LYAPUNOV INEQUALITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH PRABHAKAR DERIVATIVE

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(*Communicated by D. Hinton*)

Abstract. In this paper, we consider a fractional boundary value problem including the Prabhakar fractional derivative. We obtain associated Green function for this fractional boundary value problem and get a Lyapunov-type inequality for it.

1. Introduction

In year 1893, the Russian mathematician Aleksandr Mikhailovich Lyapunov obtained a nontrivial solution for the following boundary value problem with the real and continuous function $q(t)$, [20]

$$y''(t) + q(t)y(t) = 0, \qquad a < t < b, \tag{1}$$

$$y(a) = y(b) = 0,$$

and get its corresponding inequality that was called *the Lyapunov inequality* after him

$$\int_a^b |q(u)|du > \frac{4}{b-a}. \tag{2}$$

Later, with developments in theory of fractional calculus many authors tried to express the differential inequalities with fractional derivatives [2, 3, 8, 4, 14, 24, 29, 30]. For the Lyapunov inequality in fractional differential equations, Ferreira [6, 7] showed for the fractional differential equations

$$(_aD^\mu y)(t) + q(t)y(t) = 0, \qquad y(a) = y(b) = 0, \qquad 1 < \mu \leqslant 2, \tag{3}$$

$$(_a^C D^\mu y)(t) + q(t)y(t) = 0, \qquad y(a) = y(b) = 0, \qquad 1 < \mu \leqslant 2, \tag{4}$$

in the sense of Riemann-Liouville and Caputo derivatives the following inequalities hold, respectively

$$\int_a^b |q(u)|du > \Gamma(\mu)\left(\frac{4}{b-a}\right)^{\mu-1}, \tag{5}$$

$$\int_a^b |q(u)|du > \Gamma(\mu)\frac{\mu^\mu}{[(\mu-1)(b-a)]^{\mu-1}}. \tag{6}$$

Also, Jleli and Samet modified the above inequalities for fractional differential equations with mixed boundary conditions [15, 16]. Now in this paper, as a generalization of fractional derivatives, we consider the following fractional boundary value problem including the Prabhakar fractional derivative

$$(D_{\rho,\mu,\omega,a+}^\gamma y)(t) + q(t)y(t) = 0, \quad a < t < b, \quad 1 < \mu \leqslant 2,\ \gamma,\rho,\omega \in \mathbb{R}^+, \tag{7}$$

with the boundary conditions

$$y(a) = y(b) = 0, \tag{8}$$

where $y \in C[a,b]$ (the class of all continuous functions). For this purpose, we intend to find the associated Green function of the fractional boundary value problem (7) in terms of the generalized Mittag-Leffler functions $E_{\rho,\mu}^\gamma(z)$. We state some properties of this Green function and obtain the Lyapunov inequality for the fractional boundary value problem (7). In particular case, we reduce the Green function and the Lyapunov inequality of the fractional boundary value problem (7) to the Green function and Lyapunov inequality of fractional boundary value problem (3).

2. Preliminaries

2.1. The generalized Mittag-Leffler function

In year 1971, Prabhakar introduced the generalized Mittag-Leffler function (Mittag-Leffler function with three parameters) on his study on singular integral equations as follows [25]

$$E_{\rho,\mu}^\gamma(z) = \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\rho k + \mu)}\frac{z^k}{k!}, \quad \gamma,\rho,\mu \in \mathbb{C}, \Re(\rho) > 0, \tag{9}$$

where $(\gamma)_k$ is the Pochhammer symbol [5]

$$(\gamma)_0 = 1, \qquad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1), \qquad k = 1,2,\cdots.$$

For $\gamma = 1$, we get the two-parameter Mittag-Leffler function $E_{\rho,\mu}(z)$ defined by

$$E_{\rho,\mu}(z) := E_{\rho,\mu}^1(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \mu)}, \quad \rho,\mu \in \mathbb{C}, \Re(\rho) > 0, \tag{10}$$

and for $\gamma = \mu = 1$, this function coincides with the classical Mittag-Leffler function $E_\rho(z)$ [22, 23]

$$E_\rho(z) := E_{\rho,1}^1(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + 1)}, \quad \rho \in \mathbb{C}, \Re(\rho) > 0. \tag{11}$$

Also, for $\gamma = 0$ we have

$$E_{\rho,\mu}^0(z) = \frac{1}{\Gamma(\mu)}. \tag{12}$$

For the generalized Mittag-Leffler function (9), many researchers have established many contributions in theory of fractional calculus with applications in mathematical physics and Cauchy-type initial and boundary value problems [9, 10, 11, 12, 13, 17, 18, 19, 21], [26, 27, 28].

