DIAGONAL RECURRENCE RELATIONS, INEQUALITIES, AND MONOTONICITY RELATED TO THE STIRLING NUMBERS OF THE SECOND KIND

FENG QI

(Communicated by A. Laforgia)

Abstract. In the paper, the author derives several "diagonal" recurrence relations, constructs some inequalities, finds monotonicity, and poses a conjecture related to the Stirling numbers of the second kind.

1. Introduction

In mathematics, the Stirling numbers arise in a variety of combinatorics problems. They are introduced in the eighteen century by James Stirling. There are two kinds of the Stirling numbers: the Stirling numbers of the first and second kinds. Some properties and recurrence relations of the Stirling numbers of these two kinds are collected in, for example, [1, Chapter V].

The Stirling number of the second kind $S(n,k)$ is the number of ways of partitioning a set of n elements into k nonempty subsets. It may be computed by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

and may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \tag{1.1}$$

The Stirling numbers of the second kind $S(n,k)$ satisfy the following "triangular", "vertical", and "horizontal" recurrence relations:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n,k \geqslant 1;$$

$$S(n,k) = \sum_{k-1 \leqslant \ell \leqslant n-1} \binom{n-1}{\ell} S(\ell, k-1),$$

$$S(n,k) = \sum_{k \leqslant \ell \leqslant n} S(\ell-1, k-1) k^{n-\ell};$$

$$S(n,k) = \sum_{0 \leqslant \ell \leqslant n-k} (-1)^{\ell} \langle k+1 \rangle_{\ell} S(n+1, k+\ell+1),$$

$$k! S(n,k) = k^n - \sum_{\ell=1}^{k-1} (k)_{\ell} S(n,\ell),$$

where

$$\langle x \rangle_{\ell} = x(x+1)\cdots(x+\ell-1), \quad \langle x \rangle_0 = 1$$

and

$$(x)_{\ell} = x(x-1)\cdots(x-\ell+1), \quad (x)_0 = 1.$$

See [1, pp. 208–209, Theorems A, B, and C].

In this paper, we will derive several "diagonal" recurrence relations, construct some inequalities, and find monotonicity related to $S(n,k)$. By the way, we will also pose a conjecture on monotonicity and logarithmic concavity of sequences related to the Stirling numbers of the second kind $S(n,k)$.

2. Several "diagonal" recurrence relations of $S(n,k)$

In [1, p. 209], two "vertical" and two "horizontal" recurrence relations for $S(n,k)$ were listed. Relative to the words "triangular", "vertical", and "horizontal", we may call the following formulas (2.1) and (2.2) the "diagonal" recurrence relations for the Stirling numbers of the second kind $S(n,k)$.

THEOREM 2.1. *For $n > k \geqslant 0$, we have*

$$S(n,k) = \binom{n}{k} \sum_{\ell=1}^{n-k} (-1)^{\ell} \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \sum_{i=0}^{\ell} (-1)^i \binom{n-k+\ell}{\ell-i} S(n-k+i, i) \qquad (2.1)$$

$$= (-1)^n \sum_{i=2k-n}^{k-1} (-1)^i \binom{n}{i} \binom{i-1}{2k-n-1} S(n-i, k-i), \qquad (2.2)$$

where the conventions that

$$\binom{0}{0} = 1, \quad \binom{-1}{-1} = 1, \quad \text{and} \quad \binom{p}{q} = 0 \qquad (2.3)$$

for $p \geqslant 0 > q$ are adopted.

