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A CLASS OF CONTINUED FRACTION INEQUALITIES

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Abstract. Given a finite sequence of positive real numbers, we construct terminating continued fractions whose partial denominators are formed by the arrangement of the numbers according to simple rules. This does not impose any restriction on the generality of our results and all simple continued fractions can be recast and formed according to these rules. After showing that the resulting finite continued fractions are multivariate convex or concave functions of the given sequence, we derive a class of inequalities using results from the theory of majorization. The main result of this paper is expressed in the form of inequalities connecting certain types of finite continued fractions and Fibonacci numbers.

1. Introduction

The literature on continued fractions is ripe with inequalities relating their convergents to their numerical value or partial denominators [1, 2, 3, 4]. The inequalities presented in this article, however, are of a different nature. We are not concerned with the accuracy of the convergents of a particular Diophantine approximation or the deviation from the limit as is the case in the analytic studies of continued fractions [5, 6]. Our results are in the form of inequalities for the maximum and minimum values attained by continued fractions when the partial denominators are selected from the permutations of the elements of a finite sequence.

Our treatment of the subject starts with establishing a lemma on the convexity or concavity of finite continued fractions when treated as multivariate functions. Of interest here are finite continued fractions symbolically expressed as

$$C_n = [b_1; b_2, \dots, b_n] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}, \tag{1}$$

where the partial denominators b_i are all positive and real numbers. Associated with the C_n is the vector $(p_1, p_2, \dots, p_n)$, from which the values of the b_i are determined according to simple rules where either p_i or its reciprocal p_i^{-1} is assigned to the b_i.

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We generate two types of finite continued fractions using two simple rules distinguished by the notations $C_n^{\dagger}$ and $C_n^{\ddagger}$. Specifically, we write

$$C_n^{\dagger}|_{(p_1,\ldots,p_n)} = [p_1^{-1}; p_2, p_3^{-1}, \ldots, p_n^{1-2(n \bmod 2)}], \tag{2}$$

whereas, for $C_n^{\ddagger}$ we define

$$C_n^{\ddagger}|_{(p_1,\ldots,p_n)} = [p_1; p_2^{-1}, p_3, \ldots, p_n^{-1+2(n \bmod 2)}]. \tag{3}$$

In the following Section, techniques from the theory of convex functions are used to show that $C_n^{\dagger}$ is a multivariate convex function and $C_n^{\ddagger}$ is a multivariate concave function with respect to the p_i when these symbols are treated as variables. The theory of majorization is then employed to derive bounds on the extremal values of $C_n^{\dagger}$ and $C_n^{\ddagger}$ over all permutations of the components of the vector $(p_1, \ldots, p_n)$. This leads to a variety of inequalities governing finite continued fractions and, eventually, yields the main result in connection with the Fibonacci numbers. Henceforth, we use the terms vector and sequence interchangeably.

2. Convexity of continued fractions

In the following, we state and prove a useful lemma concerning the convexity of finite continued fractions (1) when constructed according to our rules.

LEMMA 1. *Given the sequence $(p_1, \ldots, p_n)$ of positive real variables, the terminating continued fraction $C_n^{\dagger}$ is a convex function, while $C_n^{\ddagger}$ is a concave function in the variables p_i, $i = 1, 2, \ldots, n$, where $C_n^{\dagger}$ and $C_n^{\ddagger}$ are formed according to (2) and (3) respectively.*

Proof. The proof is by induction on n. The cases where $n = 1$ or $n = 2$ are trivial. To gain a deeper insight, we start with the case where $n = 3$ and consider

$$C_3^{\dagger}|_{(p_a,p_b,p_c)} = [p_a^{-1}; p_b, p_c^{-1}]. \tag{4}$$

It is to be shown that $C_3^{\dagger}$ is convex with respect to the three positive variables. This can be established by forming the expression

$$f(\alpha) = \left[\frac{1}{\alpha p_a' + \beta p_a''}, \alpha p_b' + \beta p_b'', \frac{1}{\alpha p_c' + \beta p_c''}\right], \tag{5}$$

where

$$0 \leqslant \alpha \leqslant 1 \quad \text{and} \quad \alpha + \beta = 1.$$

A useful result from [7, p. 190] then ascertains that in order to show the validity of the lemma we only need to establish that $f(\alpha)$ is convex with respect to $\alpha \in [0, 1]$. Since

