

## POSITIVITY OF SUMS AND INTEGRALS FOR CONVEX FUNCTIONS OF HIGHER ORDER OF $n$ VARIABLES

ASIF R. KHAN, JOSIP PEČARIĆ AND SANJA VAROŠANEC

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*Abstract.* We provide one general discrete identity for  $\sum \cdots \sum P_{k_1 \cdots k_n} f(x_{1k_1}, \dots, x_{nk_n})$  and one general integral identity for  $\Lambda(f) = \int \cdots \int P(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \cdots dx_n$  of Popoviciu type. We obtain necessary and sufficient conditions under which these sum and integral are non-negative for higher order convex functions of  $n$  variables. These identities and inequalities generalize various established results. We also state new generalized Lagrange type and Cauchy type mean value theorems. We obtain an Ostrowski type result as a special case of our main integral identity and we also establish a bound on remainder term of our main integral identity in terms of  $L_p$ -norm by using Hölder's inequality. Finally we apply the functional  $\Lambda(f)$  on the family of some exponentially convex functions and discuss some of its major properties.

### 1. Introduction and preliminaries

In start we give some notations and definitions necessary for the understanding of the paper as follows (see [4], [5] and [8]):  $I = [a, b] \subset \mathbb{R}$ ,  $J = [c, d] \subset \mathbb{R}$ ,  $I_j = [a_j, b_j] \subset \mathbb{R}$  for  $j \in \{1, \dots, n\}$ .

DEFINITION 1. The  $n$ th order divided difference of a function  $f : I \rightarrow \mathbb{R}$  at distinct points  $x_i, x_{i+1}, \dots, x_{i+n} \in I \subset \mathbb{R}$  for some  $i \in \mathbb{N}$  is defined recursively by:

$$[x_j; f] = f(x_j), \quad j \in \{i, \dots, i+n\}$$

$$[x_i, \dots, x_{i+n}; f] = \frac{[x_{i+1}, \dots, x_{i+n}; f] - [x_i, \dots, x_{i+n-1}; f]}{x_{i+n} - x_i}.$$

It may easily be verified that

$$[x_i, \dots, x_{i+n}; f] = \sum_{k=0}^n \frac{f(x_{i+k})}{\prod_{j=i, j \neq i+k}^{i+n} (x_{i+k} - x_j)}.$$

REMARK 1. Let us denote  $[x_i, \dots, x_{i+n}; f]$  by  $\Delta_{(n)} f(x_i)$ . The value  $[x_i, \dots, x_{i+n}; f]$  is independent of the order of points  $x_i, x_{i+1}, \dots, x_{i+n}$ . We can extend this definition by including the cases in which two or more points coincide by taking respective limits.

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DEFINITION 2. A function  $f : I \rightarrow \mathbb{R}$  is called *convex of order  $n$*  or  *$n$ -convex* if for all choices of  $(n + 1)$  distinct points  $x_i, \dots, x_{i+n}$  we have  $\Delta_{(n)}f(x_i) \geq 0$ . Further, we say that if  $n$ th order derivative  $f_{(n)}$  exists, then  $f$  is  *$n$ -convex* if and only if  $f_{(n)} \geq 0$ .

Now we extend the definition of divided difference up to order  $(n_1, \dots, n_k)$ .

DEFINITION 3. Let  $f : I_1 \times \dots \times I_k \rightarrow \mathbb{R}$  be a function. Then the *divided difference of order  $(n_1, \dots, n_k)$*  of a function  $f$  at distinct points  $x_{ji_j}, \dots, x_{j(i_j+n_j)} \in I_j$ , for  $j \in \{1, \dots, k\}$  is given as

$$\begin{aligned} \Delta_{(n_1, \dots, n_k)}f(x_{1i_1}, \dots, x_{ki_k}) &= \left[ \begin{array}{ccc} x_{1i_1}, \dots, x_{1(i_1+n_1)} \\ \vdots & \vdots & \vdots \\ x_{ki_k}, \dots, x_{k(i_k+n_k)} \end{array} \right] f \\ &= [x_{1i_1}, \dots, x_{1(i_1+n_1)}; [x_{2i_2}, \dots, x_{2(i_2+n_2)}; [\dots; [x_{ki_k}, \dots, x_{k(i_k+n_k)}; f]]]] \end{aligned}$$

DEFINITION 4. We say that a function  $f : I_1 \times \dots \times I_k \rightarrow \mathbb{R}$  is *convex of order  $(n_1, \dots, n_k)$*  or  *$(n_1, \dots, n_k)$ -convex* if  $\Delta_{(n_1, \dots, n_k)}f(x_{1i_1}, \dots, x_{ki_k}) \geq 0$ , where  $x_{ji_j}, \dots, x_{j(i_j+n_j)} \in I_j$ , for  $j \in \{1, \dots, k\}$  and  $\Delta_{(n_1, \dots, n_k)}$  represents divided difference of order  $(n_1, \dots, n_k)$ . Further, if all partial derivatives  $\frac{\partial^{n_1+\dots+n_k}f}{\partial x_1^{n_1} \dots \partial x_k^{n_k}}$  (denoted by  $f_{(n_1, \dots, n_k)}$ ) exist, then  $f$  is  *$(n_1, \dots, n_k)$ -convex* if and only if  $f_{(n_1, \dots, n_k)} \geq 0$ .

For other results about convex functions of higher order we refer to the book [8].

DEFINITION 5. The  *$(n, m)$  order finite difference* of a function  $f$  for  $x \in I$ ,  $y \in J$  and  $h, k \in \mathbb{R}$  is defined as

$$\begin{aligned} \Delta_{h,k}^{(n,m)}f(x,y) &= \Delta_h^{(n)}(\Delta_k^{(m)}f(x,y)) = \Delta_k^{(m)}(\Delta_h^{(n)}f(x,y)) \\ &= \sum_{i=0}^n \sum_{j=0}^m (-1)^{n+m-i-j} \binom{n}{i} \binom{m}{j} f(x+ih, y+jk) \end{aligned}$$

provided that  $x+ih \in I$  for  $i \in \{0, \dots, n\}$  and  $y+jk \in J$  for  $j \in \{0, \dots, m\}$ . Moreover, we say that a function  $f : I \times J \rightarrow \mathbb{R}$  is *convex of order  $(n, m)$*  or  *$(n, m)$ -convex* if  $\Delta_{h,k}^{(n,m)}f(x,y) \geq 0$  holds for each  $x \in I$ ,  $y \in J$  and  $h, k \in \mathbb{R}$ .

DEFINITION 6. *Divided and finite differences of order  $(n, m)$  of a sequence  $(a_{ij})$*   $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m\}$  are defined as  $\Delta_{(n,m)}a_{ij} = \Delta_{(n,m)}f(x_i, y_j)$  and  $\Delta_{1,1}^{(n,m)}a_{ij} = \Delta_{1,1}^{(n,m)}f(x_i, y_j)$  respectively, where  $x_i = i$ ,  $y_j = j$  and  $f : \{1, \dots, n\} \times \{1, \dots, m\} \rightarrow \mathbb{R}$  is the function  $f(i, j) = a_{ij}$ . Moreover, we say that a sequence  $(a_{ij})$  is *convex of order  $(n, m)$*  or  *$(n, m)$ -convex* if  $\Delta_{(n,m)}a_{ij} \geq 0$  holds for  $n, m \geq 0$  and  $i, j \in \mathbb{N}$ .

Let us introduce some further notations as follows. For some fixed integer  $a$  and  $m \in \mathbb{N}$ :  $a^{(m)} = a(a-1) \dots (a-m+1)$ ,  $a^{(0)} = 1$ .

For some real sequence  $(a_n)$ ,  $n \in \mathbb{N}$  and  $m \in \{2, 3, \dots\}$ :

$$\Delta^{(1)}a_n = \Delta a_n = a_{n+1} - a_n, \quad \Delta^{(m)}a_n = \Delta(\Delta^{(m-1)}a_n).$$

Also for  $n$  distinct real numbers  $x_i$ ,  $i \in \{1, \dots, n\}$  and  $m \geq 0$ :

$$(x_k - x_i)^{(m+1)} = (x_k - x_i)(x_k - x_{i+1}) \cdots (x_k - x_{i+m}), \quad (x_k - x_i)^{(0)} = 1.$$

This paper discusses the Popoviciu type characterization of positivity of sums and integrals for higher order convex functions of  $n$  variables. We divide this paper into five main parts. After ‘‘Introduction and Preliminaries’’ section, in the second and in the third sections respectively, we obtain one discrete identity for the sum  $\sum \cdots \sum P_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n})$  and one integral identity for  $\Lambda(f) = \int \cdots \int P(x_1, \dots, x_n) \times f(x_1, \dots, x_n) dx_1 \cdots dx_n$  of Popoviciu-type. These results are in fact generalizations of the results given in [5], [7], [10] and [11]. In both the sections, we also obtain necessary and sufficient conditions under which these sum and integral are nonnegative for higher order convex functions of  $n$  variables. The fourth section presents the mean value theorems, while in the last section we apply the functional  $\Lambda(f)$  on the family of certain exponentially convex functions and we discuss some of its major properties.

### 2. Discrete identity and inequality for functions of $n$ variables

Let us state a result from [6] as follows.