LEMMA 1. *The Laplace transforms of generalized Mittag-Leffler function (9) has the following form* [25]:

$$\mathscr{L}\left[x^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega x^{\rho})\right](s) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}, \qquad |\omega s^{-\rho}| < 1, \qquad (13)$$

for $\gamma,\rho,\mu,\omega,s \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(s) > 0$.

LEMMA 2. *Let* $\gamma,\rho,\mu,\omega \in \mathbb{C}$, $\Re(\rho) > 0$. *Then for any* $n \in \mathbb{N}$, *differentiation of the generalized Mittag-Leffler function (9) is given by* [18]

$$\left(\frac{d}{dx}\right)^{n}\left[x^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega x^{\rho})\right] = x^{\mu-n-1}E_{\rho,\mu-n}^{\gamma}(\omega x^{\rho}). \qquad (14)$$

2.2. Prabhakar derivative and integral

DEFINITION 1. (Prabhakar integral). *Let* $f \in L^{1}[0,b]$, $0 < x < b \leqslant \infty$. *The Prabhakar integral operator including the generalized Mittag-Leffler function (9) is defined as follows* [9]

$$E_{\rho,\mu,\omega,0+}^{\gamma}f(x)dx = \int_{0}^{x}(x-u)^{\mu-1}E_{\rho,\mu}^{\gamma}\left(\omega(x-u)^{\rho}\right)f(u)du, \qquad x > 0, \qquad (15)$$

where $\rho,\mu,\omega,\gamma \in \mathbb{C}$, $\Re(\rho)$, $\Re(\mu) > 0$.

REMARK 1. We note that for $\gamma = 0$, the Prabhakar integral operator (15) coincides with the Riemann-Liouville fractional integral of order μ

$$E_{\rho,\mu,\omega,0+}^{0}f = I_{0+}^{\mu}f, \qquad (16)$$

where the Riemann-Liouville fractional integral is defined as

$$I_{0+}^{\mu}f(x) = \frac{1}{\Gamma(\mu)}\int_{0}^{x}(x-t)^{\mu-1}f(t)dt, \qquad \mu \in \mathbb{C}, \Re(\mu) > 0. \qquad (17)$$

DEFINITION 2. (Prabhakar derivative). *Let* $f \in L^{1}[0,b]$, $0 < x < b \leqslant \infty$. *The Prabhakar derivative is defined by* [9]

$$D_{\rho,\mu,\omega,0+}^{\gamma}f(x) = \frac{d^{m}}{dx^{m}}E_{\rho,m-\mu,\omega,0+}^{-\gamma}f(x), \qquad (18)$$

where $\rho,\mu,\omega,\gamma \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\mu) > 0$, $m-1 < \Re(\mu) < m$.

REMARK 2. It is obvious that the Prabhakar derivative (18) generalizes the Riemann-Liouville fractional derivative

$$D_{0+}^{\mu}f(x) = \frac{d^m}{dx^m}(I_{0+}^{m-\mu})(x), \qquad \mu \in \mathbb{C}, \ \mathfrak{R}(\mu) > 0, \qquad m-1 < \mathfrak{R}(\mu) < m. \quad (19)$$

LEMMA 3. *The Laplace transform of Prabhakar integral (15) is given by*

$$\mathscr{L}\{E_{\rho,\mu,\omega,0+}^{\gamma}f(x);s\} = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}F(s), \quad (20)$$

where

$$F(s) = \mathscr{L}\{f(x);s\} = \int_0^{\infty} e^{-sx}f(x)dx, \qquad s \in \mathbb{C}. \quad (21)$$

Proof. According to Definition 1, we have

$$E_{\rho,\mu,\omega,0+}^{\gamma}f(x)dx = \int_0^x (x-u)^{\mu-1}E_{\rho,\mu}^{\gamma}\left(\omega(x-u)^{\rho}\right)f(u)du = \left(f * e_{\rho,\mu,\omega}^{\gamma}\right)(x), \quad (22)$$

where $*$ is the convolution integral and $e_{\rho,\mu,\omega}^{\gamma}(x) = x^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega x^{\rho})$. Therefore, by using the Laplace transform

$$\mathscr{L}\{(f*g)(x);s\} = \mathscr{L}\{f(x);s\}\mathscr{L}\{g(x);s\}, \quad (23)$$

and applying the relation (13), the proof of lemma is completed. □

LEMMA 4. *For $m-1 < \mu \leqslant m$, the Laplace transform of Prabhakar derivative (18) has the form*

$$\mathscr{L}\{D_{\rho,\mu,\omega,0+}^{\gamma}f(x);s\} = s^{\mu}(1-\omega s^{-\rho})^{\gamma}F(s) - \sum_{k=0}^{m-1} s^k (D_{\rho,\mu-k-1,\omega,0+}^{\gamma}f)(0), \quad (24)$$

where $F(s)$ is the Laplace transform of $f(x)$.