$f(\alpha)$ is a smooth function in α, we can verify the convexity of f by checking the sign of its second partial derivative with respect to α, i.e.,

$$\frac{\partial^2 f}{\partial \alpha^2} = \frac{2(p'_a - p''_a)^2}{(\alpha p'_a + \beta p''_a)^2} + \frac{2(p'_b + p'_c - p''_b - p''_c)^2}{(\alpha(p'_b + p'_c) + \beta(p''_b + p''_c))^2} \geqslant 0. \tag{6}$$

It follows that $f(\alpha)$ is a convex function in α and consequently $C_3^\dagger$ a convex function in p_a, p_b and p_c as well. In a similar manner we can show that

$$g(\alpha) = \left[\alpha p'_a + \beta p''_a, \frac{1}{\alpha p'_b + \beta p''_b}, \alpha p'_c + \beta p''_c\right] \tag{7}$$

is concave in p_a, p_b and p_c, by verifying that

$$\frac{\partial^2 g}{\partial \alpha^2} \leqslant 0. \tag{8}$$

This establishes that $C_3^\ddagger$ is a concave function in the three variables p_a, p_b and p_c. Next set $n \geqslant 4$ and assume that $C_{n-1}^\ddagger$ together with all shorter continued fractions constructed according to the same rule are concave in the p_i on $(0,\infty)$. Also suppose that $C_{n-1}^\dagger$ and all shorter continued fractions of the same type are convex in the p_i on $(0,\infty)$. Note that the reciprocal of a positive and concave function is convex with respect to the same variables. Thus $\frac{1}{C_{n-1}^\ddagger}$ is convex in $n-1$ variables.

We now show the validity of the inductive step by first proving the convexity of $C_n^\dagger$ in the p_i, $i = 1, \ldots, n$. The recursive expression for $C_n^\dagger$ in terms of the p_i can be written as

$$C_n^\dagger|_{(p_1,\ldots,p_n)} = \frac{1}{p_1} + \frac{1}{C_{n-1}^\ddagger|_{(p_2,\ldots,p_n)}}. \tag{9}$$

Note that $\frac{1}{p_1}$ is convex with respect to p_1. Also, from the foregoing comments, the reciprocal of the shorter continued fraction $C_{n-1}^\ddagger$ is convex in $p_2, \ldots, p_n$. Consequently, $C_n^\dagger$, being the sum of two convex functions, is convex with respect to $p_1, \ldots, p_n$ as required.

It remains to show that $C_n^\ddagger$ is a concave function in the p_i. Write

$$C_n^\ddagger|_{(p_1,\ldots,p_n)} = p_1 + \frac{1}{\frac{1}{p_2} + \frac{1}{C_{n-2}^\ddagger|_{(p_3,\ldots,p_n)}}}. \tag{10}$$

Consider the expression

$$\alpha p'_1 + \beta p''_1 + \frac{1}{\frac{1}{\alpha p'_2 + \beta p''_2} + \frac{1}{C_{n-2}^\ddagger\big|_{p_i = \alpha p'_i + \beta p''_i}}}. \tag{11}$$

By assumption, $C_{n-2}^\ddagger$ is concave in the $n-2$ variable $p_3, \ldots, p_n$. Hence $\frac{\partial^2 C_{n-2}^\ddagger}{\partial \alpha^2} \leqslant 0$. The second partial derivative of the expression in (11) with respect to α can be expressed symbolically and evaluated to be nonpositive, i.e.,

$$\frac{(\alpha p_2' + \beta p_2'')^2 \left(C_{n-2}^{\ddagger}+\alpha p_2'+\beta p_2''\right) \frac{\partial^2 C_{n-2}^{\ddagger}}{\partial \alpha^2} - 2\left((p_2'-p_2'')C_{n-2}^{\ddagger}-(\alpha p_2'+\beta p_2'')\frac{\partial C_{n-2}^{\ddagger}}{\partial \alpha}\right)^2}{\left(C_{n-2}^{\ddagger}+\alpha p_2'+\beta p_2''\right)^3} \leqslant 0. \tag{12}$$

