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INEQUALITIES FOR AVERAGES OF QUASICONVEX AND SUPERQUADRATIC FUNCTIONS

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Abstract. For $n \in \mathbb{Z}_+$ we consider the difference

$$B_{n-1}(f) - B_n(f) := \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right)$$

where the sequences $\{a_i\}$ and $\{a_i - a_{i-1}\}$ are increasing. Some lower bounds are derived when f is 1-quasiconvex and when f is a closely related superquadratic function. In particular, by using some fairly new results concerning the so called "Jensen gap", these bounds can be compared. Some applications and related results about

$$A_{n+1}(f) - A_n(f) := \frac{1}{a_n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}}\sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right)$$

are also included.

1. Introduction

In this paper we illustrate and combine results for averages obtained in [1] related to superquadratic functions with results on γ-quasiconvexity introduced and discussed in [2], [3], [4] and [5]. By this we get refinements of bounds of differences of averages when the function involved is 1-quasiconvex as well as superquadratic.

1-quasiconvex and superquadratic functions are closely related and therefore it is of interest to compare their bounds. By using recent results we have succeeded also to get such comparisons in this paper.

We start with quoting two definitions and five lemmas that are the basic properties and definitions of superquadratic functions and 1-quasiconvex functions and which we use in the sequel:

DEFINITION 1. [1] A function $\varphi : [0,b) \to \mathbb{R}$ is superquadratic provided that for all $0 \leqslant x < b$ there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) \geqslant C_\varphi(x)(y-x) + \varphi(|y-x|)$$

for every y, $0 \leqslant y < b$.

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DEFINITION 2. [5] A function $f:[0,b)\to\mathbb{R}$ that satisfies $f=x\varphi$, where φ is a convex function is called 1-quasiconvex function.

LEMMA 1. [1] *Let* x_i, $0\leqslant x_i\leqslant b$, $0<b\leqslant\infty$, $0\leqslant\alpha_i\leqslant 1$, $i=1,\dots,m$, $\sum_{i=1}^{m}\alpha_i=1$, $\bar{x}=\sum_{i=1}^{m}\alpha_i x_i$, *and let* $f:[0,b)\to\mathbb{R}$, $0<b\leqslant\infty$ *be a superquadratic function. Then*

$$\sum_{i=1}^{m}\alpha_i f(x_i)-f(\bar{x})\geqslant\sum_{i=1}^{m}\alpha_i f(|x_i-\bar{x}|) \tag{1.1}$$

holds.

Moreover, when f *is nonnegative, (1.1) is a refinement of Jensen's inequality* $\sum_{i=1}^{m}\alpha_i f(x_i)\geqslant f(\bar{x})$.

In particular, for $m=2$ *we get from inequality (1.1) that when the superquadratic function* f *satisfies* $f=x\varphi$ *and* $0\leqslant\alpha\leqslant 1$

$$\begin{aligned} &\alpha f(x_1)+(1-\alpha)f(x_2)-f(\alpha x_1+(1-\alpha)x_2) \\ \geqslant\ &\alpha f((1-\alpha)|x_1-x_2|)+(1-\alpha)f(\alpha|x_1-x_2|) \\ =\ &\alpha(1-\alpha)(|x_1-x_2|)(\varphi(\alpha|x_1-x_2|)+\varphi((1-\alpha)|x_1-x_2|)) \end{aligned} \tag{1.2}$$

holds.

LEMMA 2. [5] *Let* x_i, $0\leqslant x_i\leqslant b$, $0<b\leqslant\infty$, $0\leqslant\alpha_i\leqslant 1$, $1=1,\dots,m$, $\sum_{i=1}^{m}\alpha_i=1$, $\bar{x}=\sum_{i=1}^{m}\alpha_i x_i$, *and let* $\varphi:[0,b)\to\mathbb{R}$, $0<b\leqslant\infty$ *be a differentiable convex function, and* f *be 1-quasiconvex, where* $f=x\varphi$. *Then*

$$\sum_{i=1}^{m}\alpha_i f(x_i)-f(\bar{x})\geqslant\sum_{i=1}^{m}\varphi'(\bar{x})\alpha_i(x_i-\bar{x})^2, \tag{1.3}$$

holds. Moreover, when φ *is increasing (1.3) is a refinement of Jenen's inequality.*

In particular, for $m=2$ *when* $0\leqslant\alpha\leqslant 1$ *we get that*

$$\begin{aligned} &\alpha f(x_1)+(1-\alpha)f(x_2)-f(\alpha x_1+(1-\alpha)x_2) \\ \geqslant\ &\varphi'(\alpha x_1+(1-\alpha)x_2)\alpha(1-\alpha)(x_1-x_2)^2. \end{aligned} \tag{1.4}$$

LEMMA 3. [4] *Let* $\varphi:[0,b)\to\mathbb{R}$ *be a differentiable convex function satisfying*

$$\varphi(0)=\lim_{x\to 0^+}x\varphi'(x)=0.$$

Then $x\varphi'(x)\geqslant\varphi(x)$, *when* $x\in[0,b)$.