**PROPOSITION 1.** *Let  $p_k \in \mathbb{R}$  for  $k \in \{1, \dots, N\}$ . Then for any real sequence  $(a_k)$ ,  $k \in \{1, \dots, N\}$  the following identity holds*

$$\begin{aligned} \sum_{k=1}^N p_k a_k &= \sum_{i=0}^{m-1} \sum_{k=i+1}^N p_k (k-1)^{(i)} \frac{\Delta^{(i)} a_1}{i!} \\ &+ \sum_{i=m+1}^N \left( \sum_{k=i}^N p_k (k-i+m-1)^{(m-1)} \right) \frac{\Delta^{(m)} a_{i-m}}{(m-1)!}. \end{aligned} \tag{1}$$

A result analogous to (1) for real functions was proved by Popoviciu [11] which is stated as:

**PROPOSITION 2.** *Let  $p_k \in \mathbb{R}$  for  $k \in \{1, \dots, N\}$ . If  $f : I \rightarrow \mathbb{R}$  is a function and  $x_k$ ,  $k \in \{1, \dots, N\}$  be mutually distinct points from  $I$ , then the following identity holds*

$$\begin{aligned} \sum_{k=1}^N p_k f(x_k) &= \sum_{i=0}^{m-1} \left( \sum_{k=i+1}^N p_k (x_k - x_1)^{(i)} \right) \Delta_{(i)} f(x_1) \\ &+ \sum_{i=m+1}^N \left( \sum_{k=i}^N p_k (x_k - x_{i-m+1})^{(m-1)} \right) \Delta_{(m)} f(x_{i-m})(x_i - x_{i-m}). \end{aligned} \tag{2}$$

The next theorem is from [5].

PROPOSITION 3. *Let the assumptions of Proposition 2 be valid and let  $x_1 < x_2 < \dots < x_N$ . Then the inequality*

$$\sum_{k=1}^N p_k f(x_k) \geq 0$$

*holds for every convex function  $f$  of order  $r, r + 1, \dots, m$  for  $r \in \{0, \dots, m\}$  if and only if*

$$\sum_{k=i+1}^N p_k (x_k - x_1)^{(i)} = 0, \quad i \in \{0, \dots, r - 1\}, \tag{3}$$

$$\sum_{k=i+1}^N p_k (x_k - x_1)^{(i)} \geq 0, \quad i \in \{r, \dots, m - 1\}, \tag{4}$$

$$\sum_{k=i}^N p_k (x_k - x_{i-m+1})^{(m-1)} \geq 0, \quad i \in \{m + 1, \dots, N\}.$$

For  $r = 0$  (or  $r = m$ ), condition (3) (or (4)) can be omitted.

REMARK 2. The result for the special case  $f(x_k) = a_k$  can be found in [9], see also [8, p. 257].

For two variable case some identities and inequalities can be found in [5]. Now we further extend this Popoviciu type identities and inequalities for  $n$  variables.

For our main theorems of this section we define further notations as follows.

Let for  $r \in \{0, \dots, n\}$ ,  $j \in \{1, \dots, n\}$ ,  ${}^n C_r(i_j, m_j)$  be the set of all  $n$ -tuples in which on the  $k$ th place we put  $m_k$  or  $i_k$  and  $r$  places are filled with constants from the set  $\{m_1, \dots, m_n\}$  while on the other  $n - r$  places we put variables from the set  $\{i_1, \dots, i_n\}$ . For example:

$$\begin{aligned} {}^n C_1(i_j, m_j) &= \{(m_1, i_2, \dots, i_n), (i_1, m_2, \dots, i_n), \dots, (i_1, i_2, \dots, i_{n-1}, m_n)\}, \\ {}^n C_2(i_j, m_j) &= \{(m_1, m_2, i_3, \dots, i_n), (m_1, i_2, m_3, i_4, \dots, i_n), \dots, (m_1, i_2, \dots, i_{n-1}, m_n), \\ &\quad (i_1, m_2, m_3, i_4, \dots, i_n), \dots, (i_1, m_2, i_3, \dots, i_{n-1}, m_n), \dots, \\ &\quad (i_1, i_2, \dots, i_{n-2}, m_{n-1}, m_n)\}. \end{aligned}$$

Note that the number of elements of the class  ${}^n C_r(i_j, m_j)$  are equal to the binomial coefficient  $\binom{n}{r}$ . We introduce  $\bar{\Delta}$  involving variables  $i_1, \dots, i_n$  and constants  $m_1, \dots, m_n$  as follows. For  $(i_1, \dots, i_n) \in {}^n C_0(i_j, m_j)$ , we have

$$\begin{aligned} \bar{\Delta}(i_1, \dots, i_n) &= \sum_{i_n=0}^{m_n-1} \dots \sum_{i_1=0}^{m_1-1} \sum_{k_1=i_1+1}^{N_1} \dots \sum_{k_n=i_n+1}^{N_n} p_{k_1 \dots k_n} \prod_{j=1}^n (x_{j k_j} - x_{j 1})^{(i_j)} \\ &\quad \times \Delta_{(i_1, \dots, i_n)} f(x_{11}, \dots, x_{n1}), \end{aligned}$$

For  $(i_1, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n) \in {}^n C_1(i_j, m_j)$ , we have

$$\begin{aligned} &\bar{\Delta}(i_1, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n) \\ &= \sum_{i_n=0}^{m_n-1} \dots \sum_{i_{t+1}=0}^{m_{t+1}-1} \sum_{i_t=m_t+1}^{N_t} \sum_{i_{t-1}=0}^{m_{t-1}-1} \dots \sum_{i_1=0}^{m_1-1} \sum_{k_1=i_1+1}^{N_1} \dots \sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}} \sum_{k_t=i_t}^{N_t} \sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}} \dots \end{aligned}$$

$$\begin{aligned} &\times \sum_{k_n=i_n+1}^{N_n} p_{k_1 \dots k_n} \left( \prod_{j=1, j \neq t}^n (x_{jk_j} - x_{j1})^{(i_j)} \right) (x_{tk_t} - x_{t(i_t-m_t+1)})^{(m_t-1)} \\ &\times (x_{ti_t} - x_{t(i_t-m_t)}) \Delta_{(i_1, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n)} f(x_{11}, \dots, x_{(t-1)1}, x_{t(i_t-m_t)}, x_{(t+1)1}, \dots, x_{n1}). \end{aligned}$$

In general, for  $(i_1, \dots, i_{s-1}, m_s, i_{s+1}, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n) \in {}^n C_r(i_j, m_j)$ , we have

$$\begin{aligned} &\overline{\Delta}(i_1, \dots, i_{s-1}, m_s, i_{s+1}, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n) \\ &= \sum_{i_n=0}^{m_n-1} \dots \sum_{i_{t+1}=0}^{m_{t+1}-1} \sum_{i_t=m_t+1}^{N_t} \sum_{i_{t-1}=0}^{m_{t-1}-1} \dots \sum_{i_{s+1}=0}^{m_{s+1}-1} \sum_{i_s=m_s+1}^{N_s} \sum_{i_{s-1}=0}^{m_{s-1}-1} \dots \sum_{i_1=0}^{m_1-1} \\ &\times \sum_{k_1=i_1+1}^{N_1} \dots \sum_{k_{s-1}=i_{s-1}+1}^{N_{s-1}} \sum_{k_s=i_s}^{N_s} \sum_{k_{s+1}=i_{s+1}+1}^{N_{s+1}} \dots \sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}} \sum_{k_t=i_t}^{N_t} \sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}} \dots \sum_{k_n=i_n}^{N_n} \\ &\times p_{k_1 \dots k_n} \prod_{j=1, j \notin I_r}^n (x_{jk_j} - x_{j1})^{(i_j)} \prod_{j \in I_r} (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} (x_{ji_j} - x_{j(i_j-m_j)}) \\ &\times \Delta_{(i_1, \dots, i_{s-1}, m_s, i_{s+1}, \dots, i_{t-1}, m_t, i_{t+1}, \dots, i_n)} \\ &\times f(x_{11}, \dots, x_{(s-1)1}, x_{s(i_s-m_s)}, x_{(s+1)1}, \dots, x_{(t-1)1}, x_{t(i_t-m_t)}, x_{(t+1)1}, \dots, x_{n1}) \end{aligned}$$

where  $I_r$  is a set of all  $r$  indices  $s, \dots, t$  of used constants  $m_s, \dots, m_t$ .

Finally, for  $(m_1, \dots, m_n) \in {}^n C_n(i_j, m_j)$ , we have

$$\begin{aligned} &\overline{\Delta}(m_1, \dots, m_n) \\ &= \sum_{i_n=m_n+1}^{N_n} \dots \sum_{i_1=m_1+1}^{N_1} \sum_{k_1=i_1}^{N_1} \dots \sum_{k_n=i_n}^{N_n} p_{k_1 \dots k_n} \Delta_{(m_1, \dots, m_n)} f(x_{1(k_1-m_1)}, \dots, x_{n(k_n-m_n)}) \\ &\times \prod_{j=1}^n \left( (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} (x_{ji_j} - x_{j(i_j-m_j)}) \right). \end{aligned}$$

The following theorem gives an identity for sum  $\sum \dots \sum p_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n})$  involving  $n$  variables.

