)(z)|^2 \rho(1-|z|^2)dA(z) \lesssim \|f\|_{D_\rho}^2. \end{aligned}$$

That is $\mu_{T(g)}$ is a D_ρ-Carleson measure. □

The next lemma can be found in [15, Lemma 2.2].

LEMMA 9. *Let $\{a_k\}_{k=0}^\infty$ be a d-lattice in $\mathbb{D}$ ($d>0$). For $f \in H(\mathbb{D})$, there exists a disjoint decomposition $\{D_k\}_{k=0}^\infty$ of $\mathbb{D}$, i.e., $\mathbb{D} = \cup_k D_k$, such that $|D_k| \asymp (1-|a_k|^2)^2$ and*

$$|(I-A)(f)(z)| \lesssim dT(f)(z),$$

where I is identity operator, T as in Lemma 8 and

$$A(f)(z) = \sum_{k=0}^{\infty} f(a_k)|D_k| \frac{(1-|a_k|^2)^\alpha}{(1-\overline{a_k}z)^{2+\alpha}}, \quad \alpha \geqslant 0.$$

Now we prove the main result of this section.

THEOREM 3. *Let $0<\delta \leqslant \gamma < 1$ and $\alpha > \frac{1+\gamma}{2}$. Suppose that ρ is of upper type γ and lower type δ. For any sequence $\{a_k\}_{k=1}^\infty$ which is d-lattice ($d>0$) in $\mathbb{D}$, we have:*

(1). *For sequences $\{c_k\}_{k=0}^\infty$, if $\sum_{k=0}^\infty |c_k|^2 \delta_{a_k}$ is a D_ρ-Carleson measure and*

$$f(z) = \sum_{k=0}^{\infty} c_k \frac{(1-|a_k|^2)^{1+\alpha}}{\sqrt{\rho(1-|a_k|^2)}(1-\overline{a_k}z)^{2+\alpha}}. \tag{4.3}$$

then $f \in A_\rho^2$ and μ_f is a D_ρ-Carleson measure, where δ_{a_k} is a point value measure.

(2). *If $f \in H(\mathbb{D})$ and μ_f is a D_ρ-Carleson measure, then f can be written as (4.3) and*

$$\left\| \sum_{k=0}^{\infty} |c_k|^2 \delta_{a_k} \right\|_\rho \lesssim \|\mu_f\|_\rho.$$

Proof. (1). From the assumption on the sequence $\{c_k\}_{k=0}^{\infty}$, we know that $\{c_k\}_{k=0}^{\infty} \in l^2$. Using Lemmas 6 and 7, we have $f \in A_\rho^2$. To prove that μ_f is a D_ρ-Carleson measure, by Theorem 1, it is sufficient to prove that $h_{\alpha,f} : D_\rho \to L_\rho^2$ is bounded. For $g \in D_\rho$, $z \in \mathbb{D}$, let

$$G(w) = \frac{g(w)}{(1-\overline{z}w)^{2+\alpha}}, \quad w \in \mathbb{D}.$$

It is easy to caculate that $G \in A_\rho^2$. Thus, $G = P_\alpha(G)$ and

$$\begin{aligned}\overline{h_{\alpha,f}(g)(z)} &= \int_{\mathbb{D}} \frac{f(w)\overline{g(w)}}{(1-\overline{w}z)^{2+\alpha}}(1-|w|^2)^\alpha dA(w)\\ &= \sum_{k=0}^{\infty} c_k \frac{(1-|a_k|^2)^{1+\alpha}}{\sqrt{\rho(1-|a_k|^2)}}\overline{P_\alpha(G)(a_k)}\\ &= \sum_{k=0}^{\infty} c_k \frac{(1-|a_k|^2)^{1+\alpha}\overline{g(a_k)}}{\sqrt{\rho(1-|a_k|^2)}(1-\overline{a_k}z)^{2+\alpha}}.\end{aligned}$$

Using Lemma 6, (4.2) and assumption of the theorem, it follows that

$$\|h_{\alpha,f}(g)\|_{L_\rho^2}^2 \lesssim \sum_{k=0}^{\infty} |c_k g(a_k)|^2 \lesssim \left\| \sum_{k=0}^{\infty} |c_k|^2 \delta_{a_k} \right\|_\rho \|g\|_{D_\rho}^2.$$