Proof. The equation (1.1) may be rearranged as

$$\left(\frac{e^x - 1}{x}\right)^k = \sum_{n=0}^{\infty} \frac{S(n+k, k)}{\binom{n+k}{k}} \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}.$$

Consequently, as coefficients of the power series expansion of the function $\left(\frac{e^x-1}{x}\right)^k$,

$$\frac{S(n+k,k)}{\binom{n+k}{k}} = \lim_{x\to 0} \frac{d^n}{dx^n}\left[\left(\frac{e^x-1}{x}\right)^k\right],$$

that is, for $n \geqslant k \geqslant 0$,

$$S(n,k) = \binom{n}{k}\lim_{x\to 0}\frac{d^{n-k}}{dx^{n-k}}\left[\left(\int_1^e u^{x-1}\,du\right)^k\right]. \tag{2.4}$$

In combinatorics, the Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$, may be defined by

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum_{\substack{1\leqslant i\leqslant n \\ \ell_i\in\{0\}\cup\mathbb{N} \\ \sum_{i=1}^n i\ell_i=n \\ \sum_{i=1}^n \ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \geqslant k \geqslant 0$, and the well-known Faà di Bruno formula may be described in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ by

$$\frac{d^n}{dx^n}f\circ g(x) = \sum_{k=0}^n f^{(k)}(g(x))B_{n,k}\big(g'(x),g''(x),\ldots,g^{(n-k+1)}(x)\big).$$

See [1, p. 134, Theorem A] and [1, p. 139, Theorem C]. In [2, Theorem 1] and [10, Example 4.2], it was derived that

$$B_{n,k}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!}\sum_{i=0}^k(-1)^{k-i}\binom{n+k}{k-i}S(n+i,i).$$

Consequently, we may obtain the following conclusions:

1. When $1 \leqslant m \leqslant k$,

$$\frac{d^m}{dx^m}\left[\left(\int_1^e u^{x-1}\,du\right)^k\right]$$

$$= \sum_{\ell=1}^m \frac{k!}{(k-\ell)!}\left(\int_1^e u^{x-1}\,du\right)^{k-\ell}$$

$$\times B_{m,\ell}\left(\int_1^e u^{x-1}\ln u\,du, \int_1^e u^{x-1}(\ln u)^2\,du,\ldots,\int_1^e u^{x-1}(\ln u)^{m-\ell+1}\,du\right)$$

$$\to \sum_{\ell=1}^m \frac{k!}{(k-\ell)!}B_{m,\ell}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{m-\ell+2}\right), \quad x\to 0 \tag{2.5}$$

$$= \sum_{\ell=1}^m \frac{\binom{k}{\ell}}{\binom{m+\ell}{m}}\sum_{i=0}^\ell(-1)^i\binom{m+\ell}{i}S(m+\ell-i,\ell-i)$$

$$= \sum_{\ell=1}^m(-1)^\ell\frac{\binom{k}{\ell}}{\binom{m+\ell}{m}}\sum_{i=0}^\ell(-1)^i\binom{m+\ell}{m+i}S(m+i,i);$$

2. Similarly, when $m > k$, we have

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\left(\int_1^e u^{x-1}\,\mathrm{d}u\right)^k\right] \to \sum_{\ell=1}^k (-1)^\ell \frac{\binom{k}{\ell}}{\binom{m+\ell}{m}} \sum_{i=0}^\ell (-1)^i \binom{m+\ell}{m+i} S(m+i,i) \quad (2.6)$$

as $x \to 0$.

Since the convention that $\binom{k}{m} = 0$ for $m > k$, the equation (2.5) holds for all $m \geqslant 1$ and includes (2.6). Substituting (2.5) into (2.4) produces

$$S(n,k) = \binom{n}{k} \sum_{\ell=1}^{n-k} (-1)^\ell \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \sum_{i=0}^\ell (-1)^i \binom{n-k+\ell}{n-k+i} S(n-k+i,i)$$

for $n - k \geqslant 1$. The formula (2.1) follows.