**THEOREM 1.** *Let  $f : I_1 \times \dots \times I_n \rightarrow \mathbb{R}$  be a function. Let  $p_{k_1 \dots k_n} \in \mathbb{R}$  and let  $x_{jk_j} \in I_j$  be distinct real numbers for  $k_j \in \{1, \dots, N_j\}$ ,  $j \in \{1, \dots, n\}$ , where  $I_j = [a_j, b_j] \subset \mathbb{R}$ . Then we have*

$$\sum_{k_1=1}^{N_1} \dots \sum_{k_n=1}^{N_n} p_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n}) = \sum_{r=0}^n \sum_{(p_1, \dots, p_n) \in {}^n C_r(i_j, m_j)} \overline{\Delta}(p_1, \dots, p_n). \tag{5}$$

*Proof.* We start with considering

$$\sum_{k_1=1}^{N_1} \dots \sum_{k_n=1}^{N_n} p_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n}) = \sum_{k_1=1}^{N_1} \dots \sum_{k_{n-1}=1}^{N_{n-1}} \left[ \sum_{k_n=1}^{N_n} \mathcal{Q}_{k_n}^{(1,1)} F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) \right]$$

where  $Q_{k_n}^{(1,1)} = p_{k_1 \dots k_n}$  and  $F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) = f(x_{1k_1}, \dots, x_{nk_n})$  where  $Q_{k_n}^{(1,1)}$  represents that this function only depends on  $k_n$  and independent of other  $n-1$  variables. Similarly  $F_{x_{nk_n}}^{(1,1)}$  represents that this is only a function of variable  $x_{nk_n}$  and independent of other  $n-1$  variables. So using Proposition 2 we get,

$$\begin{aligned} & \sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n}) \\ &= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-1}=1}^{N_{n-1}} \left[ \sum_{i_n=0}^{m_n-1} \left( \sum_{k_n=i_n+1}^{N_n} Q_{k_n}^{(1,1)}(x_{nk_n} - x_{n1})^{(i_n)} \Delta_{(i_n)} F_{x_{nk_n}}^{(1,1)}(x_{n1}) \right. \right. \\ & \quad \left. \left. + \sum_{i_n=m_n+1}^{N_n} \sum_{k_n=i_n}^{N_n} Q_{k_n}^{(1,1)}(x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \Delta_{(m_n)} F_{x_{nk_n}}^{(1,1)}(x_{n(i_n-m_n)})(x_{ni_n} - x_{n(i_n-m_n)}) \right) \right] \\ &= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \left[ \sum_{k_{n-1}=1}^{N_{n-1}} \left( \sum_{k_n=i_n+1}^{N_n} p_{k_1 \dots k_n}(x_{nk_n} - x_{n1})^{(i_n)} \Delta_{(i_n)} f(x_{1k_1}, \dots, x_{(n-1)k_{n-1}}, x_{n1}) \right) \right. \\ & \quad \left. + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=m_n+1}^{N_n} \left[ \sum_{k_{n-1}=1}^{N_{n-1}} \left( \sum_{k_n=i_n}^{N_n} p_{k_1 \dots k_n}(x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \right. \right. \right. \\ & \quad \left. \left. \times (x_{ni_n} - x_{n(i_n-m_n)}) \right) \Delta_{(m_n)} f(x_{1k_1}, \dots, x_{(n-1)k_{n-1}}, x_{n(i_n-m_n)}) \right] \right] \\ &= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \left[ \sum_{k_{n-1}=1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)} F_{x_{(n-1)k_{n-1}}}^{(2,1)} \right] + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=m_n+1}^{N_n} \left[ \sum_{k_{n-1}=1}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)} F_{x_{n-1}k_{n-1}}^{(2,2)} \right] \end{aligned}$$

where

$$\begin{aligned} Q_{k_{n-1}}^{(2,1)} &= \sum_{k_n=i_n+1}^{N_n} p_{k_1 \dots k_n}(x_{nk_n} - x_{n1})^{(i_n)}, \\ Q_{k_{n-1}}^{(2,2)} &= \sum_{k_n=i_n}^{N_n} p_{k_1 \dots k_n}(x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)}(x_{ni_n} - x_{n(i_n-m_n)}), \\ F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)k_{n-1}}) &= \Delta_{(i_n)} f(x_{1k_1}, \dots, x_{(n-1)k_{n-1}}, x_{n1}), \\ F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)k_{n-1}}) &= \Delta_{(m_n)} f(x_{1k_1}, \dots, x_{(n-1)k_{n-1}}, x_{n(i_n-m_n)}). \end{aligned}$$

Note that, this time we assume  $Q_{k_{n-1}}^{(2,1)}$  to be only dependent on  $k_{n-1}$ , whereas  $F_{x_{(n-1)k_{n-1}}}^{(2,1)}$  is considered to be a function of variable  $x_{(n-1)k_{n-1}}$  as far as  $Q_{k_{n-1}}^{(2,2)}$  is concerned, it only depends on  $k_{n-1}$  and  $F_{x_{(n-1)k_{n-1}}}^{(2,2)}$  is a function of variable  $x_{(n-1)k_{n-1}}$ . So, again applying Proposition 2, we have

$$\begin{aligned} & \sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1 \dots k_n} f(x_{1k_1}, \dots, x_{nk_n}) \\ &= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \left[ \sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \right] \end{aligned}$$

$$\begin{aligned}
 & \times \Delta_{(i_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)1}) \\
 & + \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} \\
 & \times \Delta_{(i_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}) \\
 & + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=m_{n+1}}^{N_n} \left[ \sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \right. \\
 & \times \Delta_{(i_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)1}) \\
 & + \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} \\
 & \left. \times \Delta_{(m_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}) \right] \\
 = & \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=0}^{m_{n-1}-1} \sum_{i_{n-1}=0}^{m_{n-1}-1} \left[ \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \right. \\
 & \times \Delta_{(i_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)1}) \left. + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=0}^{m_{n-1}-1} \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \right. \\
 & \times \left[ \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} \right. \\
 & \left. \times \Delta_{(m_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}) \right] \\
 & + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_{n+1}}^{N_n} \sum_{i_{n-1}=0}^{m_{n-1}-1} \left[ \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \right. \\
 & \times \Delta_{(i_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)1}) \left. + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_{n+1}}^{N_n} \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \right. \\
 & \times \left[ \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} \right. \\
 & \left. \times \Delta_{(m_{n-1})} F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}) \right] \\
 = & \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=0}^{m_{n-1}-1} \sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_{n+1}}^{N_n} p_{k_1 \cdots k_n}(x_{nk_n} - x_{n1})^{(i_n)} \\
 & \times (x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \Delta_{(i_{n-1}, i_n)} f(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)1}, x_{n1}) \\
 & + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=0}^{m_{n-1}-1} \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} \sum_{k_n=i_{n+1}}^{N_n} p_{k_1 \cdots k_n}(x_{nk_n} - x_{n1})^{(i_n)}
 \end{aligned}$$

$$\begin{aligned}
 & \times (x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} (x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}) \\
 & \times \Delta_{(m_{n-1}, i_n)} f(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)(i_{n-1}-m_{n-1})}, x_{n1}) \\
 & + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_n+1}^{N_n} \sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1 \dots k_n} \\
 & \times (x_n k_n - x_n(i_n - m_n + 1))^{(m_n - 1)} (x_n i_n - x_n(i_n - m_n)) (x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \\
 & \times \Delta_{(i_{n-1}, m_n)} f(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)1}, x_n(i_n - m_n)) \\
 & + \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_n+1}^{N_n} \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1 \dots k_n} \\
 & \times (x_n k_n - x_n(i_n - m_n + 1))^{(m_n - 1)} (x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)} \\
 & \times \Delta_{(m_{n-1}, m_n)} f(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)(i_{n-1}-m_{n-1})}, x_n(i_n - m_n)) \\
 & \times (x_n i_n - x_n(i_n - m_n)) (x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}).
 \end{aligned}$$

Continuing in the similar fashion we finally get identity (5).  $\square$

REMARK 3. If we set  $n = 2$  in previous theorem, then we get following corollary which can be found in [5].

COROLLARY 1. Let  $f : I \times J \rightarrow \mathbb{R}$  be a function and let  $p_{ij} \in \mathbb{R}$  for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ . Then the following identity holds

$$\begin{aligned}
 & \sum_{i=1}^N \sum_{j=1}^M p_{ij} f(x_i, y_j) \\
 = & \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k+1}^M p_{sr} (x_s - x_1)^{(t)} (y_r - y_1)^{(k)} \Delta_{(t,k)} f(x_t, y_1) \\
 & + \sum_{k=0}^{m-1} \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k+1}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} (y_r - y_1)^{(k)} \Delta_{(n,k)} f(x_{t-n}, y_1) (x_t - x_{t-n}) \\
 & + \sum_{k=m+1}^M \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k}^M p_{sr} (x_s - x_1)^{(t)} (y_r - y_{k-m+1})^{(m-1)} \Delta_{(t,m)} f(x_1, y_{k-m}) (y_k - y_{k-m}) \\
 & + \sum_{k=m+1}^M \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} (y_r - y_{k-m+1})^{(m-1)} \\
 & \times \Delta_{(n,m)} f(x_{t-n}, y_{k-m}) (x_t - x_{t-n}) (y_k - y_{k-m}) \tag{6}
 \end{aligned}$$

where  $(x_i, y_j) \in I \times J$  are distinct points for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ .

REMARK 4. If we put  $x_i = i$ ,  $y_j = j$  and  $f(x_i, y_j) = f(i, j) = a_{ij}$  in Corollary 1, then we get the following result.



COROLLARY 2. Let  $a_{ij}, p_{ij} \in \mathbb{R}$  for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ . Then the following identity holds

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^M p_{ij} a_{i,j} \\ &= \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k+1}^M p_{sr} \binom{s-1}{t} \binom{r-1}{k} \Delta^{(t,k)} a_{11} \\ &+ \sum_{k=0}^{m-1} \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k+1}^M p_{sr} \binom{s-t+n-1}{n-1} \binom{r-1}{k} \Delta^{(n,k)} a_{(t-n)(1)} \\ &+ \sum_{k=m+1}^M \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k}^M p_{sr} \binom{s-1}{t} \binom{r-k+m-1}{m-1} \Delta^{(t,m)} a_{1(k-m)} \\ &+ \sum_{k=m+1}^M \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k}^M p_{sr} \binom{s-t+n-1}{n-1} \binom{r-k+m-1}{m-1} \Delta^{(n,m)} a_{(t-n)(k-m)}. \end{aligned}$$

REMARK 5. If in Corollary 1 we simply put  $f(x_i, y_j) = f(x_i)g(y_j)$ , then we obtain the similar statement for two functions  $f$  and  $g$  as follows.

