Mathematical
Inequalities
& Applications
Volume 19, Number 1 (2016), 369–379 doi:10.7153/mia-19-28

KANTOROVICH PROBLEMS UNDER YOUNG TYPE CONSTRAINTS

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(*Communicated by C. P. Niculescu*)

Abstract. We explain how the Hermite-Hadamard inequality agrees with the primal Monge-Kantorovich problems. We also focus onto a particular case of the dual Kantorovich problem, by considering a version significantly affected by some Young type constraints.

1. Introduction

As it is well-known, the optimal transport is a simple, meaningful, natural and therefore universal concept. See Cédric Villani [8]. In this paper we study the classical transportation problems with additional constraints (of Young type), proving that whenever our constraints are not too restrictive, it is possible to describe the solution in simple mathematical terms.

The framework is as follows. Let $(\mathscr{X},\mu)$ and $(\mathscr{Y},\nu)$ be two separable, complete, probability metric spaces. Let $c:\mathscr{X}\times\mathscr{Y}\to\mathbb{R}_{+}\cup\{+\infty\}$ be a lower semicontinuous cost function.

The so-called Monge transportation problem consists in the determination of a Borel measurable map $T:\mathscr{X}\to\mathscr{Y}$ that realizes the infimum

$$\inf_{T\#\mu=\nu}\int_{\mathscr{X}}c\left(x,T\left(x\right)\right)\mathrm{d}\mu\left(x\right), \tag{Monge}$$

where $T\#\mu$ represents the push-forward of μ through a Borel map T, defined by

$$\left(T\#\mu\right)\left(A\right)=\mu\left(T^{-1}\left(A\right)\right)$$

for every Borel subset A of $\mathscr{Y}$. In fact, this is the abstract formulation in modern terminology of a problem put in 1781 by Monge in the context of excavations and embankments (1781). In other words one looks for the cheapest way of transporting some given mass distribution onto another one. But the problem has been ill posed, because there does not always exist T which satisfies $T\#\mu=\nu$. Even when such a map T exists, it can happen that the problem has no minimizer.

Mathematics subject classification (2010): 49N15, 82C70, 28A35, 90B06, 26A51.
Keywords and phrases: Cost function, Monge-Kantorovich problem, duality, Kantorovich potential, cost convexity, Young's inequality, Hermite-Hadamard inequality.

We recall that a coupling of μ and ν is a measure π on $\mathscr{X}\times\mathscr{Y}$ which admits μ and ν as marginals on $\mathscr{X}$ and $\mathscr{Y}$ respectively, that is,

$$\pi(A\times\mathscr{Y})=\mu(A) \text{ and } \pi(\mathscr{X}\times B)=\nu(B),$$

for all measurable sets $A\subset\mathscr{X}$, $B\subset\mathscr{Y}$. We can interpret $\pi(A\times B)$ as the amount of mass transported from A to B. For instance if this quantity is directly proportional to $\nu(B)$, then π coincides with the tensor product $\mu\otimes\nu$. The measures π obtained by coupling μ and ν are called transference plans and satisfy the condition

$$\int_{\mathscr{X}\times\mathscr{Y}}(f(x)+g(y))\,\mathrm{d}\pi(x,y)=\int_{\mathscr{X}}f(x)\,\mathrm{d}\mu(x)+\int_{\mathscr{Y}}g(y)\,\mathrm{d}\nu(y) \tag{1.1}$$

for all bounded and continuous functions f,g. The reader may to consult [6] and [8] for further background on this topic. We denote by $\Pi(\mu,\nu)$ the set of all couplings of the measures μ and ν.

The so-called (primal) Monge-Kantorovich problem of optimal mass transportation (1942) consists in the determination of a minimizer π in $\Pi(\mu,\nu)$ for the optimal transport cost

$$\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathscr{X}\times\mathscr{Y}}c(x,y)\,\mathrm{d}\pi(x,y). \tag{Monge-Kantorovich}$$

The infimum is known as Kantorovich-Wasserstein distance. The set $\Pi(\mu,\nu)$ is not empty (as we have seen, it contains the tensor product $\mu\otimes\nu$), hence the problem is well posed. Due to the property of lower semicontinuity of the function c the infimum is achieved, that is, optimal transference plans π exist, usually not uniquely determined. Also, if T is a solution for the Monge problem, then we may define a plan $\bar{\pi}\in\Pi(\mu,\nu)$ such that the mass is not split during the transportation process, as

$$\bar{\pi}(E)=(Id,T)\#\mu(E)=\mu(\{x:(x,T(x))\in E\}) \tag{1.2}$$

for every Borel subset E of $\mathscr{X}\times\mathscr{Y}$. Obviously, in most cases the optimal couplings in $\Pi(\mu,\nu)$ need not be generated by any one-to-one mapping T. Still, if μ is atomless and a map T generates an optimal coupling, then T is also an optimal transport map for the Monge problem.