LEMMA 4. [5] *Let* $\varphi:[0,b)\to\mathbb{R}_+$, $0<b\leqslant\infty$, *be a differentiable convex increasing function satisfying*

$$\varphi(0)=\lim_{x\to 0^+}x\varphi'(x)=0.$$

Then the 1-quasiconvex function f, *where* $f=x\varphi$ *is also superquadratic and convex.*

For a function f where $f = x\varphi$ which is simultaneously 1-quasiconvex and superquadratic we obtain in the following lemma a comparison between lower bounds of the so called "Jensen's gap" in (1.1) and (1.3), (see [5, Proposition 4]).

LEMMA 5. *Let $\varphi : [0,b) \to \mathbb{R}_+$, $0 < b \leqslant \infty$, be a differentiable increasing convex function, $\varphi(0) = \lim_{x\to 0^+} x\varphi'(x) = 0$ and let the function f satisfy $f = x\varphi$. Then the lower bound of "Jensen's gap" obtained by the superquadracity of $f = x\varphi$ is weaker than the lower bound of "Jensen's gap" obtained by the 1-quasiconvexity of f, when $0 \leqslant x_i \leqslant 2\overline{x}$, $0 < b \leqslant \infty$, $0 \leqslant \alpha_i \leqslant 1$, $1 = 1,\dots,m$, $\sum_{i=1}^m \alpha_i = 1$, $\overline{x} = \sum_{i=1}^m \alpha_i x_i$, that is, the inequalities*

$$\begin{aligned}
&\sum_{i=1}^m \alpha_i f(x_i) - f(\overline{x}) \\
&\geqslant \sum_{i=1}^m \varphi'(\overline{x})\alpha_i |x_i - \overline{x}|^2 \geqslant \sum_{i=1}^m \alpha_i f(|x_i - \overline{x}|) = \sum_{i=1}^m \alpha_i |x_i - \overline{x}|\,\varphi(|x_i - \overline{x}|) \geqslant 0
\end{aligned}$$

hold. In particular, when $m = 2$ and $0 \leqslant \alpha \leqslant 1$ then

$$\begin{aligned}
&\alpha f(x_1) + (1-\alpha) f(x_2) \\
&\geqslant \varphi'(\alpha x_1 + (1-\alpha)x_2)\,\alpha(1-\alpha)(x_1 - x_2)^2 \\
&\geqslant \alpha(1-\alpha)(|x_1 - x_2|)\,(\varphi(\alpha|x_1 - x_2|) + \varphi((1-\alpha)|x_1 - x_2|)) \geqslant 0,
\end{aligned} \tag{1.5}$$

and in this case a sufficient condition for $0 \leqslant x_i \leqslant 2\overline{x}$, $i = 1,2$ to hold is that $x_2 \leqslant 2x_1 < b$ when $0 < x_1 < x_2$.

One basic idea for the investigarions in this paper is to show the fact that for functions that are simultaneously superquadratic and 1-quasiconvex, inequality (1.5) can be used for comparing bounds of the difference

$$B_{n-1}(f) - B_n(f) := \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right), \tag{1.6}$$

and also the bounds of the difference

$$A_{n+1}(f) - A_n(f) := \frac{1}{a_n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}}\sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right). \tag{1.7}$$

These differences were previously investigated in a more general setting (see [1]) and the following results that we use in the sequel were obtained there:

LEMMA 6. [1, Theorem 5.1] *Let $\{a_i\}$ be a sequence such that $a_i > 0$ is increasing and $a_i - a_{i-1}$, $i = 1,\dots$, is decreasing and let $a_0 = 0$. Let $f : [0,b) \to \mathbb{R}_+$ be an increasing function. Let also the interval $[0,b)$ include all the $\frac{a_i}{a_n}$ for $i \leqslant n$. Then, when $n \geqslant 2$ the difference $A_{n+1}(f) - A_n(f)$ satisfies*

$$\begin{aligned}
&A_{n+1}(f) - A_n(f) \\
&\geqslant \frac{1}{a_{n-1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{a_{i+1}}{a_{n+1}}\right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_{n+1}}\right) - f\left(\frac{a_i}{a_n}\right)\right).
\end{aligned} \tag{1.8}$$

LEMMA 7. [1, Theorem 5.4] *Let* $\{a_i\}$ *be a sequence where* $a_i > 0,\ a_i - a_{i-1},\ i = 1,\ldots,$ *are increasing and* $a_0 = 0$. *Let* $f : [0,b) \to \mathbb{R}_+$ *be an increasaing function. Let also the interval* $[0,b)$ *include all the* $\frac{a_i}{a_{n-1}}$ *for* $i \leqslant n$. *Then, when* $n \geqslant 2$ *the difference* $B_{n-1}(f) - B_n(f)$ *satisfies*

$$B_{n-1}(f) - B_n(f) \geqslant \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_{i-1}}{a_{n-1}}\right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_{n-1}}\right) - f\left(\frac{a_i}{a_n}\right) \right). \tag{1.9}$$

The next example is one of the main motivations for us to introduce this research:

EXAMPLE 1. In the introduction of [1] and [7] it was noted that if f is convex, then

$$\frac{1}{n} \sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right) \geqslant \frac{1}{n+1} \sum_{i=0}^{n} f\left(\frac{i}{n}\right). \tag{1.10}$$

In particular, if $f(x) = x^p,\ x \geqslant 0,\ p \geqslant 1$, then this inequality can be rewritten as

$$\left(\frac{(n+1)\sum_{i=1}^{n-1} i^p}{n \sum_{i=1}^{n} i^p} \right)^{\frac{1}{p}} \geqslant \frac{n-1}{n}, \quad n \geqslant 2. \tag{1.11}$$

Such types of inequalities are discussed in several publications (see for instance [6] - [30]).