This completes the proof. □

3. Majorization inequalities

The reader is referred to [8] for a comprehensive account of the theory of majorization . If the sequence $(r_1,\ldots,r_n)$ is majorized by the sequence $(p_1,\ldots,p_n)$, we write

$$(r_1,\ldots,r_n) \prec (p_1,\ldots,p_n), \tag{13}$$

and this implies that

$$\sum_{i=1}^{k-1} r_{(i)} \geqslant \sum_{i=1}^{k-1} p_{(i)}, \quad k=1,\ldots,n-1, \quad \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} p_i \tag{14}$$

where the components of the vectors $(r_1,\ldots,r_n)$ and $(p_1,\ldots,p_n)$, when sorted in increasing order, are denoted in the forms

$$r_{(1)} \leqslant r_{(2)} \leqslant \cdots \leqslant r_{(n)} \quad \text{and} \quad p_{(1)} \leqslant p_{(2)} \leqslant \cdots \leqslant p_{(n)} \tag{15}$$

after re-indexing. It is a well-known result that such vectors, ordered according to the partial ordering of majorization, are related by a doubly stochastic matrix $\mathbf{A}$, namely,

$$(r_1,\ldots,r_n) = \mathbf{A}\cdot(p_1,\ldots,p_n).$$

Matrix $\mathbf{A}$ can be expressed as a convex combination of $n\times n$ permutation matrices by writing [7]

$$\mathbf{A} = \lambda_1\mathbf{Q}_1 + \ldots + \lambda_m\mathbf{Q}_m, \tag{16}$$

where $\lambda_1+\ldots+\lambda_m = 1$ and $\lambda_i \geqslant 0$, $i=1,\ldots,m$. With the r_i expressed in terms of p_i, the convex function $C_n^{\dagger}$ satisfies the inequality

$$C_n^{\dagger}|_{(r_1,\ldots,r_n)} \leqslant \sum_{i=1}^{m} \lambda_i C_n^{\dagger}|_{\mathbf{Q}_i\cdot(p_1,\ldots,p_n)}. \tag{17}$$

It is then an immediate result that

$$C_n^{\dagger}|_{(r_1,\ldots,r_n)} \leqslant \max_{\sigma} C_n^{\dagger}|_{(p_{\sigma(1)},\ldots,p_{\sigma(n)})}, \tag{18}$$

where σ belongs to the symmetric group of permutations over n elements.

Likewise, we can show that

$$C_n^{\ddagger}|_{(r_1,\ldots,r_n)} \geqslant \min_{\sigma} C_n^{\ddagger}|_{(p_{\sigma(1)},\ldots,p_{\sigma(n)})}. \tag{19}$$

The right sides of (18) and (19) are the maxima and minima of continued fractions $C_n^{\dagger}$ and $C_n^{\ddagger}$, respectively, attained over all possible permutations of the p_i. By using an inductive argument, it can be proved that

$$\max_{\sigma} C_n^{\dagger}|_{(p_{\sigma(1)},p_{\sigma(2)},\ldots,p_{\sigma(n)})} = C_n^{\dagger}|_{(p_{(1)},p_{(2)},\ldots,p_{(n)})}, \tag{20}$$

and similarly,

$$\min_{\sigma} C_n^{\ddagger}|_{(p_{\sigma(1)},p_{\sigma(2)},\ldots,p_{\sigma(n)})} = C_n^{\ddagger}|_{(p_{(1)},p_{(2)},\ldots,p_{(n)})}. \tag{21}$$

Interestingly, these extremal values are both obtained by using the same ordered forms of the vector under consideration.