COROLLARY 3. Let  $f : I \rightarrow \mathbb{R}$  and  $g : J \rightarrow \mathbb{R}$  be two functions and let  $p_{ij} \in \mathbb{R}$  for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ . Then the following identity holds

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^M p_{ij} f(x_i)g(y_j) \\ &= \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k+1}^M p_{sr} (x_s - x_1)^{(t)} \Delta_{(t)} f(x_1) (y_r - y_1)^{(k)} \Delta_{(k)} g(y_1) \\ &+ \sum_{k=0}^{m-1} \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k+1}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} \Delta_{(n)} f(x_{t-n}) (x_t - x_{t-n}) (y_r - y_1)^{(k)} \Delta_{(k)} g(y_1) \\ &+ \sum_{k=m+1}^M \sum_{t=0}^{n-1} \sum_{s=t+1}^N \sum_{r=k}^M p_{sr} (x_s - x_1)^{(t)} \\ &\times \Delta_{(t)} f(x_1) (y_r - y_{k-m+1})^{(m-1)} \Delta_{(m)} g(y_{k-m}) (y_k - y_{k-m}) \\ &+ \sum_{k=m+1}^M \sum_{t=n+1}^N \sum_{s=t}^N \sum_{r=k}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} \Delta_{(n)} f(x_{t-n}) (x_t - x_{t-n}) \\ &\times (y_r - y_{k-m+1})^{(m-1)} \Delta_{(m)} g(y_{k-m}) (y_k - y_{k-m}) \end{aligned}$$

where  $(x_i, y_j) \in I \times J$  are distinct points for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ .

REMARK 6. If we put  $f(x_i) = a_i$  and  $g(y_j) = b_j$  in Corollary 3, then we retrieve an identity given in [10] for sequences  $(a_i)$  and  $(b_j)$ .

**THEOREM 2.** *Let the assumptions of Theorem 1 be valid. Then the inequality*

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1 \cdots k_n} f(x_{1k_1}, \dots, x_{nk_n}) \geq 0 \quad (7)$$

holds for every  $(m_1, \dots, m_n)$ -convex function on  $I_1 \times \cdots \times I_n$  if and only if

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1 \cdots k_n} \prod_{j=1}^n (x_{jk_j} - x_{j1})^{(i_j)} = 0, \quad (8)$$

$$\forall i_1 \in \{0, \dots, m_1 - 1\}, \dots, i_n \in \{0, \dots, m_n - 1\},$$

$$\sum_{k_1=i_1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1 \cdots k_n} (x_{1k_1} - x_{1(i_1-m_1+1)})^{(m_1-1)} \prod_{j=2}^n (x_{jk_j} - x_{j1})^{(i_j)} = 0, \quad (9)$$

$$\forall i_1 \in \{m_1 + 1, \dots, N_1\}, i_2 \in \{0, \dots, m_2 - 1\}, \dots, i_n \in \{0, \dots, m_n - 1\},$$

⋮

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1 \cdots k_n} \prod_{j=1}^{n-1} (x_{jk_j} - x_{j1})^{(i_j)} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} = 0, \quad (10)$$

$$\forall i_1 \in \{0, \dots, m_1 - 1\}, \dots, i_{n-1} \in \{0, \dots, m_{n-1} - 1\}, i_n \in \{m_n + 1, \dots, N_n\},$$

⋮

$$\sum_{k_1=i_1+1}^{N_1} \sum_{k_2=i_2}^{N_2} \cdots \sum_{k_n=i_n}^{N_n} p_{k_1 \cdots k_n} (x_{1k_1} - x_{11}) \prod_{j=2}^n (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} = 0, \quad (11)$$

$$\forall i_1 \in \{0, \dots, m_1 - 1\}, i_2 \in \{m_2 + 1, \dots, N_2\}, \dots, i_n \in \{m_n + 1, \dots, N_n\},$$

$$\sum_{k_1=i_1}^{N_1} \cdots \sum_{k_n=i_n}^{N_n} p_{k_1 \cdots k_n} \prod_{j=1}^n (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} \geq 0, \quad (12)$$

$$\forall i_1 \in \{m_1 + 1, \dots, N_1\}, \dots, i_n \in \{m_n + 1, \dots, N_n\}.$$

*Proof.* If (8), (9), ..., (10), ..., (11) hold, then all these sums are zero in (5) and the required inequality (7) holds by using (12).

Conversely, let (7) hold for every convex function  $f$  of order  $(m_1, \dots, m_n)$ . Let us consider the following functions

$$f^1(x_{1k_1}, \dots, x_{nk_n}) = \prod_{j=1}^n (x_{jk_j} - x_{j1})^{(i_j)} \quad \text{and} \quad f^2 = -f^1,$$

for  $i_1 \in \{0, \dots, m_1 - 1\}, \dots, i_n \in \{0, \dots, m_n - 1\}$ . Since these functions are convex of order  $(m_1, \dots, m_n)$ , so by (7) the inequalities

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1 \cdots k_n} f^k(x_{1k_1}, \dots, x_{nk_n}) \geq 0 \quad \text{for} \quad k \in \{1, 2\}$$

hold and we get required inequality (8). In the same way if we consider the following  $(m_1, \dots, m_n)$ -convex functions

$$f^3(x_{1k_1}, \dots, x_{nk_n}) = \begin{cases} (x_{1k_1} - x_{1(i_1-m_1+1)})^{(m_1-1)} \prod_{j=2}^n (x_{jk_j} - x_{j1})^{(i_j)}, & x_{1(i_1-1)} < x_{1k_1}, \\ 0, & x_{1(i_1-1)} \geq x_{1k_1}, \end{cases}$$

and  $f^4 = -f^3$ , where  $i_1 \in \{m_1 + 1, \dots, N_1\}$ ,  $i_2 \in \{0, \dots, m_2 - 1\}$ , ...,  $i_n \in \{0, \dots, m_n - 1\}$ , then we get the required equality (9).

Similarly, if we consider in (7) the following  $(m_1, \dots, m_n)$ -convex functions

$$f^5(x_{1k_1}, \dots, x_{nk_n}) = \begin{cases} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \prod_{j=1}^{n-1} (x_{jk_j} - x_{j1})^{(i_j)}, & x_{n(i_n-1)} < x_{nk_n}, \\ 0, & x_{n(i_n-1)} \geq x_{nk_n}, \end{cases}$$

and  $f^6 = -f^5$ , where  $i_1 \in \{0, \dots, m_1 - 1\}$ , ...,  $i_{n-1} \in \{0, \dots, m_{n-1} - 1\}$ ,  $i_n \in \{m_n + 1, \dots, N_n\}$ , then we get the required equality (10) and so on.

The last inequality (12) is followed by considering the following  $(m_1, \dots, m_n)$ -convex function in (7)

$$f^7(x_{1k_1}, \dots, x_{nk_n}) = \begin{cases} \prod_{j=1}^n (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)}, & x_{1(i_1-1)} < x_{1k_1}, \dots, x_{n(i_n-1)} < x_{nk_n}, \\ 0, & \text{otherwise} \end{cases}$$

where  $i_1 \in \{m_1 + 1, \dots, N_1\}$ , ...,  $i_n \in \{m_n + 1, \dots, N_n\}$ . □

**COROLLARY 4.** Let  $f : I \times J \rightarrow \mathbb{R}$  be a function and let  $p_{ij} \in \mathbb{R}$  for  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, M\}$ . For real numbers  $x_1 < x_2 < \dots < x_N$ ,  $x_i \in I$ ,  $y_1 < y_2 < \dots < y_M$ ,  $y_j \in J$ , the inequality

$$\sum_{i=1}^N \sum_{j=1}^M p_{ij} f(x_i, y_j) \geq 0$$

holds for every  $(n, m)$ -convex function on  $I \times J$  if and only if

$$\begin{aligned} \sum_{s=t+1}^N \sum_{r=k+1}^M p_{sr} (x_s - x_1)^{(t)} (y_r - y_1)^{(k)} &= 0, & k \in \{0, \dots, m-1\} \\ & & t \in \{0, \dots, n-1\} \\ \sum_{s=t}^N \sum_{r=k+1}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} (y_r - y_1)^{(k)} &= 0, & k \in \{0, \dots, m-1\} \\ & & t \in \{n+1, \dots, N\} \\ \sum_{s=t+1}^N \sum_{r=k}^M p_{sr} (x_s - x_1)^{(t)} (y_r - y_{k-m+1})^{(m-1)} &= 0, & k \in \{m+1, \dots, M\} \\ & & t \in \{0, \dots, n-1\} \\ \sum_{s=t}^N \sum_{r=k}^M p_{sr} (x_s - x_{t-n+1})^{(n-1)} (y_r - y_{k-m+1})^{(m-1)} &\geq 0, & k \in \{m+1, \dots, M\} \\ & & t \in \{n+1, \dots, N\}. \end{aligned}$$

**REMARK 7.** The case when  $f(x_i, y_j) = a_{ij}$  for  $i \in \{1, \dots, N\}$ ,  $j \in \{1, \dots, M\}$  and  $m = n = 1$  was considered in [7]. The case when  $f(x_i, y_j) = a_i b_j$ , where  $(a_i)$  for  $i \in \{1, \dots, N\}$  is an  $n$ -convex sequence and  $(b_j)$  for  $j \in \{1, \dots, M\}$  is an  $m$ -convex sequence was researched in [10]. Also the case  $f(x_i, y_j) = a_i b_j$  for monotonic  $n$ -tuples **a** and **b** was considered by Popoviciu in [11] (see also [5]).

### 3. Integral identity and inequality for higher order differentiable functions of $n$ variables

As we done in previous section, for the present section also we introduce some notations to simplify the statement of our main theorems as follows.