We now recall the basics of Kantorovich duality. A pair of continuous (f,g) of a continuous and bounded function $f:\mathscr{X}\to\mathbb{R}$ and a continuous and bounded function $g:\mathscr{Y}\to\mathbb{R}$ is called competitive if $f(x)+g(y)\leqslant c(x,y)$ for all $(x,y)\in\mathscr{X}\times\mathscr{Y}$. The determination of a couple of so-called potentials which maximize

$$\sup_{(f,g)\text{ competitive}}\left\{\int_{\mathscr{X}}f(x)\,\mathrm{d}\mu(x)+\int_{\mathscr{Y}}g(y)\,\mathrm{d}\nu(y)\right\} \tag{Kantorovich}$$

is called the dual Kantorovich problem (1942). Moreover,

$$c(x,y)=\sup_{(f,g)\text{ competitive}}\{f(x)+g(y)\}\ \pi-\text{a.e.},$$

where π is a solution of the primal problem.

A proper function $f:\mathscr{X}\rightarrow[-\infty,\infty)$ is called c-concave (see for instance [6, Definition 2.33]) if there exists a proper function $g:\mathscr{Y}\rightarrow[-\infty,\infty)$ such that for all $x\in\mathscr{X}$ we have

$$f(x)=\inf_{y\in\mathscr{Y}}\{c(x,y)-g(y)\}. \tag{1.3}$$

Its c-transform (c-conjugate) is f^{c}, defined by

$$f^{c}(y)=\inf_{x\in\mathscr{X}}\{c(x,y)-f(x)\}.$$

For some structural results concerning the cost convexity, the reader is referred to our paper [3]. Every solution (f,g) for the dual Kantorovich problem contains c-concave conjugate functions, that is $g=f^{c}$. Also, there is no duality gap, that is

$$\sup_{(f,g)\text{ competitive}}\left\{\int_{\mathscr{X}}f(x)\,\mathrm{d}\mu(x)+\int_{\mathscr{Y}}g(y)\,\mathrm{d}\nu(y)\right\}$$
$$=\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathscr{X}\times\mathscr{Y}}c(x,y)\,\mathrm{d}\pi(x,y). \tag{1.4}$$

For the particular case $c(x,y)=xy$ we get from (1.3) the usual definition of concave functions.

The following result is interesting in itself.

PROPOSITION 1. *Let $\pi\in\Pi(\mu,\nu)$. Then*

$$\pi((\mathscr{X}\times\mathscr{Y})\setminus(A\times B))\leqslant\mu(\mathscr{X}\setminus A)+\nu(\mathscr{Y}\setminus B)$$

for all measurable sets $A\subset\mathscr{X}$, $B\subset\mathscr{Y}$. Equality holds if and only if

$$\pi((\mathscr{X}\setminus A)\times(\mathscr{Y}\setminus B))=0.$$

Proof. We successively derive that

$$\pi((\mathscr{X}\times\mathscr{Y})\setminus(A\times B))\leqslant\pi((\mathscr{X}\setminus A)\times\mathscr{Y})+\pi(\mathscr{X}\times(\mathscr{Y}\setminus B))=\mu(\mathscr{X}\setminus A)+\nu(\mathscr{Y}\setminus B).$$

The equality case is obvious. $\square$

For the reader's convenience we recall the Hermite-Hadamard inequality which will be closer investigated in the next section through primal Monge-Kantorovich problems.

If μ is a Borel probability measure on an interval $[a,b]$ with barycenter

$$b_{\mu}=\int_{a}^{b}x\,\mathrm{d}\mu(x),$$

then for every continuous convex function $f:[a,b]\rightarrow\mathbb{R}$ we have

$$f(b_{\mu})\leqslant\int_{a}^{b}f(x)\,\mathrm{d}\mu(x)\leqslant\frac{b-b_{\mu}}{b-a}f(a)+\frac{b_{\mu}-a}{b-a}f(b).$$

DEFINITION 1. A strictly increasing continuous function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$ is called *Young function* (Y-function).

Obviously the inverse function of a Y-function is again a Y-function.