Proof. Applying the Laplace transform operator on the Prabhakar derivative (18) and using the following formula [1]

$$\mathscr{L}\{\frac{d^m}{dx^m}f(x);s\} = s^m\mathscr{L}\{f(x),s\} - \sum_{k=0}^{m-1} s^k f^{(m-k-1)}(0), \quad (25)$$

we have

$$\mathscr{L}\{D_{\rho,\mu,\omega,0+}^{\gamma}f(x);s\} = \mathscr{L}\{\frac{d^m}{dx^m}E_{\rho,m-\mu,\omega,0+}^{-\gamma}f(x);s\}$$

$$= s^m\mathscr{L}\{E_{\rho,m-\mu,\omega,0+}^{-\gamma}f(x);s\} - \sum_{k=0}^{m-1} s^k \left(\frac{d^{m-k-1}}{dx^{m-k-1}}E_{\rho,m-\mu,\omega,0+}^{-\gamma}f\right)(0).$$

Now, according to Lemma 3, we can deduce the relation (24) easily. □

LEMMA 5. *If* $f(x) \in C(a,b) \cap L(a,b)$, *then*

$$D^\gamma_{\rho,\mu,\omega,a+} E^\gamma_{\rho,\mu,\omega,a+} f(x) = f(x), \tag{26}$$

and if $f(x), D^\gamma_{\rho,\mu,\omega,a+} f(x) \in C(a,b) \cap L(a,b)$, *then for* $c_j \in \mathbb{R}$ *and* $m-1 < \mu \leqslant m$, *we have*

$$
\begin{aligned}
E^\gamma_{\rho,\mu,\omega,a+} D^\gamma_{\rho,\mu,\omega,a+} f(x) &= f(x) + c_1(x-a)^{\mu-1} E^\gamma_{\rho,\mu}\big(\omega(x-a)^\rho\big) \\
&\quad + c_2(x-a)^{\mu-2} E^\gamma_{\rho,\mu-1}\big(\omega(x-a)^\rho\big) \\
&\quad + \ldots + c_m(x-a)^{\mu-m} E^\gamma_{\rho,\mu-m+1}\big(\omega(x-a)^\rho\big).
\end{aligned} \tag{27}
$$

Proof. Using the relations (20) and (24) for the Laplace transform of left hand side of (27), we obtain (in case $a = 0$)

$$\mathscr{L}\{E^\gamma_{\rho,\mu,\omega,0+} D^\gamma_{\rho,\mu,\omega,0+} f(x); s\} = F(s) - \sum_{k=0}^{m-1} s^{k-\mu}(1-\omega s^{-\rho})^{-\gamma} \big(D^\gamma_{\rho,\mu-k-1,\omega,0+} f\big)(0). \tag{28}$$

Now, by applying the inverse Laplace transform and modification for $a \neq 0$, we get the right hand side of relation (27). □

3. Main Theorems

THEOREM 1. *Let* $1 < \mu \leqslant 2$, $\gamma, \rho, \omega \in \mathbb{R}^+$, $y \in C[a,b] \cap L[a,b]$, *then the fractional boundary value problem*

$$(D^\gamma_{\rho,\mu,\omega,a+} y)(t) + q(t)y(t) = 0, \qquad y(a) = y(b) = 0, \qquad a < t < b, \tag{29}$$

is equivalent to the integral equation

$$y(t) = \int_a^b G(t,u)q(u)y(u)du, \tag{30}$$

where the Green function G is given by

$$
G(t,u) = \begin{cases}
\dfrac{(t-a)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(t-a)^\rho)}{(b-a)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(b-a)^\rho)}(b-u)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(b-u)^\rho) \\
\qquad\qquad - (t-u)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(t-u)^\rho),\ a \leqslant u \leqslant t \leqslant b, \\
\dfrac{(t-a)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(t-a)^\rho)}{(b-a)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(b-a)^\rho)}(b-u)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(b-u)^\rho),\ a \leqslant t \leqslant u \leqslant b.
\end{cases} \tag{31}
$$