(2). Let $\{a_k\}_{k=0}^{\infty}$ be a d-lattice in $\mathbb{D}$ and $f \in H(\mathbb{D})$. From Lemma 9, there exists a disjoint decomposition $\{D_k\}$ of $\mathbb{D}$, such that

$$|D_k| \asymp (1-|a_k|^2)^2.$$

For $g \in D_\rho$, the assumption on f implies $fg \in A_\rho^2$. From [13, page 328] and the fact that

$$\rho(1-|a_k|^2) \asymp \rho(1-|z|^2), \quad z \in D_k,$$

we know

$$\{f(a_k)g(a_k)(1-|a_k|^2)\sqrt{\rho(1-|a_k|^2)}\}_{k=0}^{\infty} \in l^2.$$

Since $|fg|^2$ is subharmonic, using the submean value property, we have

$$|f(a_k)g(a_k)|^2 \lesssim \frac{1}{\rho(1-|a_k|^2)|D_k|} \int_{D_k} |f(z)g(z)|^2 \rho(1-|z|^2) dA(z).$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} \rho(1-|a_k|^2)|D_k||f(a_k)g(a_k)|^2 &\lesssim \sum_{k=0}^{\infty} \int_{D_k} |f(z)g(z)|^2 \rho(1-|z|^2) dA(z)\\ &\lesssim \int_{\mathbb{D}} |f(z)g(z)|^2 \rho(1-|z|^2) dA(z)\\ &\lesssim \|\mu_f\|_\rho \|g\|_{D_\rho}^2.\end{aligned}$$

Thus, $\sum_{k=0}^{\infty}\left|f(a_k)\frac{\sqrt{\rho(1-|a_k|^2)}}{(1-|a_k|^2)}|D_k|\right|^2\delta_{a_k}$ is a D_ρ-Carleson measure and

$$\left\|\sum_{k=0}^{\infty}\left|f(a_k)\frac{\sqrt{\rho(1-|a_k|^2)}}{(1-|a_k|^2)}|D_k|\right|^2\delta_{a_k}\right\|_\rho \lesssim \|\mu_f\|_\rho . \tag{4.4}$$

Then, from (1) and (4.4), we get that $\mu_{A(f)}$ is a D_ρ-Carleson measure. Using Lemma 9 again, we have

$$|(I-A)(f)(z)| \lesssim dT(f)(z).$$

If d sufficiently small, using Lemma 8, we obtain

$$\|I-A\| \leqslant 1/2.$$

So A^{-1} exists and

$$\|A^{-1}\| \leqslant \sum_{n=0}^{\infty}\|(I-A)^n\| \leqslant 2.$$

Combining this with the definition of the operator A, we can rewrite the function f, that is

$$\begin{aligned} f(z) &= (AA^{-1}f)(z) \\ &\asymp \sum_{k=0}^{\infty}(A^{-1}f)(a_k)|D_k|\frac{(1-|a_k|^2)^\alpha}{(1-\overline{a_k}z)^{2+\alpha}} \\ &\asymp \sum_{k=0}^{\infty}(A^{-1}f)(a_k)|D_k|\frac{\sqrt{\rho(1-|a_k|^2)}}{(1-|a_k|^2)}\frac{(1-|a_k|^2)^{1+\alpha}}{\sqrt{\rho(1-|a_k|^2)}(1-\overline{a_k}z)^{2+\alpha}}. \end{aligned}$$

Let

$$c_k = (A^{-1}f)(a_k)\frac{\sqrt{\rho(1-|a_k|^2)}}{(1-|a_k|^2)}|D_k|.$$

Using (4.4) and the boundedness of A^{-1} give

$$\left\|\sum_{k=0}^{\infty}|c_k|^2\delta_{a_k}\right\|_\rho \lesssim \left\|\mu_{A^{-1}f}\right\|_\rho \lesssim \|A^{-1}\|\,\|\mu_f\|_\rho .$$

The proof is completed. □

REMARK 2. Our proofs of theorems above depend on the condition $\alpha > \frac{1+\gamma}{2}$, we failed to prove those results without the condition.

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