REMARK 1. As applications of our results in this paper we obtain a strictly better inequality then (1.11) for each $p \geqslant 2$, thus improving and complementing several results mentioned above (see Examples 5 and 6).

As further applications we point out a strict improvement of (1.10) and similar inequalities for each $p \geqslant 2$ (see Examples 2, 3 and 4).

Functions which are superquadratic and 1-quasiconvex are related in certain senses. For example the function $f(x) = x^p,\ x \geqslant 0,\ p \geqslant 2$ is both 1-quasiconvex and superquadratic, whereas $f(x) = x^p,\ x \geqslant 0,\ 1 \leqslant p \leqslant 2$ is neither 1-quasiconvex nor superquadratic see e.g., Lemma 4, but note that f can be of course 1-quasiconvex without the condition $f(0) = 0$.

Hence it is natural to try to derive lower bounds for both differences (1.6) and (1.7) and to use our previous mentioned results (especially Lemma 5) to compare these bounds. This is exactly what we have succeeded to do in this paper.

The main results are stated and proved in Section 2 and the motivating examples which, in particular, give refinements of a number of inequalities (related to convexity) are given in Section 3.

2. Main results

Our first main result reads:

THEOREM 1. *Let $\varphi : [0,b) \to \mathbb{R}_+,\ 0<b\leqslant\infty$ be differentiable convex increasing function, and let $f=x\varphi$. Let the sequence $\{a_i\}$ be such that $a_0=0,\ a_i,\ a_{i+1}-a_i,$ $i=1,\ldots$ are increasing. Then, for $n\geqslant 2$ we get for the 1-quasiconvex function f that*

$$\begin{aligned}&B_{n-1}(f)-B_n(f)\\ \geqslant&\sum_{i=1}^{n-1}\frac{(a_i-a_{i-1})^2a_i(a_n-a_i)}{a_{n+1}a_n^2a_{n-1}^2}\varphi'\left(\frac{a_i(a_n+a_{i-1}-a_i)}{a_na_{n-1}}\right)\\ \geqslant&\sum_{i=1}^{n-1}\frac{(a_i-a_{i-1})^2a_i(a_n-a_i)}{a_{n+1}a_n^2a_{n-1}^2}\varphi'\left(\frac{a_i}{a_n}\right)\geqslant 0.\end{aligned}\tag{2.1}$$

If in addition φ' is convex, then

$$\begin{aligned}&B_{n-1}(f)-B_n(f)\\ \geqslant&\sum_{i=1}^{n-1}\frac{(a_i-a_{i-1})^2a_i(a_n-a_i)}{a_{n+1}a_n^2a_{n-1}^2}\varphi'\left(\frac{\sum_{i=1}^{n-1}a_i^2(a_n-a_i)(a_i-a_{i-1})^2}{a_n\sum_{i=1}^{n-1}a_i(a_n-a_i)(a_i-a_{i-1})^2}\right)\geqslant 0.\end{aligned}\tag{2.2}$$

Proof. By denoting

$$x_{i,1}=\frac{a_{i-1}}{a_{n-1}},\quad x_{i,2}=\frac{a_i}{a_{n-1}},\quad \alpha_{i,1}=\frac{a_i}{a_n},\quad \alpha_{i,2}=\frac{a_n-a_i}{a_n},\tag{2.3}$$

$$\alpha_{i,1}\,x_{i,1}+\ \alpha_{i,2}x_{i,2}=\overline{x_i}=\frac{a_i}{a_n}\left(\frac{a_n+a_{i-1}-a_i}{a_{n-1}}\right),\quad i=1,\ldots,n-1,$$

and by using that $f=x\varphi$ we can derive that

$$\begin{aligned}&B_{n-1}(f)-B_n(f)\\ \geqslant&\frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n}f\left(\frac{a_{i-1}}{a_{n-1}}\right)+\frac{a_n-a_i}{a_n}f\left(\frac{a_i}{a_{n-1}}\right)-f\left(\frac{a_i}{a_n}\right)\right)\\ \geqslant&\frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(f\left(\frac{a_i}{a_n}\left(\frac{a_n+a_{i-1}-a_i}{a_{n-1}}\right)\right)-f\left(\frac{a_i}{a_n}\right)\right)\\ &+\frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\varphi'\left(\frac{a_i}{a_n}\left(\frac{a_n+a_{i-1}-a_i}{a_{n-1}}\right)\right)\frac{a_i(a_n-a_i)(a_i-a_{i-1})^2}{a_n^2a_{n-1}^2}\\ \geqslant&\frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\varphi'\left(\frac{a_i}{a_n}\right)\frac{a_i(a_n-a_i)(a_i-a_{i-1})^2}{a_n^2a_{n-1}^2}\geqslant 0.\end{aligned}\tag{2.4}$$