Interchanging two sums in (2.1) and then computing the inner sum result in

$$S(n,k) = \binom{n}{k} \sum_{i=1}^{n-k} (-1)^i \left[\sum_{\ell=i}^{n-k} (-1)^\ell \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \binom{n-k+\ell}{\ell-i}\right] S(n-k+i,i)$$

$$= \sum_{i=1}^{n-k} (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^{n-k} (-1)^\ell \binom{k-i}{k-\ell}\right] S(n-k+i,i)$$

$$= \begin{cases} \displaystyle\sum_{i=1}^{n-k} (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^{n-k} (-1)^\ell \binom{k-i}{k-\ell}\right] S(n-k+i,i), & k < n \leqslant 2k \\ \displaystyle\sum_{i=1}^{k} (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^{k} (-1)^\ell \binom{k-i}{k-\ell}\right] S(n-k+i,i), & n > 2k \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{i=2k-n}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^{i-(2k-n)} (-1)^\ell \binom{i}{\ell}\right] S(n-i,k-i), & k < n \leqslant 2k \\ \displaystyle\sum_{i=0}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^{i} (-1)^\ell \binom{i}{\ell}\right] S(n-i,k-i), & n > 2k \end{cases}$$

$$= \sum_{i=2k-n}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^{i-(2k-n)} (-1)^\ell \binom{i}{\ell}\right] S(n-i,k-i)$$

$$= (-1)^{n-2k} \sum_{i=2k-n}^{k-1} (-1)^i \binom{n}{i} \binom{i-1}{i-(2k-n)} S(n-i,k-i).$$

The formula (2.2) follows. The proof of Theorem 2.1 is complete. $\square$

REMARK 2.1. In [7, Theorem 1], a "diagonal" recurrence relation

$$s(n,k) = \sum_{m=1}^n \sum_{\ell=k-m}^{k-1} (-1)^{k+m-\ell} \binom{n}{\ell} \binom{\ell}{k-m} s(n-\ell,k-\ell) \quad (2.7)$$

for $n \geqslant k \geqslant 1$ was discovered for the Stirling numbers of the first kind $s(n,k)$ which may be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1.$$

As did in the derivation of (2.2), we may also interchange two sums in (2.1) and compute the inner sum as follows:

$$s(n,k) = (-1)^k \sum_{\ell=k-n}^{k-1} (-1)^\ell \binom{n}{\ell} \left[\sum_{m=k-\ell}^{n} (-1)^m \binom{\ell}{k-m} \right] s(n-\ell,k-\ell)$$

$$= (-1)^{n-k} \sum_{\ell=k-n}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell,k-\ell), \quad n \geqslant k \geqslant 1,$$

that is,

$$s(n,k) = (-1)^{n-k} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell,k-\ell), \quad n \geqslant k \geqslant 1, \qquad (2.8)$$

where the conventions listed in (2.3) are also adopted. The recurrence relation (2.8) may also be called as a "diagonal" recurrence relation for the Stirling numbers of the first kind $s(n,k)$.

The relations (2.7) and (2.8) are different from two "vertical" and two "horizontal" recurrence relations collected in [1, p. 215].

3. Inequalities and monotonicity related to $S(n,k)$

After establishing and discussing "diagonal" recurrence relations for the Stirling numbers of the fist and second kinds $S(n,k)$ and $s(n,k)$, we now construct and deduce, with the help of the formula (2.4) and in light of properties of absolutely monotonic functions, some inequalities and monotonicity related to the Stirling numbers of the second kind $S(n,k)$.

THEOREM 3.1. *Let $m \geqslant 1$ be a positive integer and let $|a_{ij}|_m$ denote a determinant of order m with elements a_{ij}.*

1. If a_i for $1 \leqslant i \leqslant m$ are non-negative integers, then

$$\left| \frac{S(a_i+a_j+k,k)}{\binom{a_i+a_j+k}{k}} \right|_m \geqslant 0 \quad and \quad \left| (-1)^{a_i+a_j} \frac{S(a_i+a_j+k,k)}{\binom{a_i+a_j+k}{k}} \right|_m \geqslant 0$$

hold true for all given $k \in \mathbb{N}$.