Proof. Applying the operator $E^\gamma_{\rho,\mu,\omega,a+}$ on fractional differential equation (29) and using Lemma 5, for real constants c_1 and c_2 we have

$$
\begin{aligned}
y(t) &= -\big(E^\gamma_{\rho,\mu,\omega,a+} qy\big)(t) + c_1(t-a)^{\mu-1} E^\gamma_{\rho,\mu}\big(\omega(t-a)^\rho\big) \\
&\quad + c_2(t-a)^{\mu-2} E^\gamma_{\rho,\mu-1}\big(\omega(t-a)^\rho\big),
\end{aligned} \tag{32}
$$

or equivalently

$$y(t) = -\int_a^t (t-u)^{\mu-1} E_{\rho,\mu}^{\gamma}\big(\omega(t-u)^{\rho}\big)q(u)y(u)du + c_1(t-a)^{\mu-1}E_{\rho,\mu}^{\gamma}\big(\omega(t-a)^{\rho}\big)$$
$$+ c_2(t-a)^{\mu-2}E_{\rho,\mu-1}^{\gamma}\big(\omega(t-a)^{\rho}\big). \tag{33}$$

Now, by employing the boundary conditions we can obtain the coefficients c_1 and c_2 as follows

$$y \in C[a,b], \ y(a) = 0 \Leftrightarrow c_2 = 0,$$

$$y(b) = 0 \Leftrightarrow c_1 = \frac{1}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}\big(\omega(b-a)^{\rho}\big)} \int_a^b (b-u)^{\mu-1} E_{\rho,\mu}^{\gamma}\big(\omega(b-u)^{\rho}\big)q(u)y(u)du.$$

Therefore, the unique solution of (29) is

$$y(t) = \int_a^t \left[\frac{(t-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-u)^{\rho}) \right.$$
$$\left. - (t-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-u)^{\rho}) \right] q(u)y(u)du$$
$$+ \int_t^b \left[\frac{(t-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-u)^{\rho}) \right]$$
$$= \int_a^b G(t,u)q(u)y(u)du. \quad \square$$

THEOREM 2. *The Green function (31) satisfies the following conditions*

1. *For all $a \leqslant t, \ u \leqslant b, \ G(t,u) \geqslant 0$.*

2. *$\max_{t \in [a,b]} G(t,u) = G(u,u)$ for $u \in [a,b]$.*

3. *The maximum of $G(u,u)$ is given at $u = \frac{a+b}{2}$ and has the value*

$$\max_{u \in [a,b]} G(u,u) = G\left(\frac{a+b}{2},\frac{a+b}{2}\right) = \left(\frac{b-a}{4}\right)^{\mu-1} \frac{E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}{E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}. \tag{34}$$

Proof. We set two functions

$$g_1(t,u) = \frac{(t-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-u)^{\rho})$$
$$- (t-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-u)^{\rho}), \qquad a \leqslant u \leqslant t \leqslant b,$$

$$g_2(t,u) = \frac{(t-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-u)^{\rho}), \qquad a \leqslant t \leqslant u \leqslant b.$$

It is clear that $g_2(t,u) \geqslant 0$. So to prove 1, we should show that $g_1(t,u) \geqslant 0$, or equivalently

$$\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-u)^{\rho}) \geqslant (t-u)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-u)^{\rho}).$$

Therefore it is sufficient to show

(i) $\frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}(b-u)^{\mu-1} \geqslant (t-u)^{\mu-1}$,

(ii) $\frac{E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}E^{\gamma}_{\rho,\mu}(\omega(b-u)^{\rho}) \geqslant E^{\gamma}_{\rho,\mu}(\omega(t-u)^{\rho})$.

Proof of (i):

$$\frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}(b-u)^{\mu-1} \geqslant (t-u)^{\mu-1}$$

$$\Leftrightarrow \frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}(b-u)^{\mu-1} \geqslant \frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}\left(b - \left(a + \frac{(u-a)(b-a)}{t-a}\right)\right)^{\mu-1}$$

$$\Leftrightarrow a + \frac{(u-a)(b-a)}{t-a} \geqslant u$$

$$\Leftrightarrow \frac{a(t-a) + (u-a)(b-a)}{t-a} \geqslant u$$

$$\Leftrightarrow a(t-b) + u(b-t) \geqslant 0$$

$$\Leftrightarrow u \geqslant a.$$

According to inequality $(t-a)(b-u) \geqslant (b-a)(t-u)$ and Taylor expansion of the generalized Mittag-Leffler function $E^{\gamma}_{\rho,\mu}(z)$, for $1 < \mu \leqslant 2$, $\gamma,\rho,\omega,z \in \mathbb{R}^+$ the proof of (ii) is completed.