For variables  $i_1, \dots, i_n$  and constants  $m_1 + 1, \dots, m_n + 1$  we define  $\tilde{\Delta}$  in the following way:

$$\begin{aligned} &\tilde{\Delta}(i_1, \dots, i_n) \\ &= \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(i_1, \dots, i_n)}(a_1, \dots, a_n) \prod_{j=1}^n \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1, \\ &\tilde{\Delta}(i_1, \dots, i_{k-1}, m_k, i_{k+1}, \dots, i_n) \\ &= \sum_{i_1=0}^{m_1} \cdots \sum_{i_{k-1}=0}^{m_{k-1}} \sum_{i_{k+1}=0}^{m_{k+1}} \cdots \sum_{i_n=0}^{m_n} \int_{a_k}^{b_k} \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{x_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) \\ &\quad \times f_{(i_1, \dots, i_{k-1}, m_k+1, i_{k+1}, \dots, i_n)} \frac{(y_k - x_k)^{m_k}}{m_k!} \prod_{j=1, j \neq k}^n \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 dx_k. \end{aligned}$$

Similarly, we can define  $\tilde{\Delta}$  for any  $n$ -tuple from  ${}^n C_r(i_j, m_j)$  for some  $j \in \{1, \dots, n\}$  and finally we define

$$\begin{aligned} \tilde{\Delta}(m_1, \dots, m_n) &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) \\ &\quad \times f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n) \prod_{j=1}^n \frac{(y_j - x_j)^{m_j}}{m_j!} dy_n \cdots dy_1 dx_n \cdots dx_1. \end{aligned}$$

Now we are ready to state our main theorems of this section.

**THEOREM 3.** *Let  $p, f : I_1 \times \cdots \times I_n \rightarrow \mathbb{R}$  be integrable functions and let  $f \in C^{(m_1+1, \dots, m_n+1)}(I_1 \times \cdots \times I_n)$ . Then the following identity holds*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n \cdots dx_1 = \sum_{r=0}^n \sum_{(p_1, \dots, p_n) \in {}^n C_r(i_j, m_j+1)} \tilde{\Delta}(p_1, \dots, p_n). \tag{13}$$

*Proof.* We consider the Taylor expansion:

$$\begin{aligned} f(x_1, \dots, x_n) &= \sum_{i_n=0}^{m_n} f_{(0, \dots, 0, i_n)}(x_1, \dots, x_{n-1}, a_n) \frac{(x_n - a_n)^{i_n}}{i_n!} \\ &\quad + \int_{a_n}^{x_n} f_{(0, \dots, 0, m_n+1)}(x_1, \dots, x_{n-1}, y_n) \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n. \end{aligned}$$

Multiply the above formula with  $p(x_1, \dots, x_n)$  and integrate it over  $[a_n, b_n]$  by variable  $x_n$ . Then we have

$$\begin{aligned} & \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n \\ &= \sum_{i_n=0}^{m_n} f_{(0, \dots, 0, i_n)}(x_1, \dots, x_{n-1}, a_n) \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n \\ & \quad + \int_{a_n}^{b_n} \left( \int_{a_n}^{x_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_n+1)}(x_1, \dots, x_{n-1}, y_n) \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n \right) dx_n. \end{aligned} \tag{14}$$

Let us use the following Taylor expansions:

$$\begin{aligned} & f_{(0, \dots, 0, i_n)}(x_1, \dots, x_{n-1}, a_n) \\ &= \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0, \dots, 0, i_{n-1}, i_n)}(x_1, \dots, x_{n-2}, a_{n-1}, a_n) \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \\ & \quad + \int_{a_{n-1}}^{x_{n-1}} f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dy_{n-1}, \\ & f_{(0, \dots, 0, m_n+1)}(x_1, \dots, x_{n-1}, y_n) \\ &= \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \\ & \quad + \int_{a_{n-1}}^{x_{n-1}} f_{(0, \dots, 0, m_{n-1}+1, m_n+1)}(x_1, \dots, x_{n-2}, y_{n-1}, y_n) \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dy_{n-1}. \end{aligned}$$

Put these two formulae in (14) and integrate over  $[a_{n-1}, b_{n-1}]$  by variable  $x_{n-1}$ . Then we have

$$\begin{aligned} & \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n dx_{n-1} \\ &= \int_{a_{n-1}}^{b_{n-1}} \left[ \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0, \dots, 0, i_{n-1}, i_n)}(x_1, \dots, x_{n-2}, a_{n-1}, a_n) \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \right. \\ & \quad \times \left. \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n \right] dx_{n-1} \\ & \quad + \int_{a_{n-1}}^{b_{n-1}} \left[ \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{x_{n-1}} f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \right. \\ & \quad \times \left. \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dy_{n-1} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n \right] dx_{n-1} \end{aligned}$$

$$\begin{aligned}
& + \int_{a_{n-1}}^{b_{n-1}} \left[ \int_{a_n}^{b_n} \int_{a_n}^{x_n} p(x_1, \dots, x_n) \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0, \dots, 0, i_{n-1}, m_{n-1}+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \right. \\
& \times \left. \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n dx_n \right] dx_{n-1} \\
& + \int_{a_{n-1}}^{b_{n-1}} \left[ \int_{a_n}^{b_n} \int_{a_n}^{x_n} p(x_1, \dots, x_n) \int_{a_{n-1}}^{x_{n-1}} f_{(0, \dots, 0, m_{n-1}+1, m_{n-1}+1)}(x_1, \dots, x_{n-2}, y_{n-1}, y_n) \right. \\
& \times \left. \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_{n-1} dy_n dx_n \right] dx_{n-1}.
\end{aligned}$$

In the first summand we change the order of summation, use linearity of integral and get

$$\begin{aligned}
& \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, i_n)}(x_1, \dots, x_{n-2}, a_{n-1}, a_n) \\
& \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1}.
\end{aligned}$$

The second summand is rewritten as

$$\begin{aligned}
& \int_{a_{n-1}}^{b_{n-1}} \left[ \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{x_{n-1}} f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \right. \\
& \times \left. \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dy_{n-1} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n \right] dx_{n-1} \\
& = \int_{a_{n-1}}^{b_{n-1}} \left[ \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} \right. \\
& \times \left. f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dx_n dy_{n-1} \right] dx_{n-1} \\
& = \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{b_{n-1}} \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \\
& \times \frac{(x_n - a_n)^{i_n}}{i_n!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dx_n dy_{n-1} dx_{n-1} \\
& = \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{b_{n-1}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \\
& \times \frac{(x_n - a_n)^{i_n}}{i_n!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dx_n dx_{n-1} dy_{n-1}
\end{aligned}$$

where in the last equation we used the Fubini theorem for variables  $y_{n-1}$  and  $x_{n-1}$ . Let us point out that firstly, the variable  $x_{n-1}$  is changed from  $a_{n-1}$  to  $b_{n-1}$  while the variable  $y_{n-1}$  is changed from  $a_{n-1}$  to  $x_{n-1}$ . After changing the order of integration we have that variable  $y_{n-1}$  is changed from  $a_{n-1}$  to  $b_{n-1}$  while the variable  $x_{n-1}$  is changed from  $y_{n-1}$  to  $b_{n-1}$ .

Similarly, the third summand is rewritten as:

$$\begin{aligned}
 & \int_{a_{n-1}}^{b_{n-1}} \left[ \int_{a_n}^{b_n} \int_{a_n}^{x_n} p(x_1, \dots, x_n) \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \right. \\
 & \times \left. \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n dx_n \right] dx_{n-1} \\
 &= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_n}^{x_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \\
 & \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n dx_n dx_{n-1} \\
 &= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \\
 & \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dy_n dx_{n-1} \\
 &= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \\
 & \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n
 \end{aligned}$$

where we use the Fubini theorem twice, firstly for changing  $y_n$  and  $x_n$  and then for  $y_n$  and  $x_{n-1}$ .

The fourth summand is rewritten as

$$\begin{aligned}
 & \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_n}^{x_n} \int_{a_{n-1}}^{x_{n-1}} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, m_n+1)}(x_1, \dots, x_{n-2}, y_{n-1}, y_n) \\
 & \times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_{n-1} dy_n dx_n dx_{n-1} \\
 &= \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, m_n+1)}(x_1, \dots, x_{n-2}, y_{n-1}, y_n) \\
 & \times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n dy_{n-1},
 \end{aligned}$$

where we use the Fubini theorem several times. Firstly, we change  $y_n$  and  $x_n$ , then  $x_n$  and  $y_{n-1}$ , then  $y_{n-1}$  and  $y_n$ , then  $y_{n-1}$  and  $x_{n-1}$ , then  $y_n$  and  $x_{n-1}$ . Using all these results we get

$$\begin{aligned}
 & \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n dx_{n-1} \\
 &= \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, i_n)}(x_1, \dots, x_{n-2}, a_{n-1}, a_n)
 \end{aligned}$$

$$\begin{aligned}
& \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} \\
& + \sum_{i_n=0}^{m_n} \int_{a_{n-1}}^{b_{n-1}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-2}, y_{n-1}, a_n) \\
& \times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dy_{n-1} \\
& + \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, i_{n-1}, m_n+1)}(x_1, \dots, x_{n-2}, a_{n-1}, y_n) \\
& \times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n \\
& + \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) f_{(0, \dots, 0, m_{n-1}+1, m_n+1)}(x_1, \dots, x_{n-2}, y_{n-1}, y_n) \\
& \times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n dy_{n-1}.
\end{aligned}$$