2. *Let $q = (q_1, q_2, \ldots, q_n)$ be a real n-tuple of non-negative integers and let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be non-increasing n-tuples of non-negative integers such that $a \succeq_q b$, that is,*

$$\sum_{i=1}^{k} q_i a_i \geqslant \sum_{i=1}^{k} q_i b_i \quad and \quad \sum_{i=1}^{n} q_i a_i = \sum_{i=1}^{n} q_i b_i.$$

for $1 \leqslant k \leqslant n - 1$. Then the inequality

$$\prod_{i=1}^{n} \left[\frac{S(a_i + k, k)}{\binom{a_i + k}{k}} \right]^{q_i} \geqslant \prod_{i=1}^{n} \left[\frac{S(b_i + k, k)}{\binom{b_i + k}{k}} \right]^{q_i} \tag{3.1}$$

hods true for all given $k \in \mathbb{N}$.

Proof. A function f is said to be absolutely monotonic on an interval I if f has derivatives of all orders on I and $0 \leqslant f^{(n)}(x) < \infty$ for $x \in I$ and $n \geqslant 0$. See [4, Chapter XIII]. In [3] and [4, p. 367], it was recited that if f is an absolutely monotonic function on $[0, \infty)$, then

$$\left| f^{(a_i + a_j)}(x) \right|_m \geqslant 0 \tag{3.2}$$

and

$$\left| (-1)^{a_i + a_j} f^{(a_i + a_j)}(x) \right|_m \geqslant 0. \tag{3.3}$$

In [4, p. 368] and [5, p. 429], it was stated that if f is an absolutely monotonic function on $[0, \infty)$ and $a \succeq_q b$, then

$$\prod_{i=1}^{n} \left[f^{(a_i)}(x) \right]^{q_i} \geqslant \prod_{i=1}^{n} \left[f^{(b_i)}(x) \right]^{q_i}. \tag{3.4}$$

It is easy to see that

$$\left(\frac{e^x - 1}{x} \right)^{(m)} = \int_1^e u^{x-1} (\ln u)^m \, du, \quad m \geqslant 0.$$

See the paper [8] and plenty of closely-related references cited therein. This means that $\frac{e^x - 1}{x} = \int_1^e u^{x-1} \, du$ is absolutely monotonic on $(-\infty, \infty)$. As a result, the function

$$H_k(x) = \left(\frac{e^x - 1}{x} \right)^k = \left(\int_1^e u^{x-1} \, du \right)^k, \quad k \in \mathbb{N}$$

is also absolutely monotonic on $(-\infty, \infty)$ and, by the formula 2.4,

$$\lim_{x \to 0} H_k^{(\ell)}(x) = \frac{S(\ell + k, k)}{\binom{\ell + k}{k}}, \quad \ell \in \{0\} \cup \mathbb{N}.$$

Making use of inequalities (3.2), (3.3), and (3.4) and taking the limit $x \to 0$ find that

$$0 \leqslant \left| H_k^{(a_i + a_j)}(x) \right|_m \to \left| \frac{S(a_i + a_j + k, k)}{\binom{a_i + a_j + k}{k}} \right|_m,$$

$$0 \leqslant \left| (-1)^{a_i+a_j} H_k^{(a_i+a_j)}(x) \right|_m \rightarrow \left| (-1)^{a_i+a_j} \frac{S(a_i+a_j+k,k)}{\binom{a_i+a_j+k}{k}} \right|_m,$$

and

$$\prod_{i=1}^{n} \left[\frac{S(a_i+k,k)}{\binom{a_i+k}{k}} \right]^{q_i} \leftarrow \prod_{i=1}^{n} \left[H_k^{(a_i)}(x) \right]^{q_i} \geqslant \prod_{i=1}^{n} \left[H_k^{(b_i)}(x) \right]^{q_i} \rightarrow \prod_{i=1}^{n} \left[\frac{S(b_i+k,k)}{\binom{b_i+k}{k}} \right]^{q_i}.$$

The proof of Theorem 3.1 is complete. $\square$

COROLLARY 3.1. *For any given $k \in \mathbb{N}$, the infinite sequence*

$$\left\{ \frac{S(n+k,k)}{\binom{n+k}{k}} \right\}_{n \geqslant 0} \tag{3.5}$$

is logarithmically convex with respect to n.