Indeed, the first inequality in (2.4), holds according to (1.9) in Lemma 7, the second inequality in (2.4) follows from (1.4) in Lemma 2 and the last inequality in (2.4) holds because f and φ' are increasing and $\frac{(a_n+a_{i-1}-a_i)}{a_{n-1}} \geqslant 1$, $i=1,\dots,n-1$, and therefore (2.1) is satisfied

If φ' is also convex we get from (2.4) by using Jensen's inequality that (2.2) holds. The proof is complete. $\square$

Next, we state a similar result but with an additional condition guaranteeing that $f(x) = x\varphi(x)$ is superquadratic (see Lemma 4).

THEOREM 2. *Let $\varphi : [0,b) \to \mathbb{R}_+$, $0 < b \leqslant \infty$ be a differentiable convex increasing function and $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi(x)$, and let $f = x\varphi$. Let the sequence $\{a_i\}$ be such that $a_0 = 0$, $a_i > 0$, $a_{i+1} - a_i$, $i = 1,\dots,$ are increasing. Then, for $n \geqslant 2$ we get that*

$$\begin{aligned} &B_{n-1}(f) - B_n(f) \qquad (2.5)\\ &\geqslant \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_n - a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) \right) \\ &= \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1}) a_i (a_n - a_i)}{a_{n+1} a_n^2 a_{n-1}} \left(\varphi\left(\frac{a_i (a_i - a_{i-1})}{a_{n-1} a_n} \right) + \varphi\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_{n-1} a_n} \right) \right) \\ &\geqslant \sum_{i=1}^{n-1} \frac{2a_i (a_n - a_i)(a_i - a_{i-1})}{a_{n+1} a_n^2 a_{n-1}} \varphi\left(\frac{a_i - a_{i-1}}{2a_{n-1}} \right) \\ &\geqslant \sum_{i=1}^{n-1} \frac{2a_i (a_n - a_i)(a_i - a_{i-1})}{a_{n+1} a_n^2 a_{n-1}} \varphi\left(\frac{\sum_{i=1}^{n-1} a_i (a_n - a_i)(a_i - a_{i-1})^2}{\sum_{i=1}^{n-1} 2a_{n-1} a_i (a_n - a_i)(a_i - a_{i-1})} \right) \geqslant 0. \end{aligned}$$

Proof. As in Theorem 1, by Lemma 7 we conclude that the difference $B_{n-1}(f) - B_n(f)$ satisfies the inequality (1.9).

Under our conditions and according to Lemma 4 the function f where $f = x\varphi$ is superquadratic. Therefore

$$\begin{aligned} &B_{n-1}(f) - B_n(f) \qquad (2.6)\\ &\geqslant \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_{i-1}}{a_{n-1}} \right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_{n-1}} \right) - f\left(\frac{a_i}{a_n} \right) \right) \\ &\geqslant \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(f\left(\frac{a_i}{a_n} \left(\frac{a_n + a_{i-1} - a_i}{a_{n-1}} \right) \right) - f\left(\frac{a_i}{a_n} \right) \right) \\ &\quad + \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_n - a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) \right) \\ &\geqslant \frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_n - a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) \right) \geqslant 0. \end{aligned}$$

The first inequality in (2.6) is just inequality (1.9), the second inequality results from (1.2), the third inequality holds because f is increasing and $\frac{a_n+a_{i-1}-a_i}{a_{n-1}} \geqslant 1$, $i = 1,\dots,n-1$.

Thus, since $f = x\varphi$ we get that

$$
\begin{aligned}
&B_{n-1}(f) - B_n(f) \qquad (2.7)\\
\geqslant &\frac{1}{a_{n+1}} \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n} f\left(\frac{a_n - a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) + \frac{a_n - a_i}{a_n} f\left(\frac{a_i}{a_n} \left| \frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_{n-1}} \right| \right) \right)\\
= &\sum_{i=1}^{n-1} \frac{(a_i - a_{i-1}) a_i (a_n - a_i)}{a_{n+1} a_n^2 a_{n-1}} \left(\varphi\left(\frac{(a_i - a_{i-1})(a_n - a_i)}{a_{n-1} a_n} \right) + \varphi\left(\frac{(a_i - a_{i-1}) a_i}{a_{n-1} a_n} \right) \right)\\
\geqslant &\, 0.
\end{aligned}
$$

Moreover, by using the convexity of φ and Jensen's inequality we get that

$$
\varphi\left(\frac{(a_i - a_{i-1})(a_n - a_i)}{a_{n-1} a_n} \right) + \varphi\left(\frac{(a_i - a_{i-1}) a_i}{a_{n-1} a_n} \right) \geqslant 2\varphi\left(\frac{a_i - a_{i-1}}{2a_{n-1}} \right) \geqslant 0. \qquad (2.8)
$$

Hence, using again the convexity of φ and Jensen's inequality we obtain from (2.6), (2.7) and (2.8) that (2.5) holds. The proof is complete. □

In Theorem 3 we show that a lower bound of $B_{n-1}(f) - B_n(f)$ obtained by the 1-quasiconvexity of f is better than by its superquadracity.