Our approach to the dual Kantorovich problem is based on Young's inequality which asserts that every Y-function verifies an inequality of the following form,

$$ab \leqslant \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y,$$

whenever a and b are nonnegative real numbers. The equality occurs if and only if $f(a) = b$.

We shall also need the following reverse of Young's inequality obtained by A. Witkowski [9].

LEMMA 1. *Let f be continuous and strictly increasing. The following inequality holds:*

$$\min\left\{1, \frac{b}{f(a)}\right\} \int_0^a f(x)\,\mathrm{d}x + \min\left\{1, \frac{a}{f^{-1}(b)}\right\} \int_0^b f^{-1}(y)\,\mathrm{d}y \leqslant ab.$$

Equality holds iff $b = f(a)$.

The aims of this paper are twofold. Firstly, we describe how the primal Monge-Kantorovich problems help us understand the Hermite-Hadamard inequality. Secondly, we discuss the dual Kantorovich type problem subjected to some additional constraints of Young type and investigate how the solution of such a problem can generate solutions for the classical optimization problems discussed above.

2. Mass transfer and the Hermite-Hadamard inequality

The mass transfer theory is strongly related to the mechanism of the Hermite-Hadamard inequality.

EXAMPLE 1. (the quadratic cost $c(x,y) = \frac{1}{2}|x-y|^2$) Let us have a closer look at the quadratic case of the quadratic cost. The transportation cost of the mass $\nu_0 = \delta_{b_\mu}$ to $\nu_1 = \frac{b-b_\mu}{b-a}\delta_a + \frac{b_\mu - a}{b-a}\delta_b$ is

$$\begin{aligned}
\int_{[a,b]^2} c(x,y)\,\mathrm{d}\nu_0 \otimes \nu_1(x,y) &= \int_a^b c(b_\mu, y)\,\mathrm{d}\nu_1(y) \\
&= \frac{b-b_\mu}{b-a} c(b_\mu, a) + \frac{b_\mu - a}{b-a} c(b_\mu, b) \\
&= \frac{1}{2}(b_\mu - a)(b - b_\mu).
\end{aligned}$$

When we transport the measure $\nu_0 = \delta_{b_\mu}$ to μ, the transportation cost is given by

$$\int_{[a,b]^2} c(x,y)\,\mathrm{d}\nu_0 \otimes \mu(x,y) = \int_a^b \frac{1}{2}\left|x - b_\mu\right|^2 \mathrm{d}\mu(x).$$

The Hermite-Hadamard inequality applied to $f(x) = \left|x - b_\mu\right|^2$ yields

$$\int_a^b \left|x - b_\mu\right|^2 \mathrm{d}\mu(x) \leqslant \left(b_\mu - a\right)\left(b - b_\mu\right),$$

hence the transfer of the mass ν_0 to μ is cheaper than the transfer of the mass ν_0 to ν_1. Our result agrees with the one obtained in [5].

EXAMPLE 2. (the cost $c(x,y) = |x - y|$) We discuss the case, originally considered by Monge, where the cost is the euclidean distance. The transportation cost of the mass $\nu_0 = \delta_{b_\mu}$ to $\nu_1 = \frac{b - b_\mu}{b-a}\delta_a + \frac{b_\mu - a}{b-a}\delta_b$ is

$$\int_{[a,b]^2} c(x,y)\,\mathrm{d}\nu_0 \otimes \nu_1(x,y) = 2\frac{\left(b_\mu - a\right)\left(b - b_\mu\right)}{b-a}.$$

The transport of the measure $\nu_0 = \delta_{b_\mu}$ to μ costs

$$\int_{[a,b]^2} c(x,y)\,\mathrm{d}\nu_0 \otimes \mu(x,y) = \int_a^b \left|x - b_\mu\right| \mathrm{d}\mu(x).$$

Again, applying the Hermite-Hadamard inequality to $f(x) = \left|x - b_\mu\right|$ yields

$$\begin{aligned}\int_a^b \left|x - b_\mu\right| \mathrm{d}\mu(x) &\leqslant \frac{b - b_\mu}{b-a}\left|a - b_\mu\right| + \frac{b_\mu - a}{b-a}\left|b - b_\mu\right| \\ &= 2\frac{\left(b_\mu - a\right)\left(b - b_\mu\right)}{b-a}.\end{aligned}$$

Emphasis is now placed on a more extended conclusion: the Hermite-Hadamard inequality agrees this way with Monge-Kantorovich problems for any convex cost function $c(|x-y|)$. Note that in both examples we had $\Pi(\nu_0, \nu_1) = \{\nu_0 \otimes \nu_1\}$ and $\Pi(\nu_0, \mu) = \{\nu_0 \otimes \mu\}$.