Now, use the Taylor expansion again and integrate over  $[a_{n-2}, b_{n-2}]$  by variable  $x_{n-2}$ . If we proceed in the similar fashion as we done before, then we finally get:

$$\begin{aligned}
& \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n dx_{n-1} dx_{n-2} \\
= & \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \\
& \times f_{(0, \dots, 0, i_{n-2}, i_{n-1}, i_n)}(x_1, \dots, x_{n-3}, a_{n-2}, a_{n-1}, a_n) \\
& \times \frac{(x_{n-2} - a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} \\
& + \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \\
& \times f_{(0, \dots, 0, m_{n-2}+1, i_{n-1}+1, i_n)}(x_1, \dots, x_{n-3}, y_{n-2}, a_{n-1}, a_n) \\
& \times \frac{(x_{n-2} - y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} dy_{n-2} \\
& + \sum_{i_n=0}^{m_n} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \\
& \times f_{(0, \dots, 0, i_{n-2}, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-3}, a_{n-2}, y_{n-1}, a_n) \\
& \times \frac{(x_{n-2} - a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} dy_{n-1} \\
& + \sum_{i_n=0}^{m_n} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \dots, x_n) \\
& \times f_{(0, \dots, 0, m_{n-2}+1, m_{n-1}+1, i_n)}(x_1, \dots, x_{n-3}, y_{n-2}, y_{n-1}, a_n)
\end{aligned}$$



$$\begin{aligned}
 & \times \frac{(x_{n-2} - y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} dy_{n-1} dy_{n-2} \\
 & + \sum_{i_{n-1}=0}^{m_{n-1}} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_n}^{b_n} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) \\
 & \times f_{(0, \dots, 0, i_{n-2}, i_{n-1}, m_{n+1})}(x_1, \dots, x_{n-3}, a_{n-2}, a_{n-1}, y_n) \\
 & \times \frac{(x_{n-2} - a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dx_{n-2} dy_n \\
 & + \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{a_n}^{b_n} \int_{y_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) \\
 & \times f_{(0, \dots, 0, m_{n-2}+1, i_{n-1}, m_{n+1})}(x_1, \dots, x_{n-3}, y_{n-2}, a_{n-1}, y_n) \\
 & \times \frac{(x_{n-2} - y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-2} \\
 & + \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) \\
 & \times f_{(0, \dots, 0, i_{n-2}, m_{n-1}+1, m_{n+1})}(x_1, \dots, x_{n-3}, a_{n-2}, y_{n-1}, y_n) \\
 & \times \frac{(x_{n-2} - a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-1} \\
 & + \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) \\
 & \times f_{(0, \dots, 0, m_{n-2}+1, m_{n-1}+1, m_{n+1})}(x_1, \dots, x_{n-3}, y_{n-2}, y_{n-1}, y_n) \\
 & \times \frac{(x_{n-2} - y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-1} dy_{n-2}.
 \end{aligned}$$

Then we use the Taylor expansion again and we integrate the result over  $[a_{n-3}, b_{n-3}]$  by variable  $x_{n-3}$ . If we continue this process, we get required identity.  $\square$

REMARK 8. If we set  $n = 2$  in the previous theorem, we get the following corollary which can be found in [5].

COROLLARY 5. Let  $p, f : I \times J \rightarrow \mathbb{R}$  be integrable functions and let  $f \in C^{(n+1, m+1)}(I \times J)$ . Then the following identity holds

$$\begin{aligned}
 & \int_a^b \int_a^b P(x, y) f(x, y) dy dx \\
 & = \sum_{i=0}^n \sum_{j=0}^m \int_a^b \int_a^b P(s, t) f_{(i, j)}(a, a) \frac{(s-a)^i}{i!} \frac{(t-a)^j}{j!} dt ds \\
 & + \sum_{j=0}^m \int_a^b \int_x^b \int_a^b P(s, t) f_{(n+1, j)}(x, a) \frac{(s-x)^n}{n!} \frac{(t-a)^j}{j!} dt ds dx
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{i=0}^n \int_a^b \int_a^b \int_y^b P(s,t) f_{(i,m+1)}(a,y) \frac{(s-a)^i}{i!} \frac{(t-y)^m}{m!} dt ds dy \\
 &+ \int_a^b \int_a^b \int_x^b \int_y^b P(s,t) f_{(n+1,m+1)}(x,y) \frac{(s-x)^n}{n!} \frac{(t-y)^m}{m!} dt ds dy dx.
 \end{aligned}$$

REMARK 9. If in Corollary 5 we simply put  $n = m = 0$ , then we get the following corollary. In fact the following identity was considered by Pečarić in Theorem 10 of [7].

COROLLARY 6. Let  $P, f : I^2 \rightarrow \mathbb{R}$  be integrable functions and if  $f$  has continuous partial derivatives  $f_{(1,0)}$ ,  $f_{(0,1)}$  and  $f_{(1,1)}$  on  $I^2$ , then

$$\begin{aligned}
 \int_a^b \int_a^b P(x,y) f(x,y) dx dy &= f(a,a) P_1(a,a) + \int_a^b P_1(x,a) f_{(1,0)}(x,a) dx \\
 &+ \int_a^b P_1(a,y) f_{(0,1)}(a,y) dy + \int_a^b \int_a^b P_1(x,y) f_{(1,1)}(x,y) dx dy
 \end{aligned}$$

where

$$P_1(x,y) = \int_x^b \int_y^b P(s,t) dt ds,$$

$$f_{(1,0)} = \frac{\partial f}{\partial x}, \quad f_{(0,1)} = \frac{\partial f}{\partial y} \quad \text{and} \quad f_{(1,1)} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

REMARK 10. If in Corollary 5 we replace  $f(x,y)$  by  $f(x)g(y)$ , then we get the following result.

COROLLARY 7. Let  $f \in C^{(n+1)}(I)$  and  $g \in C^{(m+1)}(J)$  be two functions. Further let  $p : I \times J \rightarrow \mathbb{R}$  be an integrable function. Then the following identity holds

$$\begin{aligned}
 &\int_a^b \int_a^b P(x,y) f(x) g(y) dy dx \\
 &= \sum_{i=0}^n \sum_{j=0}^m \int_a^b \int_a^b P(s,t) f_{(i)}(a) g_{(j)}(a) \frac{(s-a)^i}{i!} \frac{(t-a)^j}{j!} dt ds \\
 &+ \sum_{j=0}^m \int_a^b \int_x^b \int_a^b P(s,t) f_{(n+1)}(x) g_{(j)}(a) \frac{(s-x)^n}{n!} \frac{(t-a)^j}{j!} dt ds dx \\
 &+ \sum_{i=0}^n \int_a^b \int_a^b \int_y^b P(s,t) f_{(i)}(a) g_{(m+1)}(y) \frac{(s-a)^i}{i!} \frac{(t-y)^m}{m!} dt ds dy \\
 &+ \int_a^b \int_a^b \int_x^b \int_y^b P(s,t) f_{(n+1)}(x) g_{(m+1)}(y) \frac{(s-x)^n}{n!} \frac{(t-y)^m}{m!} dt ds dy dx.
 \end{aligned}$$

COROLLARY 8. *Let the assumptions of Theorem 3 be valid and let  $p \equiv 1$ . Then the following identity holds*

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_n \cdots dx_1 \\ = & \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \prod_{j=1}^n \frac{(b_j - a_j)^{i_j+1}}{(i_j + 1)!} f_{(i_1, \dots, i_n)}(a_1, \dots, a_n) \\ & + \sum_{i_2=0}^{m_2} \cdots \sum_{i_n=0}^{m_n} \int_{a_1}^{b_1} \frac{(b_1 - y_1)^{m_1+1}}{(m_1 + 1)!} \prod_{j=2}^n \frac{(b_j - a_j)^{i_j+1}}{(i_j + 1)!} f_{(m_1+1, i_2, \dots, i_n)}(y_1, a_2, \dots, a_n) dy_1 \\ & + \cdots + \sum_{i_1=0}^{m_1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \frac{(b_n - y_n)^{m_n+1}}{(m_n + 1)!} \prod_{j=1}^{n-1} \frac{(b_j - a_j)^{i_j+1}}{(i_j + 1)!} \\ & \times f_{(i_1, \dots, i_{n-1}, m_n+1)}(a_1, \dots, a_{n-1}, y_n) dy_n + \cdots \\ & + \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{j=1}^n \frac{(b_j - y_j)^{m_j+1}}{(m_j + 1)!} f_{(m_1+1, \dots, m_n+1)}(y_1, \dots, y_n) dy_n \cdots dy_1. \end{aligned}$$

REMARK 11. For  $n = 2$  in the previous corollary we get Theorem 6.16 of the book [2] by simply putting  $x = a$  and  $y = c$  which is in fact an Ostrowski type result.

Now we state our next main theorem:

THEOREM 4. *Let the assumptions of Theorem 3 be valid. Then the inequality*

$$\Lambda(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_n \cdots dx_1 \geq 0 \tag{15}$$

holds for every  $(m_1 + 1, \dots, m_n + 1)$ -convex function  $f$  on  $I_1 \times \cdots \times I_n$  if and only if

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) \prod_{j=1}^n \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 = 0, \tag{16}$$

$$i_1 \in \{0, 1, \dots, m_1\}, \dots, i_n \in \{0, 1, \dots, m_n\},$$

$$\int_{x_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} p(x_1, \dots, x_n) \frac{(y_1 - x_1)^{m_1}}{m_1!} \prod_{j=2}^n \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 = 0, \tag{17}$$

$$i_2 \in \{0, 1, \dots, m_2\}, \dots, i_n \in \{0, 1, \dots, m_n\}, \forall x_1 \in [a_1, b_1],$$

⋮

$$\int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \dots, x_n) \prod_{j=1}^{n-1} \frac{(y_j - a_j)^{i_j}}{i_j!} \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 = 0, \tag{18}$$

$$i_1 \in \{0, 1, \dots, m_1\}, \dots, i_{n-1} \in \{0, 1, \dots, m_{n-1}\}, \forall x_n \in [a_n, b_n],$$

$$\vdots$$

$$\int_{a_1}^{b_1} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) \frac{(y_i - a_1)^{i_1}}{i_1!} \prod_{j=2}^n \frac{(y_j - x_j)^{m_j}}{m_j!} dy_n \cdots dy_1 = 0, \quad (19)$$

$$i_1 \in \{0, 1, \dots, m\}, \forall x_2 \in [a_1, b_1], \dots, \forall x_n \in [a_n, b_n],$$

$$\int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) \prod_{j=1}^n \frac{(y_j - x_j)^{m_j}}{m_j!} dy_n \cdots dy_1 \geq 0, \quad (20)$$

$$\forall x_1 \in [a_1, b_1], \dots, \forall x_n \in [a_n, b_n].$$

*Proof.* If (16), (17), ..., (18), ..., (19) hold, then all these sums are zero in (13) and the required inequality (15) holds by using (20).