THEOREM 3. *Let $\varphi : [0, b) \to \mathbb{R}_+$ $0 < b \leqslant \infty$ be a differentiable convex increasing function satisfying $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$ and let $f = x\varphi$. Let the positive sequences $\{a_i\}$, and $\{a_i - a_{i-1}\}$, $i = 1, 2, \ldots$, be increasing and let $a_0 = 0$. Then*

$$
\begin{aligned}
&B_{n-1}(f) - B_n(f)\\
\geqslant &\sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})(a_n - a_i) a_i}{a_{n-1}^2 a_n^2 a_{n+1}} \varphi'\left(\frac{a_i (a_n + a_{i-1} - a_i)}{a_n a_{n-1}} \right)\\
\geqslant &\sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})(a_n - a_i) a_i}{a_{n-1} a_n^2 a_{n+1}} \left(\varphi\left(\frac{a_i (a_i - a_{i-1})}{a_n a_{n-1}} \right) + \varphi\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_n a_{n-1}} \right) \right)\\
\geqslant &\sum_{i=1}^{n-1} \frac{2a_i (a_n - a_i)(a_i - a_{i-1})}{a_{n+1} a_n^2 a_{n-1}} \varphi\left(\frac{\sum_{i=1}^{n-1} a_i (a_n - a_i)(a_i - a_{i-1})^2}{\sum_{i=1}^{n-1} 2a_{n-1} a_i (a_n - a_i)(a_i - a_{i-1})} \right) \geqslant 0.
\end{aligned}
$$

hold, which means that the bound obtained by the 1-quasiconvexity of f (that includes φ') is better than the bound obtained by its superquadracity.

Proof. Under our conditions on a_i for $n \geqslant 2$ inequality (1.9) holds. Now using the notation (2.3) we see that in our case $x_{i,1} < 2\overline{x_i}$, $x_{i,2} < 2\overline{x_i}$, in other words $\frac{a_{i-1}}{a_{n-1}} < \frac{a_i}{a_{n-1}} < 2\frac{a_i(a_n + a_{i-1} - a_i)}{a_n a_{n-1}}$ when $a_n \geqslant 2(a_{n-1} - a_{n-2})$ but as it is given that $a_n - a_{n-1} \geqslant a_{n-1} - a_{n-2}$, therefore $a_n \geqslant 2a_{n-1} - a_{n-2} \geqslant 2(a_{n-1} - a_{n-2})$. Hence, the conditions to use Lemma 5 are satisfied and therefore for $n \geqslant 2$,

$$\begin{aligned}
&\frac{a_i}{a_n} f\left(\frac{a_{i-1}}{a_{n-1}}\right)+\left(\frac{a_n-a_i}{a_n}\right) f\left(\frac{a_i}{a_{n-1}}\right)-f\left(\frac{a_i}{a_n}\right) \\
&\geqslant \varphi^{\prime}\left(\frac{a_i\left(a_n+a_{i-1}-a_i\right)}{a_n a_{n-1}}\right)\left(\frac{a_i}{a_n}\right)\left(\frac{a_n-a_i}{a_n}\right)\left(\frac{a_i-a_{i-1}}{a_{n-1}}\right)^2 \\
&\geqslant \frac{a_i}{a_n} f\left(\frac{\left(a_n-a_i\right)\left(a_i-a_{i-1}\right)}{a_n a_{n-1}}\right)+\frac{a_n-a_i}{a_n} f\left(\frac{a_i\left(a_i-a_{i-1}\right)}{a_n a_{n-1}}\right) \\
&=\frac{a_i\left(a_n-a_i\right)\left(a_i-a_{i-1}\right)}{a_n^2 a_{n-1}}\left(\varphi\left(\frac{a_i\left(a_i-a_{i-1}\right)}{a_n a_{n-1}}\right)+\varphi\left(\frac{\left(a_m-a_i\right)\left(a_i-a_{i-1}\right)}{a_n a_{n-1}}\right)\right) \\
&\geqslant 0.
\end{aligned}$$

Now summing up for $i=1,\ldots,n-1$ and dividing by $a_{n+1},$ we get that all the inequalities stated in Theorem 3 hold. The proof is complete. □

Up to now we have dealt with and compared the lower bounds derived for the differences defined by (1.6). By using similar arguments analogous results for the differences defined by (1.7) can be derived too. Here we just use Lemma 6 instead of Lemma 7. Hence instead of giving these results in three new theorems we sum up them in the following more comprehensive form:

THEOREM 4. *Let $\varphi:[0,b)\rightarrow\mathbb{R}_+,\ 0<b\leqslant\infty$ be a differentiable convex increasing function and let $f=x\varphi$. Let the sequence $\{a_i\},\ i=1,\ldots,$ be increasing and such that $\{a_{i+1}-a_i\}$ is decreasing and let $a_0=0$. Then, for $n\geqslant 2$*

$$\begin{aligned}
&A_{n+1}(f)-A_n(f) \\
&\geqslant \sum_{i=1}^{n-1} \frac{\left(a_{i+1}-a_i\right)^2 a_i\left(a_n-a_i\right)}{a_{n-1} a_n^2 a_{n+1}^2} \varphi^{\prime}\left(\frac{a_i\left(a_n+a_{i+1}-a_i\right)}{a_n a_{n+1}}\right) \\
&\geqslant \sum_{i=1}^{n-1} \frac{\left(a_{i+1}-a_i\right)^2 a_i\left(a_n-a_i\right)}{a_{n-1} a_n^2 a_{n+1}^2} \varphi^{\prime}\left(\frac{a_i}{a_n}\right) \geqslant 0.
\end{aligned} \tag{2.9}$$

If φ satisfies also that $\varphi(0)=0=\lim_{x\rightarrow 0^+} x\varphi^{\prime}(x)$, then

$$\begin{aligned}
&A_{n+1}(f)-A_n(f) \\
&\geqslant \sum_{i=1}^{n-1} \frac{\left(a_{i+1}-a_i\right) a_i\left(a_n-a_i\right)}{a_{n-1} a_n^2 a_{n+1}}\left(\varphi\left(\frac{\left(a_n-a_i\right)\left(a_{i+1}-a_i\right)}{a_n a_{n+1}}\right)+\varphi\left(\frac{a_i\left(a_{i+1}-a_i\right)}{a_n\left(a_{n+1}\right)}\right)\right) \\
&\geqslant \sum_{i=1}^{n-1} \frac{2\left(a_{i+1}-a_i\right) a_i\left(a_n-a_i\right)}{a_{n-1} a_n^2 a_{n+1}} \varphi\left(\frac{a_{i+1}-a_i}{2 a_{n+1}}\right) \\
&\geqslant \sum_{i=1}^{n-1} \frac{2\left(a_{i+1}-a_i\right) a_i\left(a_n-a_i\right)}{a_{n-1} a_n^2 a_{n+1}} \varphi\left(\frac{\sum_{i=1}^{n-1}\left(a_{i+1}-a_i\right)^2 a_i\left(a_n-a_i\right)}{\sum_{i=1}^{n-1} 2 a_{n+1}\left(a_{i+1}-a_i\right) a_i\left(a_n-a_i\right)}\right) \geqslant 0.
\end{aligned} \tag{2.10}$$

Moreover, the inequalities above can be compared. In fact,

$$A_{n+1}(f)-A_n(f) \geqslant \sum_{i=1}^{n-1} \frac{(a_{i+1}-a_i)^2 a_i (a_n-a_i)}{a_{n-1}a_n^2 a_{n+1}^2} \varphi'\left(\frac{a_i(a_n+a_{i+1}-a_i)}{a_n a_{n+1}}\right) \tag{2.11}$$

$$\geqslant \sum_{i=1}^{n-1} \frac{(a_{i+1}-a_i) a_i (a_n-a_i)}{a_{n-1}a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n-a_i)(a_{i+1}-a_i)}{a_n a_{n+1}}\right)+\varphi\left(\frac{a_i(a_{i+1}-a_i)}{a_n(a_{n+1})}\right)\right) \geqslant 0$$

hold.

Further, if φ' is also convex, then

$$A_{n+1}(f)-A_n(f) \tag{2.12}$$

$$\geqslant \sum_{i=1}^{n-1} \frac{(a_{i+1}-a_i)^2 a_i (a_n-a_i)}{a_{n-1}a_n^2 a_{n+1}^2} \varphi'\left(\frac{\sum_{i=1}^{n-1} a_i^2 (a_n-a_i)(a_{i+1}-a_i)^2}{a_n \sum_{i=1}^{n-1} a_i (a_n-a_i)(a_{i+1}-a_i)^2}\right) \geqslant 0.$$

Proof. Using (1.8) in Lemma 6, the 1-quasiconvexity of f, and Lemma 2 and denoting

$$\alpha_{i,1}=\frac{a_i}{a_n}, \quad \alpha_{i,2}=\frac{a_n-a_i}{a_n}, \quad x_{i,1}=\frac{a_{i+1}}{a_{n+1}}, \quad x_{i,2}=\frac{a_i}{a_{n+1}}, \tag{2.13}$$

$$\overline{x}_i=\alpha_{i,1}x_{i,1}+\alpha_{i,2}x_{i,2}=\frac{a_i(a_n+a_{i+1}-a_i)}{a_n a_{n+1}}, \quad i=1,\dots,n-1,$$

we get from (1.3) and (2.13) that the inequalities

$$A_{n+1}(f)-A_n(f) \geqslant \frac{1}{a_{n-1}} \sum_{i=1}^{n-1}\left(f\left(\frac{a_i(a_n+a_{i+1}-a_i)}{a_n a_{n+1}}\right)-f\left(\frac{a_i}{a_n}\right)\right) \tag{2.14}$$

$$+\sum_{i=1}^{n-1} \frac{(a_{i+1}-a_i)^2 a_i (a_n-a_i)}{a_{n-1}a_n^2 a_{n+1}^2} \varphi'\left(\frac{a_i(a_n+a_{i+1}-a_i)}{a_n a_{n+1}}\right)$$

$$\geqslant \sum_{i=1}^{n-1} \frac{(a_{i+1}-a_i)^2 a_i (a_n-a_i)}{a_{n-1}a_n^2 a_{n+1}^2} \varphi'\left(\frac{a_i}{a_n}\right) \geqslant 0$$

hold. The second inequality in (2.14) holds since f and φ' are increasing and $\frac{a_n+a_{i+1}-a_i}{a_{n+1}} \geqslant 1$, $i=1,\dots,n-1$. Hence, (2.9) holds.