Proof. Letting

$$n = 2, \quad q_1 = q_2 = 1, \quad a_1 = \ell+2, \quad a_2 = \ell, \quad \text{and} \quad b_1 = b_2 = \ell+1$$

in the inequality (3.1) leads to

$$\frac{S(\ell+k+2,k)}{\binom{\ell+k+2}{k}} \frac{S(\ell+k,k)}{\binom{\ell+k}{k}} \geqslant \left[\frac{S(\ell+k+1,k)}{\binom{\ell+k+1}{k}} \right]^2, \quad \ell \geqslant 0. \tag{3.6}$$

As a result, the sequence (3.5) is logarithmically convex. The proof of Corollary 3.1 is complete. $\square$

4. Monotonicity

After establishing diagonal recurrence relations and constructing inequalities related to the Stirling numbers of the second kind $S(n,k)$, we are now in a position to create an infinite sequence in terms of the Stirling numbers of the second kind $S(n,k)$ and to prove its increasing monotonicity.

THEOREM 4.1. *For any fixed positive integers n,k with $n \geqslant k \geqslant 2$, let*

$$\mathfrak{S}_1(n,k) = S^2(n,k-1) - S(n,k-2)S(n,k). \tag{4.1}$$

Then the infinite sequence $\{\mathfrak{S}_1(n+m,k+m)\}_{m \geqslant 0}$ is strictly increasing with respect to m.

Proof. It is well known in combinatorics that the Stirling numbers of the second kind $S(n,k)$ satisfy $S(0,0) = 1$, $S(n,0) = S(0,k) = 0$ for $n,k \geqslant 1$, and the "triangular" recurrence relation

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) \tag{4.2}$$

for $n \geqslant k \geqslant 1$. See [1, p. 208]. Hence, the inequality (3.6) may be rearranged as

$$
\begin{aligned}
&\frac{S(i+k+2,k)}{\binom{i+k+2}{k}} \frac{S(i+k,k)}{\binom{i+k}{k}} \\
&= \frac{S(i+k+1,k-1) + kS(i+k+1,k)}{\binom{i+k+2}{k}} \frac{S(i+k,k)}{\binom{i+k}{k}} \\
&= \frac{S(i+k,k-2) + (2k-1)S(i+k,k-1) + k^2 S(i+k,k)}{\binom{i+k+2}{k}} \frac{S(i+k,k)}{\binom{i+k}{k}} \\
&\geqslant \left[\frac{S(i+k+1,k)}{\binom{i+k+1}{k}} \right]^2 \\
&= \left[\frac{S(i+k,k-1) + kS(i+k,k)}{\binom{i+k+1}{k}} \right]^2.
\end{aligned}
$$

Replacing $i+k$ by n in the above inequality and simplifying give

$$
\frac{S(n,k-2) + (2k-1)S(n,k-1) + k^2 S(n,k)}{\binom{n+2}{k}} \frac{S(n,k)}{\binom{n}{k}} \geqslant \left[\frac{S(n,k-1) + kS(n,k)}{\binom{n+1}{k}} \right]^2,
$$

that is, by the recurrence relation (4.2),

$$
\begin{aligned}
&\frac{S(n,k-2)S(n,k)}{\binom{n+2}{k}\binom{n}{k}} - \frac{S^2(n,k-1)}{\binom{n+1}{k}^2} \\
&\geqslant \left[\frac{1}{\binom{n+1}{k}^2} - \frac{1}{\binom{n+2}{k}\binom{n}{k}} \right] k^2 S^2(n,k) + \left[\frac{2k}{\binom{n+1}{k}^2} - \frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} \right] S(n,k-1)S(n,k),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{S(n,k-2)S(n,k) - S^2(n,k-1)}{\binom{n+2}{k}\binom{n}{k}} + \left[\frac{1}{\binom{n+2}{k}\binom{n}{k}} - \frac{1}{\binom{n+1}{k}^2} \right] S^2(n,k-1) \\
&\geqslant \left[\frac{1}{\binom{n+1}{k}^2} - \frac{1}{\binom{n+2}{k}\binom{n}{k}} \right] k^2 S^2(n,k) + \left[\frac{2k}{\binom{n+1}{k}^2} - \frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} \right] S(n,k-1)S(n,k),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\mathfrak{S}_1(n,k)}{\binom{n+2}{k}\binom{n}{k}} \leqslant \left[\frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} - \frac{2k}{\binom{n+1}{k}^2} \right] S(n,k-1)S(n,k) \\
&\qquad + \left[\frac{1}{\binom{n+2}{k}\binom{n}{k}} - \frac{1}{\binom{n+1}{k}^2} \right] [k^2 S^2(n,k) + S^2(n,k-1)],
\end{aligned}
$$