Conversely, if we consider in (15) the following  $(m_1 + 1, \dots, m_n + 1)$ -convex functions

$$g^1(y_1, \dots, y_n) = \prod_{j=1}^n \frac{(y_j - a_j)^{i_j}}{i_j!} \quad \text{and} \quad g^2 = -g^1$$

for  $i_1 \in \{0, 1, \dots, m_1\}, \dots, i_n \in \{0, 1, \dots, m_n\}$ , then we get the required equality (16). In the same way, if we consider in (15) the following  $(m_1 + 1, \dots, m_n + 1)$ -convex functions for  $i_2 \in \{0, 1, \dots, m_2\}, \dots, i_n \in \{0, 1, \dots, m_n\}, \forall x_1 \in [a_1, b_1]$

$$g^3(y_1, \dots, y_n) = \begin{cases} \frac{(y_1 - x_1)^{m_1}}{m_1!} \prod_{j=2}^n \frac{(y_j - a_j)^{i_j}}{i_j!}, & x_1 < y_1, \\ 0, & x_1 \geq y_n, \end{cases} \quad \text{and} \quad g^4 = -g^3,$$

then we get the required equality (17). Similarly, if we consider in (15) the following  $(m_1 + 1, \dots, m_n + 1)$ -convex functions for  $i_1 \in \{0, 1, \dots, m_1\}, \dots, i_{n-1} \in \{0, 1, \dots, m_{n-1}\}, \forall x_n \in [a_n, b_n]$

$$g^5(y_1, \dots, y_n) = \begin{cases} \prod_{j=1}^{n-1} \frac{(y_j - a_j)^{i_j}}{i_j!} \frac{(y_n - x_n)^{m_n}}{m_n!}, & x_n < y_n, \\ 0, & x_n \geq y_n, \end{cases} \quad \text{and} \quad g^6 = -g^5,$$

then we get the required equality (18) and so on. The last inequality (20) is followed by considering the following  $(m_1 + 1, \dots, m_n + 1)$ -convex function in (15)

$$g^7(y_1, \dots, y_n) = \begin{cases} \prod_{j=1}^n \frac{(y_j - x_j)^{m_j}}{m_j!}, & x_1 < y_1, \dots, x_n < y_n, \\ 0, & \text{otherwise,} \end{cases}$$

where  $x_1 \in [a_1, b_1], \dots, x_n \in [a_n, b_n]$ .  $\square$

REMARK 12. If we set  $n = 2$  in previous theorem, then we get result given in [5].

### 4. Mean value theorems

It is a well known fact that many results of classical real analysis are consequences of the mean value theorem. Lagrange’s and Cauchy’s mean value theorems are among the most important theorems of differential calculus. For detailed discussion on the topic we refer to [12]. Here we state some generalized mean value theorems of Lagrange and of Cauchy type.

**THEOREM 5.** *Let  $\Lambda : C^{(m_1+1, \dots, m_n+1)}(I_1 \times \dots \times I_n) \rightarrow \mathbb{R}$  be the linear functional defined in (15) and let  $p : I_1 \times \dots \times I_n \rightarrow \mathbb{R}$  be an integrable function such that the conditions (16), (17), ..., (18), ..., (19), (20) of Theorem 4 be satisfied. Then there exists  $(\xi_1, \dots, \xi_n) \in I_1 \times \dots \times I_n$  such that*

$$\Lambda(f) = f_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n)\Lambda(f_0) \tag{21}$$

where  $f_0(x_1, \dots, x_n) = \prod_{j=1}^n \frac{x_j^{m_j+1}}{(m_j+1)!}$ .

*Proof.* Since  $f_{(m_1+1, \dots, m_n+1)}$  is continuous on  $(I_1 \times \dots \times I_n)$ , so it attains its maximum and minimum values on  $(I_1 \times \dots \times I_n)$ . Let

$$L = \min_{(x_1, \dots, x_n) \in I_1 \times \dots \times I_n} f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n)$$

and

$$U = \max_{(x_1, \dots, x_n) \in I_1 \times \dots \times I_n} f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n).$$

Then the function

$$G(x_1, \dots, x_n) = Uf_0(x_1, \dots, x_n) - f(x_1, \dots, x_n)$$

gives us

$$G_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n) = U - f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n) \geq 0,$$

i.e.,  $G$  is an  $(m_1 + 1, \dots, m_n + 1)$ -convex function. Hence  $\Lambda(G) \geq 0$  by Theorem 4 and we conclude that

$$\Lambda(f) \leq U\Lambda(f_0).$$

Similarly, we have

$$L\Lambda(f_0) \leq \Lambda(f).$$

Combining the two inequalities we get

$$L\Lambda(f_0) \leq \Lambda(f) \leq U\Lambda(f_0)$$

which gives us (21).  $\square$

**THEOREM 6.** *Let all the assumptions of Theorem 5 be valid. Then there exists  $(\xi_1, \dots, \xi_n) \in I_1 \times \dots \times I_n$  such that*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n)}{g_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n)}$$

*provided that the denominator of the left-hand side is nonzero.*

*Proof.* Let  $h \in C^{(m_1+1, \dots, m_n+1)}(I_1 \times \dots \times I_n)$  be defined as

$$h = \Lambda(g)f - \Lambda(f)g.$$

Using Theorem 5 there exists  $(\xi_1, \dots, \xi_n)$  such that

$$0 = \Lambda(h) = h_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n)\Lambda(f_0)$$

or

$$\left[ \Lambda(g)f_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n) - \Lambda(f)g_{(m_1+1, \dots, m_n+1)}(\xi_1, \dots, \xi_n) \right] \Lambda(f_0) = 0$$

which gives us required result.  $\square$

**COROLLARY 9.** *Let all the assumptions of Theorem 6 be satisfied with  $m = m_1 = m_2 = \dots = m_n$ . Then there exists  $(\xi_1, \dots, \xi_n) \in I_1 \times \dots \times I_n$  such that*

$$(\xi_1 \cdots \xi_n)^{q-q'} = \frac{[(q'+1)q' \cdots (q'-m+1)]^n \Lambda((x_1 \cdots x_n)^{q+1})}{[(q+1)q \cdots (q-m+1)]^n \Lambda((x_1 \cdots x_n)^{q'+1})}$$

*for  $-\infty < q \neq q' < +\infty$  and  $q, q' \notin \{-1, 0, 1, \dots, m-1\}$ .*

*Proof.* If we put  $f(x_1, \dots, x_n) = (x_1 \cdots x_n)^{q+1}$  and  $g(x_1, \dots, x_n) = (x_1 \cdots x_n)^{q'+1}$  in Theorem 6, then we get the required result.  $\square$

**REMARK 13.** Special cases of Theorems 5, 6 and Corollary 9 for  $n = 2$  can be found in [5].

For our next theorem we recall the Hölder inequality for functional

$$A(F) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_n \cdots dx_1$$

as follows:

$$A(FG) \leq A(F^q)^{1/q} A(G^{q'})^{1/q'}$$

where  $1/q + 1/q' = 1$ ,  $q, q' > 1$ .

Let us introduce some notations for simplifications of statements as follows:

$$\begin{aligned} \bar{\Lambda}(f) &= \Lambda(f) - \sum_{r=0}^{n-1} \sum_{(p_1, \dots, p_n) \in {}^n C_r(i_j, m_j+1)} \tilde{\Delta}(p_1, \dots, p_n) \\ &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n) \\ &\quad \times \frac{(y_1 - x_1)^{m_1}}{m_1!} \cdots \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 dx_n \cdots dx_1. \end{aligned}$$

**THEOREM 7.** *Let  $p : I_1 \times \cdots \times I_n \rightarrow \mathbb{R}$  be an integrable function and let  $f \in \mathcal{C}^{(m_1+1, \dots, m_n+1)}(I_1 \times \cdots \times I_n)$ . If  $|f_{(m_1+1, \dots, m_n+1)}|^q$  is an integrable function such that*

$$\|f_{(m_1+1, \dots, m_n+1)}\|_q = \left( \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n)|^q dx_n \cdots dx_1 \right)^{1/q} < \infty,$$

then the following inequality holds

$$\begin{aligned} |\bar{\Lambda}(f)| &\leq \|f_{(m_1+1, \dots, m_n+1)}\|_q \left( \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) \right. \right. \\ &\quad \left. \left. \times \frac{(y_1 - x_1)^{m_1}}{m_1!} \cdots \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 \right|^q dx_n \cdots dx_1 \right)^{1/q'} \end{aligned}$$

where  $1/q + 1/q' = 1$ ,  $q, q' > 1$ .

**REMARK 14.** The proof of the theorem is easily followed by applying the Hölder inequality. Moreover, when we consider the case  $q \rightarrow 1$ , then  $r \rightarrow \infty$ , we get the following corollary.

