Mathematical
Inequalities
& Applications
Volume 19, Number 1 (2016), 381–384 doi:10.7153/mia-19-29

UNIMODALITY OF CERTAIN PARAMETRIC INTEGRALS

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(*Communicated by I. Franjić*)

Abstract. The unimodality of the functions represented as certain parametric integrals is considered. A crucial part of the proof is based on Kemperman's necessary and sufficient condition for the unimodality of all mixtures of a given family of functions.

Let p be any real number. Let $\mathscr{F}$ stand for the set of all concave (strictly) increasing continuous functions $f\colon [0,1] \to \mathbb{R}$ with $f(0) = 0$. Let us say that a function $F\colon (0,\infty) \to \mathbb{R}$ is (strictly) unimodal if F is increasing on $(0, y_F]$ and decreasing on $[y_F, \infty)$, for some $y_F \in (0,\infty)$.

Now we are prepared to state the result presented in this note:

THEOREM 1. *The following two conditions are equivalent to each other:*

(I) *for all $f \in \mathscr{F}$ the function $F_{p,f}\colon (0,\infty) \to \mathbb{R}$ defined by the formula*

$$F_{p,f}(y) := \int_0^1 \frac{y^p \,\mathrm{d}x}{(y + f(x))^2}$$

is unimodal;

(II) $p \in [1,2)$.

One may consider $F_{p,f}$ a mixture of functions of the form

$$(0,\infty) \ni y \mapsto \frac{y^p}{(y+a)^2} \quad \text{with} \quad a \in (0,\infty), \tag{1}$$

which are unimodal if $p \in (0,2)$. Of course, not all mixtures of unimodal functions are unimodal. Kemperman [3] provided necessary and sufficient conditions for the unimodality of *all* mixtures of a given family of functions. However, because of the condition $f \in \mathscr{F}$ imposed on the functions f in Theorem 1, $F_{p,f}$ is not an arbitrary mixture of the functions of the form (1). So, some work will be needed in order to make adequate use of Kemperman's result. Namely, it will be shown that without loss of generality (w.l.o.g.) $F_{p,f}$ may be taken to be an arbitrary mixture of functions of the form (3).

Mathematics subject classification (2010): Primary 26A48, 26D15; Secondary 26A51, 47G10.
Keywords and phrases: Unimodality, mixtures, parametric integrals, convex functions, concave functions, monotonicity.

Proof of Theorem 1. In this proof, it is assumed that $f \in \mathscr{F}$ and $y \in (0,\infty)$.

Clearly, the function $F_{p,f}$ is monotonically decreasing if $p \in (-\infty,0]$ and monotonically increasing if $p \in [2,\infty)$. So, it remains to consider

$$p \in (0,2). \tag{2}$$

Next, to prove that $F_{p,f}$ is unimodal, it is enough to show that $F_{p,f}$ is non-strictly unimodal in the sense that for some $y_{p,f} \in (0,\infty)$ the function $F_{p,f}$ is non-decreasing on $(0,y_{p,f}]$ and non-increasing on $[y_{p,f},\infty)$. Indeed, $F_{p,f}$ is clearly real-analytic on $(0,\infty)$ and hence can be constant on an nonempty open subinterval of $(0,\infty)$ only if $F_{p,f}$ is constant on $(0,\infty)$. But $F_{p,f}(0+) = 0$, whereas $F_{p,f} > 0$ on $(0,\infty)$. Thus, to prove that $F_{p,f}$ is unimodal, it is enough to show that it is the pointwise limit of a sequence of non-strictly unimodal functions on $(0,\infty)$.

Consider the inverse function $g := f^{-1}\colon [0,b] \to [0,1]$, where

$$b := f(1) \in (0,\infty).$$

Then g is an increasing convex function with $g(0) = 0$. Let g' be the left derivative of g, with $g'(0) := 0$. Then $g'\colon [0,b] \to \mathbb{R}$ is a non-decreasing function, which is left-continuous on $(0,b]$. So, for all $t \in [0,b]$

$$g(t) = \int_0^t g'(u)\,\mathrm{d}u = \int_0^t \mathrm{d}u \int_{[0,u)} \mathrm{d}g'(a) = \int_{[0,t)} \mathrm{d}g'(a) \int_a^t \mathrm{d}u = \int_{[0,b)} \mathrm{d}g'(a)\,(t-a)_+,$$