REMARK 1. Concerning the previous example we see that when μ coincides with the normalized Lebesgue measure $\lambda/(b-a)$ we get

$$\int_{[a,b]^2} |x-y|\,\mathrm{d}\nu_0 \otimes \nu_1(x,y) = 2\int_{[a,b]^2} |x-y|\,\mathrm{d}\nu_0 \otimes \mu(x,y) = \frac{b-a}{2},$$

which has a nice and very intuitive interpretation: the cost of the transfer from ν_0 to $\lambda/(b-a)$ equals the cost of the transfer from $\lambda/(b-a)$ to ν_1. Indeed, a minimizer (among joint measures $\Pi(\lambda/(b-a), \nu_1)$) for the optimal transport cost

$$\inf_{\Pi(\lambda/(b-a),\nu_1)} \int_{[a,b]^2} |x-y|\,\mathrm{d}\pi(x,y)$$

is given by

$$\pi = \begin{cases} 2\lambda/(b-a) \otimes \frac{1}{2}\delta_a \text{ on } \left[a, \frac{a+b}{2}\right] \times [a,b] \\ 2\lambda/(b-a) \otimes \frac{1}{2}\delta_b \text{ on } \left[\frac{a+b}{2}, b\right] \times [a,b] \end{cases}.$$

One has

$$\begin{aligned} \int_{[a,b]^2} |x-y| \, \mathrm{d}\pi(x,y) &= \int_{\left[a,\frac{a+b}{2}\right]\times[a,b]} |x-y| \, \mathrm{d}\pi(x,y) + \int_{\left[\frac{a+b}{2},b\right]\times[a,b]} |x-y| \, \mathrm{d}\pi(x,y) \\ &= \frac{b-a}{4}. \end{aligned}$$

We stress that $\Pi(\lambda/(b-a), \nu_1) \neq \{\lambda/(b-a) \otimes \nu_1\}$.

REMARK 2. For the cost function $c(x,y) = |x-y|$, the cost of the transfer from ν_1 to ν_0 (which, of course, has the same value as the cost of the transfer from ν_0 to ν_1) equals the cost of the transfer from ν_1 to $\lambda/(b-a)$ accordingly to the transport plan $\nu_1 \otimes \lambda/(b-a)$:

$$\int_{[a,b]^2} |x-y| \, \mathrm{d}\nu_1 \otimes \lambda/(b-a) = \frac{b-a}{2}.$$

3. The dual Young-Kantorovich problem

Let $c : [0,a] \times [0,b] \to \mathbb{R}_+$ and f be a Y-function. The function f is called *Y-competitive* relative to the cost c if $f(x) + f^{-1}(y) \leqslant c(x,y)$ for all $(x,y) \in [0,a] \times [0,b]$. Furthermore, we take the cost c such that the set of Y-competitive functions is not empty.

If we put

$$\mathscr{X} = A = [0,a], \quad \mathscr{Y} = B = [0,b], \quad \pi \in \Pi(\lambda/a, \lambda/b)$$

we obtain from (1.1)

$$\int_{[0,a]\times[0,b]} \left(f(x) + f^{-1}(y)\right) \mathrm{d}\pi(x,y) = \frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b f^{-1}(y)\,\mathrm{d}y.$$

We shall replace the usual dual Kantorovich optimization problem over the set of competitive pairs of functions, by the optimization over the smaller set which consists of Y-competitive functions.

DEFINITION 2. The dual Young-Kantorovich optimization problem (Young type constraints added to Kantorovich problem): Find a maximizer for

$$\sup_{f \text{ Y-competitive}} \left\{ \frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b f^{-1}(y)\,\mathrm{d}y \right\}.$$

We will have

$$\sup_{f \text{ Y-competitive}} \left\{ \frac{1}{a} \int_0^a f(x)\,\mathrm{d}x + \frac{1}{b} \int_0^b f^{-1}(y)\,\mathrm{d}y \right\} \leqslant \inf_{\pi \in \Pi(\lambda/a, \lambda/b)} \int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y).$$

LEMMA 2. *Let f be a Y-function. It holds*

$$\frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b f^{-1}(y)\,\mathrm{d}y \geqslant \left(\frac{1}{a} - \frac{1}{b}\right)\int_0^a f(x)\,\mathrm{d}x + a,$$

for all $a, b > 0$. Equality holds iff $b = f(a)$.