**COROLLARY 10.** *Let all the assumptions of the Theorem 7 be valid. Then the inequality*

$$|\bar{\Lambda}(f)| \leq M \prod_{i=1}^n (b_i - a_i) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f_{(m_1+1, \dots, m_n+1)}(x_1, \dots, x_n) dx_n \cdots dx_1$$

holds, where

$$M = \text{ess sup} \left( \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \dots, x_n) \frac{(y_1 - x_1)^{m_1}}{m_1!} \cdots \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 \right).$$

**REMARK 15.** For the case  $p \equiv 1$ , we get the following corollary.

**COROLLARY 11.** *Let all the assumptions of the Theorem 7 be valid and if  $p \equiv 1$ . Then we have*

$$|\bar{\Lambda}(f)| \leq \prod_{i=1}^n \frac{(b_i - a_i)^{m_i+2}}{(m_i + 2)!} \|f_{(m_1+1, \dots, m_n+1)}\|_q.$$

### 5. Exponential convexity

Let  $J \subset \mathbb{R}$  be an open interval. We give some definitions in start of this section:

DEFINITION 7. [1] A function  $\psi : J \rightarrow \mathbb{R}$  is exponentially convex on  $J$  if it is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j \psi(x_i + x_j) \geq 0$$

$\forall n \in \mathbb{N}$  and all choices  $\xi_i, \xi_j \in \mathbb{R}; i, j = 1, \dots, n$  such that  $x_i + x_j \in J; i, j \in \{1, \dots, n\}$ .

EXAMPLE 1. [3] For constant  $c \geq 0$  and  $k \in \mathbb{R}$ ,  $x \mapsto ce^{kx}$  is an example of exponentially convex function.

The following proposition and two corollaries are given in [3].

PROPOSITION 4. Let  $\psi : J \rightarrow \mathbb{R}$ , the following propositions are equivalent:

- (i)  $\psi$  is exponentially convex on  $J$ .
- (ii)  $\psi$  is continuous and  $\sum_{i,j=1}^n \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0$ , for all  $\xi_i, \xi_j \in \mathbb{R}$  and every  $x_i, x_j \in J; i, j \in \{1, \dots, n\}$ .

COROLLARY 12. If  $\psi$  is an exponentially convex function on  $J$ , then the matrix

$$\left[ \psi\left(\frac{x_i + x_j}{2}\right) \right]_{i,j=1}^n$$

is a positive semi-definite matrix. Particularly

$$\det \left[ \psi\left(\frac{x_i + x_j}{2}\right) \right]_{i,j=1}^n \geq 0,$$

$\forall n \in \mathbb{N}, x_i, x_j \in J; i, j \in \{1, \dots, n\}$ .

COROLLARY 13. If  $\psi : J \rightarrow (0, \infty)$  is an exponentially convex function, then  $\psi$  is a log-convex function i.e. for every  $x, y \in J$  and every  $\lambda \in [0, 1]$ , we have

$$\psi(\lambda x + (1 - \lambda)y) \leq \psi^\lambda(x) \psi^{1-\lambda}(y).$$

Let  $I = [a, b]; a, b$  be positive real numbers and let  $D = \{\varphi^{(p)} : I^n \rightarrow \mathbb{R} : p \in \mathbb{R}\}$  be a family of functions defined as:

$$\varphi^{(p)}(x_1, \dots, x_n) = \begin{cases} \frac{(x_1 \dots x_n)^p}{(p(p-1) \dots (p-m))^n}, & p \notin \{0, 1, 2, \dots, m\}; \\ \frac{(x_1 \dots x_n)^p (\log(x_1 \dots x_n))^n}{(-1)^{m-p} n! (p!(m-p))!}, & p \in \{0, 1, 2, \dots, m\}. \end{cases}$$



Clearly  $\varphi_{(m+1, \dots, m+1)}^{(p)}(x_1, \dots, x_n) = (x_1 \dots x_n)^{p-m-1} = e^{(p-m-1)\ln(x_1 \dots x_n)}$  for  $(x_1, \dots, x_n) \in I^n$  so  $\varphi^{(p)}$  is an  $(m+1, \dots, m+1)$ -convex function and  $p \mapsto \varphi_{(m+1, \dots, m+1)}^{(p)}(x_1, \dots, x_n)$  is exponentially convex. From Corollary 13 we know that every exponentially convex function is log-convex. So, now we are in position to state our next theorem.

**THEOREM 8.** *Let  $\Lambda : C^{(m+1, \dots, m+1)}(I^n) \rightarrow \mathbb{R}$  be a linear functional as defined in (15) and let the conditions (16), (17), ..., (18), ..., (19), (20) of Theorem 4 for function  $P$  are satisfied and  $\varphi^{(p)}$  be a function defined above. Then the following statements hold:*

- (i)  $p \mapsto \Lambda(\varphi^{(p)})$  is continuous on  $\mathbb{R}$ .
- (ii)  $p \mapsto \Lambda(\varphi^{(p)})$  is an exponentially convex function on  $\mathbb{R}$ .
- (iii) If  $p \mapsto \Lambda(\varphi^{(p)})$  is a positive function on  $\mathbb{R}$ , then  $p \mapsto \Lambda(\varphi^{(p)})$  is log-convex function on  $\mathbb{R}$ .
- (iv) For every  $k \in \mathbb{N}$  and  $p_1, \dots, p_k \in \mathbb{R}$ , the matrix  $\left[ \Lambda(\varphi^{(\frac{p_i+p_j}{2})}) \right]_{i,j=1}^k$  is a positive semi-definite. Particularly

$$\det \left[ \Lambda(\varphi^{(\frac{p_i+p_j}{2})}) \right]_{i,j=1}^k \geq 0.$$

- (v) If  $p \mapsto \Lambda(\varphi^{(p)})$  is differentiable on  $\mathbb{R}$ . Then for every  $s, t, u, v \in \mathbb{R}$ , such that  $s \leq u$  and  $t \leq v$ , we have

$$\mathfrak{M}_{s,t}(x_1, \dots, x_n) \leq \mathfrak{M}_{u,v}(x_1, \dots, x_n) \tag{22}$$

where

$$\mathfrak{M}_{s,t}(x_1, \dots, x_n) = \begin{cases} \left( \frac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp \left( \frac{\frac{d}{ds} \Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})} \right), & s = t \end{cases}$$

for  $\varphi^{(s)}, \varphi^{(t)} \in D$ .

*Proof.* (i) For fixed  $n \in \mathbb{N} \cup \{0\}$ , using the L'Hôpital rule  $n$ -times and applying limit, we get

$$\begin{aligned} \lim_{p \rightarrow 0} \Lambda(\varphi^{(p)}) &= \lim_{p \rightarrow 0} \frac{\int_a^b \dots \int_a^b P(x_1, \dots, x_n) (x_1 \dots x_n)^p dx_n \dots dx_1}{(p(p-1) \dots (p-m))^n} \\ &= \frac{\int_a^b \dots \int_a^b P(x_1, \dots, x_n) (\log(x_1 \dots x_n))^n dx_n \dots dx_1}{(-1)^{m_n} (m!)^n} = \Lambda(\varphi^{(0)}). \end{aligned}$$

In the same way we can get  $\lim_{p \rightarrow k} \Lambda(\varphi^{(p)}) = \Lambda(\varphi^{(k)})$   $k \in \{1, \dots, m\}$

(ii) For  $p_i \in \mathbb{R}$ ,  $\alpha_i \in \mathbb{R}$ ,  $i \in \{1, \dots, k\}$ , let us define the function

$$\omega(x_1, \dots, x_n) = \sum_{i,j=1}^k \alpha_i \alpha_j \varphi^{\left(\frac{p_i+p_j}{2}\right)}(x_1, \dots, x_n).$$

Since the function  $p \mapsto \varphi_{(m+1, \dots, m+1)}^{(p)}$  is exponentially convex, we have

$$\omega_{(m+1, \dots, m+1)} = \sum_{i,j=1}^k \alpha_i \alpha_j \varphi_{(m+1, \dots, m+1)}^{\left(\frac{p_i+p_j}{2}\right)} \geq 0,$$

which implies that  $\omega$  is  $(m+1, \dots, m+1)$ -convex function on  $I^n$  and therefore we have  $\Lambda(\omega) \geq 0$ . Hence  $\sum_{i,j=1}^k \alpha_i \alpha_j \Lambda(\varphi^{\left(\frac{p_i+p_j}{2}\right)}) \geq 0$ . We conclude that the function  $p \rightarrow$

$\Lambda(\varphi^{(p)})$  is an exponentially convex function on  $\mathbb{R}$ .

(iii) It is direct consequence of (ii).

(iv) This is consequence of Corollary 12.

(v) From the definition of convex function  $\phi$ , we have the following inequality [8, p. 2]

$$\frac{\phi(s) - \phi(t)}{s - t} \leq \frac{\phi(u) - \phi(v)}{u - v}, \quad (23)$$

$\forall s, t, u, v \in J \subset \mathbb{R}$  such that  $s \leq u$ ,  $t \leq v$ ,  $s \neq t$ ,  $u \neq v$ .

Since by (iii),  $\Lambda(\varphi^{(p)})$  is log-convex, so set  $\phi(x) = \log \Lambda(\varphi^{(x)})$  in (23) we have

$$\frac{\log \Lambda(\varphi^{(s)}) - \log \Lambda(\varphi^{(t)})}{s - t} \leq \frac{\log \Lambda(\varphi^{(u)}) - \log \Lambda(\varphi^{(v)})}{u - v} \quad (24)$$

for  $s \leq u$ ,  $t \leq v$ ,  $s \neq t$ ,  $u \neq v$ , which is equivalent to (22). The cases for  $s = t$ , and / or  $u = v$  are easily followed from (24) by taking respective limits.  $\square$

REMARK 16. Here we notice that Theorem 8 generalizes Theorem 5.6 of [5].

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Asif R. Khan  
Department of Mathematical Sciences  
University of Karachi  
Karachi, Pakistan  
e-mail: asifrk@uok.edu.pk

Josip Pečarić  
Faculty of Textile Technology  
University of Zagreb  
Zagreb, Croatia  
e-mail: pecaric@element.hr

Sanja Varošanec  
Department of Mathematics  
University of Zagreb  
Zagreb, Croatia  
e-mail: varosans@math.hr