by Fubini's theorem, where $x_+ := 0 \vee x$. Hence, by the change of variables $t = f(x) \iff x = g(t)$ and again Fubini's theorem,

$$\begin{aligned} y^{-p}F_{p,f}(y) &= \int_{[0,b]} \frac{\mathrm{d}g(t)}{(y+t)^2} = \int_{[0,b)} \mathrm{d}g'(a) \int_{[0,b]} \frac{\mathrm{d}_t(t-a)_+}{(y+t)^2} = \int_{[0,b)} \mathrm{d}g'(a) \int_a^b \frac{\mathrm{d}t}{(y+t)^2} \\ &= \int_{[0,b)} \mathrm{d}g'(a)\, \frac{b-a}{(y+b)(y+a)} \end{aligned}$$

for $y \in (0,\infty)$. By rescaling $z := y/b$, w.l.o.g. $b = 1$.

So, for each real p, the unimodality of $F_{p,f}$ for all the functions $f \in \mathscr{F}$ is equivalent to the unimodality of all (say discrete) mixtures (that is, all linear combinations with nonnegative coefficients) of the functions of the form

$$(0,\infty) \ni y \mapsto g_{p,a}(y) := \frac{y^p}{(y+1)(y+a)} \quad \text{with } a \in (0,1]. \tag{3}$$

Accordingly, assume in the sequel that $(p,a,y) \in (0,2) \times (0,1] \times (0,\infty)$. Note that

$$g_{1;p,a}(y) := g'_{p,a}(y)(1+y)^2(a+y)^2y^{1-p} = (a+y)(py+p-y) - y(y+1) \tag{4}$$

equals $g'_{p,a}(y)$ in sign. Moreover, $g'_{1;p,a}(y) = 2(p-2)y + (p-1)(1+a)$ decreases in $y \in (0,\infty)$ from $(p-1)(1+a)$ to $-\infty$. So, $g_{1;p,a}$ decreases on $(0,\infty)$ if $p \in (0,1]$, and $g_{1;p,a}$ switches from increase on $(0,y_{1;p,a}]$ to decrease on $[y_{1;p,a},\infty)$ if

$p \in (1,2)$, where $y_{1;p,a} := (p-1)(1+a)/(4-2p)$. At that, $g_{1;p,a}(0+) = ap > 0$ and $g_{1;p,a}(\infty-) = -\infty < 0$. It follows that, for each pair $(p,a) \in (0,2) \times (0,1]$, there is a unique (mode) $m_{p,a} \in (0,\infty)$ of the function $g_{p,a}$ such that

$$g'_{p,a}(y) \begin{cases} > 0 & \text{if } y \in (0, m_{p,a}), \\ = 0 & \text{if } y = m_{p,a}, \\ < 0 & \text{if } y \in (m_{p,a}, \infty). \end{cases} \tag{5}$$

Further, if $y = m_{p,a}$, then $g_{1;p,a}(y) = 0$. This and (4) imply $\frac{\partial}{\partial a} g_{1;p,a}(y) = py + p - y = \frac{y(y+1)}{a+y} > 0$ at $y = m_{p,a}$. So, $g_{1;p,\tilde{a}}(m_{p,a}) > 0$ and hence $g'_{p,\tilde{a}}(m_{p,a}) > 0$ and hence, by (5), $m_{p,a} \in (0, m_{p,\tilde{a}})$ for all $a \in (0,1)$ and all $\tilde{a}$ in a right neighborhood of a. Similarly, $m_{p,a} \in (m_{p,\tilde{a}}, \infty)$ for all $a \in (0,1]$ and all $\tilde{a}$ in a left neighborhood of a. Thus, the mode $m_{p,a}$ is increasing in $a \in (0,1]$.

Take now any a_1 and a_2 such that $0 \leqslant a_1 < a_2 \leqslant 1$. Then $m_{p,a_1} < m_{p,a_2}$, by the monotonicity of $m_{p,a}$ in a. Moreover, by (5) and (4), the condition

$$m_{p,a_1} < y < m_{p,a_2} \tag{6}$$

is equivalent to $g_{1;p,a_1}(y) < 0 < g_{1;p,a_2}(y)$, which (together with the condition $0 \leqslant a_1 < a_2 \leqslant 1$) in turn reduces to

$$y \in (0,\infty) \quad \& \quad \frac{2y}{1+y} < p < \frac{1+2y}{1+y} \quad \& \quad 0 < a_1 < a_{p,y} \quad \& \quad a_{p,y} < a_2 \leqslant 1, \tag{7}$$

where

$$a_{p,y} := \frac{y - py + 2y^2 - py^2}{p - y + py} \in (0,1), \tag{8}$$

and at that $p - y + py > 0$.