According to Lemma 4 when $\varphi(0)=0=\lim_{x\to 0^+} x\varphi'(x)$ it yields that f is superquadratic. Therefore we get from (1.2) in Lemma 1 and from (2.13) that

$$A_{n+1}(f)-A_n(f) \tag{2.15}$$

$$\geqslant \frac{1}{a_{n-1}} \sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{a_{i+1}}{a_{n+1}}\right)+\frac{a_n-a_i}{a_n} f\left(\frac{a_i}{a_{n+1}}\right)-f\left(\frac{a_i}{a_n}\right)\right)$$

$$\geqslant \frac{1}{a_{n-1}} \sum_{i=1}^{n-1}\left(f\left(\frac{a_i(a_n+a_{i+1}-a_i)}{a_n a_{n+1}}\right)-f\left(\frac{a_i}{a_n}\right)\right)$$

$$+\sum_{i=1}^{n-1} \frac{a_i}{a_{n-1}a_n}\left(f\left(\frac{(a_n-a_i)(a_{i+1}-a_i)}{a_n a_{n+1}}\right)+\frac{a_n-a_i}{a_{n-1}a_n} f\left(\frac{a_i(a_{i+1}-a_i)}{a_n(a_{n+1})}\right)\right)$$

$$
\begin{aligned}
&\geqslant \sum_{i=1}^{n-1} \frac{a_i}{a_{n-1}a_n}\left(f\left(\frac{(a_n-a_i)(a_{i+1}-a_i)}{a_na_{n+1}}\right)+\frac{a_n-a_i}{a_{n-1}a_n}f\left(\frac{a_i(a_{i+1}-a_i)}{a_n(a_{n+1})}\right)\right)\\
&\geqslant \sum_{i=1}^{n-1}\left(\varphi\left(\frac{(a_n-a_i)(a_{i+1}-a_i)}{a_na_{n+1}}\right)+\varphi\left(\frac{a_i(a_{i+1}-a_i)}{a_n(a_{n+1})}\right)\right)\frac{(a_{i+1}-a_i)a_i(a_n-a_i)}{a_{n-1}a_n^2a_{n+1}}\\
&\geqslant 0.
\end{aligned}
$$

Using (1.5) in Lemma 5, we get from $a_{i+1}-a_i \leqslant a_i-a_{i-1} \leqslant a_i$ that $\frac{a_{i+1}}{a_{n+1}} \leqslant 2\frac{a_i}{a_{n+1}}$ holds and therefore the bound in the first inequality in (2.9) obtained by using the 1-quasiconvexity of f is better that the bound obtained in (2.15) by using the superquadracity of f. This means that inequalities (2.11) hold.

Further, if φ' is also convex, then from (2.9) by Jensen's inequality we get that (2.12) holds.

Finally since φ is convex we get from (2.15) and Jensen's inequality that (2.10) holds. The proof is complete. □

3. Examples

In our first example we give the announced refinement of (1.10):

EXAMPLE 2. For $a_i=i,\ i=0,1,\dots$ we get from Theorem 1 by the 1-quasiconvexity of f where $f=x\varphi$, under the conditions there on φ and f that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right)-\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\\
&\geqslant \frac{1}{(n+1)n^2(n-1)^2}\sum_{i=1}^{n-1} i(n-i)\varphi'\left(\frac{i}{n}\right)\geqslant 0.
\end{aligned}
$$

If φ' is also convex on $[0,\infty)$ we get that

$$
\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right)-\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\geqslant \frac{1}{6n(n-1)}\varphi'\left(\frac{1}{2}\right)\geqslant 0
$$

because $\sum_{i=1}^{n-1} i(n-i)=\frac{n(n-1)(n+1)}{6}$ and $\sum_{i=1}^{n-1} i^2(n-i)=\frac{n^2(n-1)(n+1)}{12}$.

From Theorem 2 we get by the superquadracity of f where $f=x\varphi$ and by Jensen's inequality for the convex function φ that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right)-\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\\
&\geqslant \frac{2}{(n+1)n^2(n-1)}\sum_{i=1}^{n-1} i(n-i)\varphi\left(\frac{i}{(n-1)n}\right)\geqslant \frac{1}{3n}\varphi\left(\frac{1}{2(n-1)}\right)\geqslant 0.
\end{aligned}
$$

The first inequality is obtained in this special case because

$$
\sum_{i=1}^{n-1} a_n-a_i=\sum_{i=1}^{n-1}(n-i)=\sum_{i=1}^{n-1} i=\sum_{i=1}^{n-1} a_i.
$$

From Theorem 3 we get that the bounds obtained by the 1-quasiconvexity of f is better that by its superquadracity.