Proof of (ii): By differentiating g_1 with respect to t for every fixed u and by applying Lemma 2, we get

$$g_1'(t,u) = \frac{(b-u)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-u)^{\rho})}{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}(t-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(t-a)^{\rho})$$

$$- (t-u)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-u)^{\rho})$$

$$\leqslant \frac{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}(t-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(t-a)^{\rho})$$

$$- (t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho}) = 0,$$

that yields g_1 is a decreasing function of t. Similarly by differentiating g_2 with respect to t for every fixed u, we conclude that g_2 is a increasing function. Therefore, the maximum of G with respect to t is the value $G(u,u)$. Finally if we set

$$f(u) = G(u,u) = \frac{(u-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(u-a)^{\rho})}{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}(b-u)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-u)^{\rho}), \qquad u \in [a,b],$$

then, using the relations (9) and (14) for $u \in [a,b]$ we have

$$
\begin{aligned}
f'(u) &= \frac{1}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}\Big[(u-a)^{\mu-2}E_{\rho,\mu-1}^{\gamma}(\omega(u-a)^{\rho})(b-u)^{\mu-1} \\
&\quad \times E_{\rho,\mu}^{\gamma}(\omega(b-u)^{\rho}) - (u-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(u-a)^{\rho})(b-u)^{\mu-2}E_{\rho,\mu-1}^{\gamma}(\omega(b-u)^{\rho})\Big] \\
&= \frac{((u-a)(b-u))^{\mu-2}}{(b-a)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}\Big[(b-u)\sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k(u-a)^{\rho k}}{\Gamma(\rho k+\mu-1)k!}\sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k(b-u)^{\rho k}}{\Gamma(\rho k+\mu)k!} \\
&\quad -(u-a)\sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k(u-a)^{\rho k}}{\Gamma(\rho k+\mu)k!}\sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k(b-u)^{\rho k}}{\Gamma(\rho k+\mu-1)k!}\Big].
\end{aligned}
$$

By solving $f'(u) = 0$, we see that $f'(u) > 0$ on $(a, \frac{a+b}{2})$ and $f'(u) < 0$ on $(\frac{a+b}{2}, b)$. Hence, we deduct that $u = \frac{a+b}{2}$ is maximum point. $\square$

THEOREM 3. *Let $\mathscr{B} = C[a,b]$ be the Banach space with norm $\|y\| = \sup_{t\in[a,b]}|y(t)|$ and a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{\rho,\mu,\omega,a+}^{\gamma}y)(t) + q(t)y(t) = 0, \qquad a < t < b, \qquad y(a) = y(b) = 0,$$

exists, then for the real and continuous function q the following inequality holds

$$\int_a^b |q(u)|du > \left(\frac{4}{b-a}\right)^{\mu-1}\frac{E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}{E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}. \tag{35}$$

Proof. According to Theorem 1, a solution of the above fractional boundary value problem satisfies the integral equation

$$y(t) = \int_a^b G(t,u)q(u)y(u)du,$$

which by applying the indicated norm on both sides of it, we have

$$\|y\| \leqslant \max_{t\in[a,b]}\int_a^b |G(t,u)q(u)|du\|y\|, \tag{36}$$

or equivalently

$$1 \leqslant \max_{t\in[a,b]}\int_a^b |G(t,u)q(u)|du. \tag{37}$$

At this point, using the second property of the Green function in Theorem 2, we get

$$1 < \left(\frac{b-a}{4}\right)^{\mu-1}\frac{E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})E_{\rho,\mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}{E_{\rho,\mu}^{\gamma}(\omega(b-a)^{\rho})}\int_a^b |q(u)|du. \quad \square \tag{38}$$

COROLLARY 1. *In the special case* $\gamma = 0$, *the fractional boundary value problem* (29) *is reduced to* [6]

$$(_aD^\mu y)(t) + q(t)y(t) = 0, \qquad y(a) = y(b) = 0, \qquad 1 < \mu \leqslant 2,$$

with the Green function

$$G(t,u) = \frac{1}{\Gamma(\mu)} \begin{cases} \frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}(b-u)^{\mu-1} - (t-u)^{\mu-1}, \ a \leqslant u \leqslant t \leqslant b, \\ \frac{(t-a)^{\mu-1}}{(b-a)^{\mu-1}}(b-u)^{\mu-1}, \ a \leqslant t \leqslant u \leqslant b, \end{cases} \tag{39}$$

and the Lyapunov inequality

$$\int_a^b |q(u)|du > \Gamma(\mu) \left(\frac{4}{b-a} \right)^{\mu-1}.$$

Also, when $\mu = 2$, *we get the classical Lyapunov inequality* (2).

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