$$\mathfrak{S}_1(n,k) \leqslant \left[2k-1-2k\frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2}\right]S(n,k-1)S(n,k)$$

$$+ \left[1-\frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2}\right]\left[k^2S^2(n,k)+S^2(n,k-1)\right],$$

$$\mathfrak{S}_1(n,k) \leqslant \left[1-\frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2}\right]\left[k^2S^2(n,k)+2kS(n,k-1)S(n,k)+S^2(n,k-1)\right]$$

$$- S(n,k-1)S(n,k)$$

$$= \left[1-\frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2}\right]\left[kS(n,k)+S(n,k-1)\right]^2 - S(n,k-1)S(n,k)$$

$$= \frac{k}{(n+1)(n-k+2)}S^2(n+1,k) - S(n,k-1)S(n,k).$$

In order to prove the increasing monotonicity of the infinite sequence $\{\mathfrak{S}_1(n+m,k+m)\}_{m\geqslant 0}$, it suffices to show

$$\frac{kS^2(n+1,k)}{(n+1)(n-k+2)} - S(n,k-1)S(n,k) \leqslant S^2(n+1,k) - S(n+1,k-1)S(n+1,k+1)$$

which may be reformulated as

$$\frac{S(n+1,k-1)S(n+1,k+1)}{S^2(n+1,k)} - \frac{S(n,k-1)S(n,k)}{S^2(n+1,k)} \leqslant \frac{(n+2)(n-k+1)}{(n+1)(n-k+2)}. \qquad (4.3)$$

In [9, p. 698], it was proved by the recurrence relation in (4.2) and by induction that

$$(m-1)(n-m)S^2(n,m) > (m+1)(n-m+1)S(n,m-1)S(n,m+1) \qquad (4.4)$$

for $2 \leqslant m \leqslant n-1$, which may be rewritten as

$$\frac{S(n+1,k-1)S(n+1,k+1)}{S^2(n+1,k)} < \frac{(k-1)(n-k+1)}{(k+1)(n-k+2)}$$

for $2 \leqslant k \leqslant n$. Therefore, in order to show (4.3), it is sufficient to verify

$$\frac{S(n,k-1)S(n,k)}{S^2(n+1,k)} \geqslant \frac{(k-1)(n-k+1)}{(k+1)(n-k+2)} - \frac{(n+2)(n-k+1)}{(n+1)(n-k+2)}$$

$$= -\frac{(n-k+1)(2n+k+3)}{(k+1)(n+1)(n-k+2)}$$

which is obvious. The proof of Theorem 4.1 is complete. $\square$

5. A conjecture

Finally, motivated by Theorem 4.1, we pose the following conjecture.

CONJECTURE 5.1. *For $k, \ell, n \in \mathbb{N}$, let $\mathfrak{S}_1(n,k)$ is defined by* (4.1) *and let*

$$\mathfrak{S}_{\ell+1}(n,k) = \mathfrak{S}_\ell^2(n,k-1) - \mathfrak{S}_\ell(n,k-2)\mathfrak{S}_\ell(n,k)$$

and

$$\mathscr{S}_\ell(n,k) = \frac{\mathfrak{S}_{\ell+1}(n,k)}{\mathfrak{S}_\ell(n,k)}$$

for $n \geqslant k \geqslant \ell+2$. Then the following claims are valid.