Let now $\mathrm{D}g_{p,a}(y) := g'_{p,a}(y)(1+y)^2 y^{1-p}$ and $\mathrm{DD}g_{p,a}(y) := g''_{p,a}(y)(1+y)^3 y^{2-p}$. Then, by Remark 1 in [3], all mixtures (in a) of the functions $g_{p,a}$ are unimodal iff

$$\begin{aligned} d_{p,a_1,a_2}(y) :=& \begin{vmatrix} \mathrm{D}g_{p,a_1}(y) & \mathrm{DD}g_{p,a_1}(y) \\ \mathrm{D}g_{p,a_2}(y) & \mathrm{DD}g_{p,a_2}(y) \end{vmatrix} \frac{(a_1+y)^3(a_2+y)^3}{y(1+y)^2(a_2-a_1)} \\ =&(a_1+a_2)(p-1)y(py+p-2y) + a_1 a_2 p(py+p-y+1) \\ &+ (p-2)y^2(py+p-3y-1) \geqslant 0 \end{aligned} \tag{9}$$

whenever (6) or, equivalently, (7) holds.

Now, in view of (2), the implication (I) $\Longrightarrow$ (II) of Theorem 1 follows, because in the case $p \in (0,1)$ one has $\lim_{y\downarrow 0} \lim_{a\downarrow 0} \frac{1}{y} d_{p,a,1}(y) = (p-1)p < 0$, and for $p \in (0,1)$ the limit transition $\lim_{y\downarrow 0} \lim_{a\downarrow 0}$ is allowed by the condition (7) with $(a_1,a_2) = (a,1)$, because this condition can be rewritten for $p \in (0,1)$ as $0 < y < \frac{p}{2-p}$ & $0 < a < a_{p,y}$.

To complete the proof of the implication (II) $\Longrightarrow$ (I) of Theorem 1, assume indeed that $p \in [1,2)$ and note that $d_{p,a_1,a_2}(y)$ is affine in a_1 and in a_2. So, in view of (7), it suffices to check the inequality in (9) just for the pairs

$$(a_1,a_2)\in\big\{(0,a_{p,y}),(0,1),(a_{p,y},a_{p,y}),(a_{p,y},1)\big\},\quad \text{assuming } \frac{2y}{1+y}<p<\frac{1+2y}{1+y}.$$

But

$$\begin{aligned} d_{p,0,a_{p,y}}(y)\,\frac{p-y+py}{y^2} &= d_{p,a_{p,y},a_{p,y}}(y)\,\frac{(p-y+py)^2}{y^2(1+y)} = d_{p,a_{p,y},1}(y)\,\frac{p-y+py}{y(1+y)} \\ &= d_{*;y}(p) := -p^2(1+y)^2+p(1+y)(1+3y)-2y^2 \end{aligned}$$

is concave in p, with $d_{*;y}(\frac{2y}{1+y})=2y>0$ and $d_{*;y}(\frac{1+2y}{1+y})=y>0$. So, $d_{*;y}(p)>0$ and hence $d_{p,a_1,a_2}(y)>0$ for $(a_1,a_2)\in\big\{(0,a_{p,y}),(a_{p,y},a_{p,y}),(a_{p,y},1)\big\}$. Next,

$$d_{p,*}(y) := \tfrac{1}{y}\,d_{p,0,1}(y) = (3-p)(2-p)y^2-2(2-p)(p-1)y+(p-1)p$$

is convex in y, with $d_{p,*}(\frac{p-1}{2-p})=\frac{p-1}{2-p}$ and $d'_{p,*}(\frac{p-1}{2-p})=2(p-1)$, so that $d_{p,*}(y)>0$ and hence $d_{p,0,1}(y)>0$ if $\frac{p-1}{2-p}<y<\frac{p}{2-p}$, which latter condition is equivalent $\big($for $y\in(0,\infty)$ and $p\in(0,2)\big)$ to the condition $\frac{2y}{1+y}<p<\frac{1+2y}{1+y}$ in (7). Thus, the implication (II) $\Longrightarrow$ (I) of Theorem 1 is established as well. □

Theorem 1 is an answer to an extension of the question posted at [1]. A general treatment of unimodality and convexity can be found in [2].

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(Received June 22, 2015)

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