4. Connection to Fibonacci numbers

Assume that the p_i satisfy the constraint

$$p_1 + \cdots + p_n = n. \tag{22}$$

It is well-known that for all such sequences we always have

$$(1,1,\ldots,1) \prec (p_1,p_2,\ldots,p_n). \tag{23}$$

For the vector $(1,1,\ldots,1)$, we have

$$C_n^{\dagger}|_{(1,1,\ldots,1)} = C_n^{\ddagger}|_{(1,1,\ldots,1)} = [1;1,\ldots,1] \tag{24}$$

and hence it is possible to combine (18) and (19) and write

$$C_n^{\ddagger}|_{(p_{(1)},p_{(2)},\ldots,p_{(n)})} \leqslant [1;1,\ldots,1] \leqslant C_n^{\dagger}|_{(p_{(1)},p_{(2)},\ldots,p_{(n)})}. \tag{25}$$

This result can be stated in the form of the following theorem.

THEOREM 1. *For all positive real sequences p_i, $i=1,\ldots,n$, where $\sum_i p_i = n$, the inequality*

$$p_{(1)} + \cfrac{1}{p_{(2)}^{-1} + \cfrac{1}{p_{(3)} + \cfrac{1}{\ddots + \cfrac{1}{p_{(n)}^{-1+2(n \bmod 2)}}}}} \leqslant \frac{F_{n+1}}{F_n} \leqslant p_{(1)}^{-1} + \cfrac{1}{p_{(2)} + \cfrac{1}{p_{(3)}^{-1} + \cfrac{1}{\ddots + \cfrac{1}{p_{(n)}^{1-2(n \bmod 2)}}}}} \tag{26}$$

is always valid.

Generalization of (26) to the cases where $\sum_i p_i$ is an arbitrary positive number is not hard and yields the following theorem.

THEOREM 2. *For all positive real sequences p_i, $i = 1,\ldots,n$, the inequalities*

$$\frac{nF_{n+1}}{F_n \sum_i p_i} \leqslant p_{(1)}^{-1} + \cfrac{1}{p_{(2)} + \cfrac{1}{p_{(3)}^{-1} + \cfrac{1}{\ddots + \cfrac{1}{p_{(n)}^{1-2(n \bmod 2)}}}}} \tag{27}$$

and

$$p_{(1)} + \cfrac{1}{p_{(2)}^{-1} + \cfrac{1}{p_{(3)} + \cfrac{1}{\ddots + \cfrac{1}{p_{(n)}^{-1+2(n \bmod 2)}}}}} \leqslant \frac{F_{n+1} \sum_i p_i}{nF_n} \tag{28}$$

are always valid.

5. Concluding remarks

Even though continued fractions are not symmetric functions of their partial denominators, and thus are not Schur-convex or Schur-concave, we utilized majorization theory to derive our inequalities by taking an approach based on the maximum and minimum values attained by the function. Another point to consider is that we can further employ other known results from majorization theory to obtain many other relevant inequalities. As a final remark, we re-express (25) in view of the facts that the majorization relation (23) is generally valid for all sequences summing to n, and the vector $(1,1,\ldots,1)$ is the minimal element in the partial ordering of majorization for such vectors. Thus we can write

$$\min_{\substack{p_1,\ldots,p_n>0 \\ p_1+\cdots+p_n=n}} \max_{\sigma} p_{\sigma(1)}^{-1} + \cfrac{1}{p_{\sigma(2)} + \cfrac{1}{p_{\sigma(3)}^{-1} + \cfrac{1}{\ddots + \cfrac{1}{p_{\sigma(n)}^{1-2(n \bmod 2)}}}}} = \frac{F_{n+1}}{F_n}. \tag{29}$$

The max-min form of the left side of (25) then becomes

$$\max_{\substack{p_1,\dots,p_n>0\\ p_1+\dots+p_n=n}} \min_{\sigma} p_{\sigma(1)} + \cfrac{1}{p_{\sigma(2)}^{-1} + \cfrac{1}{p_{\sigma(3)} + \cfrac{1}{\ddots + \cfrac{1}{p_{\sigma(n)}^{-1+2(n \bmod 2)}}}}} = \frac{F_{n+1}}{F_n}. \qquad (30)$$

These relations shed new light on continued fractions in connection with min-max or max-min problems.

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