Proof. It follows directly from Young's inequality. □

PROPOSITION 2. *For all Y-competitive functions f it holds*

$$\sup_{f \text{ Y-competitive}} \left\{ \left(\frac{1}{a} - \frac{1}{b}\right)\int_0^a f(x)\,\mathrm{d}x + a \right\} \leqslant \inf_{\pi \in \Pi(\lambda/a, \lambda/b)} \int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y).$$

Proof. We use Young's inequality and Lemma 2. □

A nice counterpart of Lemma 2 with reversed inequality sign reads as follows:

THEOREM 1. *Let f be a Y-function. The following inequality holds:*

$$\frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b f^{-1}(y)\,\mathrm{d}y \leqslant \left(\frac{1}{a} - \frac{1}{f(a)}\right)\int_0^a f(x)\,\mathrm{d}x + a,$$

for all $b \leqslant f(a)$. Equality holds iff $b = f(a)$.

Proof. We apply Lemma 1 and get

$$\frac{b}{f(a)}\int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y \leqslant ab.$$

Hence

$$\begin{aligned}
&\frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b f^{-1}(y)\,\mathrm{d}y \\
&\qquad \leqslant \frac{1}{a}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{b}\left(ab - \frac{b}{f(a)}\int_0^a f(x)\,\mathrm{d}x\right) \\
&\qquad = \left(\frac{1}{a} - \frac{1}{f(a)}\right)\int_0^a f(x)\,\mathrm{d}x + a.
\end{aligned}$$

This proves the claim. □

REMARK 3. Theorem 1 becomes useful when we try to solve the following Monge-Kantorovich problem:

$$\sup_{\pi\in\Pi(\lambda/a,\lambda/b)}\int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y)$$

(this means that the function c represents the profit). Then it holds

$$\sup_{\pi\in\Pi(\lambda/a,\lambda/b)}\int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y)\leqslant\left(\frac{1}{a}-\frac{1}{f(a)}\right)\int_0^a f(x)\,\mathrm{d}x+a,$$

for all Y-functions f which satisfy $f(x)+f^{-1}(y)\geqslant c(x,y)$.

4. Dealing with the profit concept

Let $K:[0,\infty)\times[0,\infty)\to[0,\infty)$ be a Lebesgue locally integrable function. Let $c:[0,a]\times[0,b]\to\mathbb{R}_+$,

$$c(x,y)=\int_0^x\int_0^y K(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

In order to maintain the notation we continue to call it *cost* throughout this section, but its true economical meaning here is of a *profit* (gain, income) function. This is also the reason why we keep the above names of the optimization problems, even if due to the change of the meaning we switch the discussion replacing sup by inf and vice versa.

We will need the following result.

PROPOSITION 3. ([4]) *Let f be a Y-function. Then for every $x,y>0$ we have*

$$\int_0^x\int_0^y K(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant\int_0^x\left(\int_0^{f(s)}K(s,t)\,\mathrm{d}t\right)\mathrm{d}s+\int_0^y\left(\int_0^{f^{-1}(t)}K(s,t)\,\mathrm{d}s\right)\mathrm{d}t.$$

If in addition K is strictly positive almost everywhere, then the equality occurs if and only if $y=f(x)$.

Let the potentials of f and f^{-1} be

$$F(x)=\int_0^x\left(\int_0^{f(s)}K(s,t)\,\mathrm{d}t\right)\mathrm{d}s,\quad x\geqslant 0 \tag{4.1}$$

and

$$G(y)=\int_0^y\left(\int_0^{f^{-1}(t)}K(s,t)\,\mathrm{d}s\right)\mathrm{d}t,\quad y\geqslant 0. \tag{4.2}$$

The functions F and G verify the relations $F^c=G$ and $G^c=F$ (due to the equality case as specified in the statement of Proposition 3), so they are both c-convex. Here

$$G^c(x)=\sup_{y\in[0,b]}\{c(x,y)-G(y)\},$$
$$F^c(y)=\sup_{x\in[0,a]}\{c(x,y)-F(x)\}.$$

The inverse of a map with a c-convex potential has also a c-convex potential, and the two potentials are c-transforms of each other. This agrees for $K \equiv 1$ with the usual convexity and then we are dealing with Fenchel transforms.