1. *For fixed integers $\ell \in \mathbb{N}$ and $n \geqslant \ell+3$, the finite sequence $\{\mathfrak{S}_\ell(n,k)\}_{\ell+1\leqslant k\leqslant n}$ is logarithmically concave with respect to k.*

2. *For fixed integers $n \geqslant k \geqslant 3$, the finite sequence $\{\mathfrak{S}_\ell(n,k)\}_{1\leqslant\ell\leqslant k-1}$ is strictly increasing with respect to ℓ.*

3. *For fixed integers $\ell \in \mathbb{N}$ and $n \geqslant k \geqslant \ell+1$, the infinite sequence $\{\mathfrak{S}_\ell(n+m,k+m)\}_{m\geqslant 0}$ is strictly increasing with respect to m.*

4. *For fixed integers $\ell \in \mathbb{N}$ and $k \geqslant \ell+1$, the infinite sequence $\{\mathfrak{S}_\ell(n,k)\}_{n\geqslant k}$ is strictly increasing with respect to n.*

5. *For fixed integers $n \geqslant k \geqslant \ell+2$, the infinite sequence $\{\mathscr{S}_\ell(n+m,k+m)\}_{m\geqslant 0}$ is strictly increasing with respect to m.*

6. *For fixed integers $k \geqslant \ell+2$, the infinite sequence $\{\mathscr{S}_\ell(n,k)\}_{n\geqslant k}$ is strictly increasing with respect to n.*

REMARK 5.1. This paper is a revised version of the preprint [6].

REFERENCES

[1] L. COMTET, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[2] B.-N. GUO AND F. QI, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory **3** (2015), no. 1, 27–30; Available online at `http://dx.doi.org/10.12785/jant/030105`.

[3] D. S. MITRINOVIĆ AND J. E. PEČARIĆ, *On two-place completely monotonic functions*, Anzeiger Öster. Akad. Wiss. Math.-Natturwiss. Kl. **126** (1989), 85–88.

[4] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993; Available online at `http://dx.doi.org/10.1007/978-94-017-1043-5`.

[5] J. E. PEČARIĆ, *Remarks on some inequalities of A. M. Fink*, J. Math. Anal. Appl. **104** (1984), no. 2, 428–431; Available online at `http://dx.doi.org/10.1016/0022-247X(84)90006-4`.

[6] F. QI, *A recurrence formula, some inequalities, and monotonicity related to Stirling numbers of the second kind*; Available online at `http://arxiv.org/abs/1402.2040`.

[7] F. QI, *Diagonal recurrence relations for the Stirling numbers of the first kind*, Contrib. Discrete Math. **10** (2015), in press; Available online at `http://arxiv.org/abs/1310.5920`.

[8] F. QI, *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory **133** (2013), no. 7, 2307–2319; Available online at `http://dx.doi.org/10.1016/j.jnt.2012.12.015`.

[9] M. SIBUYA, *Log-concavity of Stirling numbers and unimodality of Stirling distributions*, Ann. Inst. Statist. Math. **40** (1988), no. 4, 693–714; Available online at `http://dx.doi.org/10.1007/BF00049427`.

[10] Z.-Z. ZHANG AND J.-Z. YANG, *Notes on some identities related to the partial Bell polynomials*, Tamsui Oxf. J. Inf. Math. Sci. **28** (2012), no. 1, 39–48.

(Received May 5, 2015)

Feng Qi
Institute of Mathematics
Henan Polytechnic University
Jiaozuo City, Henan Province, 454010 China
and
College of Mathematics
Inner Mongolia University for Nationalities
Tongliao City, Inner Mongolia Autonomous Region, 028043 China
and
Department of Mathematics, College of Science
Tianjin Polytechnic University
Tianjin City, 300387 China
e-mail: qifeng618@gmail.com, qifeng618@hotmail.com
https://qifeng618.wordpress.com

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com