DEFINITION 3. The dual *-Young Kantorovich (dual *-YK) optimization problem states: Find a minimizer for

$$\inf_{f \text{ Y-function}} I[f].$$

Here

$$I[f] = \frac{1}{a}\int_0^a F(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b G(y)\,\mathrm{d}y.$$

Taking into account (1.4), we note that

$$\sup_{\pi \in \Pi(\lambda/a, \lambda/b)} \int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y) \leqslant \inf_{f \text{ Y-function}} I[f]. \tag{4.3}$$

One can check that (4.3) can also be obtained from Proposition 3.

Let $\pi \in \Pi(\lambda/a, \lambda/b)$ be the solution of Monge-Kantorovich problem. Let f be a minimizer (if it exists over the constraint set of Y-functions) for the dual *-YK problem. If (4.3) holds with equality sign (in other words if the dual *-Young Kantorovich problem has no duality gap) then f satisfies

$$\int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y) = I[f] = \int_{[0,a]\times[0,b]} (F(x) + G(y))\,\mathrm{d}\pi(x,y),$$

whence

$$c(x,y) = F(x) + G(y), \qquad \pi - \text{a.e.}. \tag{4.4}$$

From Proposition 3 we see that (4.4) holds if and only if $y = f(x)$, hence

$$c(x, f(x)) = F(x) + G(f(x)), \tag{4.5}$$

that is, we arrived at the following optimality criterion.

THEOREM 2. *Let K be such that the dual *-YK optimization problem has the Young function f as solution without duality gap. Then*

i) the pair (F,G) given by (4.1) and (4.2) is a minimizer of the dual Kantorovich problem

$$\inf_{u(x)+v(y)\geqslant c(x,y)} \left\{ \frac{1}{a}\int_0^a u(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b v(y)\,\mathrm{d}y \right\}; \tag{4.6}$$

ii) the function f is a solution of the Monge problem

$$\sup_{T\#\lambda/a = \lambda/b} \int_0^a c(x, T(x))\,\mathrm{d}x; \tag{4.7}$$

iii) the coupling $(Id, f)\#\lambda/a$ is a maximizer of the Monge-Kantorovich problem

$$\sup_{\pi \in \Pi(\lambda/a, \lambda/b)} \int_{[0,a]\times[0,b]} c(x,y)\,\mathrm{d}\pi(x,y). \tag{4.8}$$

Proof. The items i) and ii) are straightforward according to the discussion above. The item iii) follows from (1.2) and ii). □

REMARK 4. Accordingly to (4.4) and (4.5), the support of the optimal transport plan π is concentrated on the graph of f. This is in fact the meaning of Theorem 2, iii). Hence solving the dual *-YK problem (such that the gap duality vanishes) reduces to the problem of finding a transport plan $\pi \in \Pi(\lambda/a, \lambda/b)$ concentrated on the graph of a Y-function.

REMARK 5. We stress that problems like (4.6), (4.7), (4.8) appear not only for optimizations which involve profit functions (see the optimal pricing policies discussed in [2]), but also during the investigation of some particular cost functions. See for example [7].

Quadratic costs are of interest in many practical applications, e.g. in cosmology, where some of the methods of reconstructing the early Universe correspond to a quadratic cost (see the Monge-Ampère-Kantorovich method in [1]). It is also known that for quadratic costs the optimal map for the Monge problem exists and is unique. Taking into account the previous theorem we can also infer that in this case the dual *-YK optimization problem has at most one solution.

EXAMPLE 3. In the particular case $K(s,t) \equiv 1$, that is, $c(x,y) = xy$ (quadratic), the dual *-YK problem admits as unique minimizer the Y-function $f : [0,a] \to [0,b]$, $f(x) = \frac{b}{a}x$. Indeed, then

$$F(x) = \frac{bx^2}{2a} \text{ and } G(y) = \frac{ay^2}{2b}.$$

Obviously it satisfies $f\#\lambda/a = \lambda/b$. For $\pi = (Id, f)\#\lambda/a$, concentrated on the graph of f, it holds

$$\int_{[0,a]\times[0,b]} xy\mathrm{d}\pi(x,y) = \int_0^a xf(x)\frac{\mathrm{d}x}{a} = \frac{ab}{3}.$$

A straightforward computation gives

$$\frac{1}{a}\int_0^a F(x)\,\mathrm{d}x + \frac{1}{b}\int_0^b G(y)\,\mathrm{d}y = \frac{ab}{3},$$

hence f solves the dual *-YK optimization problem.

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(Received June 12, 2015)

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