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n-1} f\left(\frac{i}{n-1}\right)-\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\\ \geqslant\ &\frac{1}{(n+1)\,n^2\,(n-1)^2}\sum_{i=1}^{n-1} i\,(n-i)\,\varphi'\left(\frac{i}{n}\right)\\ \geqslant\ &\frac{2}{(n+1)\,n^2\,(n-1)}\sum_{i=1}^{n-1} i\,(n-i)\,\varphi\left(\frac{i}{(n-1)\,n}\right)\geqslant 0.\end{aligned}$$

Now we will see that when φ' is also convex and φ is defined on $[0,\infty)$, the lower bound that we can derive from the convexity of φ' is again better than the lower bound we get from the convexity of φ which means that we get

$$\frac{1}{6n\,(n-1)}\varphi'\left(\frac{1}{2}\right)\geqslant\frac{1}{3n}\varphi\left(\frac{1}{2\,(n-1)}\right)\geqslant 0.$$

To see that this inequality holds, we use the inequalities $x\varphi'\geqslant\varphi$ (see Lemma 3) and $x_1\varphi\left(\frac{1}{x_1}\right)<x_2\varphi\left(\frac{1}{x_2}\right)$, $0<x_2<x_1$, which are satisfied under our conditions on φ and we find that

$$\frac{1}{6n\,(n-1)}\varphi'\left(\frac{1}{2}\right)=\frac{2}{6n\,(n-1)}\frac{1}{2}\varphi'\left(\frac{1}{2}\right)\geqslant\frac{1}{6n\,(n-1)}2\varphi\left(\frac{1}{2}\right)$$

$$\geqslant\frac{1}{6n\,(n-1)}2\,(n-1)\,\varphi\left(\frac{1}{2\,(n-1)}\right)=\frac{1}{3n}\varphi\left(\frac{1}{2\,(n-1)}\right)\geqslant 0.$$

Hence,

$$\frac{1}{6n\,(n-1)}\varphi'\left(\frac{1}{2}\right)\geqslant\frac{1}{3n}\varphi\left(\frac{1}{2\,(n-1)}\right)\geqslant 0.$$

Therefore, from the inequalities derived in our case and in view of the 1-quasiconvexity of f and the superquadracity of f we can deduce that the bound obtained by the 1-quasiconvexity of $f=x\varphi$ is better than by its superquadracity

$$\begin{aligned}&\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right)-\frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right)\\ \geqslant\ &\frac{1}{6n\,(n-1)}\varphi'\left(\frac{1}{2}\right)\geqslant\frac{1}{3n}\varphi\left(\frac{1}{2\,(n-1)}\right)\geqslant 0.\end{aligned}$$

We also present one example with another natural choice of the basic sequence.

EXAMPLE 3. Let us choose now in Theorems 1, 2 and 3 $\varphi(0)=0,\ a_0=0$ and $a_i=2i-1,\ i=1,\ldots$. Then, by the 1-quasiconvexity of $f=x\varphi$ the inequalities

$$\begin{aligned}
&\frac{1}{2n-1}\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n-3}\right)-\frac{1}{2n+1}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n-1}\right)\\
&\geqslant \frac{8}{(2n+1)(2n-1)^2(2n-3)^2}\sum_{i=1}^{n-1}(2i-1)(n-i)\varphi'\left(\frac{2i-1}{2n-1}\right)\\
&\geqslant \frac{4n(n-1)}{3(2n+1)(2n-1)(2n-3)^2}\varphi'\left(\frac{2n^2-4n-1}{(2n-1)^2}\right)\geqslant 0
\end{aligned}$$

hold, where the last inequality is derived if φ' is convex. By the superquadracity of $f=x\varphi$ we obtain also that since φ is convex

$$\begin{aligned}
&\frac{1}{2n-1}\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n-3}\right)-\frac{1}{2n+1}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n-1}\right)\\
&\geqslant \sum_{i=1}^{n-1}\frac{4(2i-1)(n-i)}{(2n+1)(2n-1)^2(2n-3)}\left(\varphi\left(\frac{4(n-i)}{(2n-3)(2n-1)}\right)+\varphi\left(\frac{2(2i-1)}{(2n-3)(2n-1)}\right)\right)\\
&\geqslant \frac{4n(n-1)}{(2n+1)(2n-1)(2n-3)}\varphi\left(\frac{1}{2n-1}\right)\geqslant 0.
\end{aligned}$$

In this case the requirement $a_n\geqslant 2(a_{n-1}-a_{n-2})$ holds for $n\geqslant 3$ and we get that the lower bound obtained by the quasiconvexity of $f=x\varphi$ is better than the lower bound obtained by its superquadracity which means that

$$\begin{aligned}
&\frac{1}{2n-1}\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n-3}\right)-\frac{1}{2n+1}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n-1}\right)\\
&\geqslant \frac{8}{(2n+1)(2n-1)^2(2n-3)^2}\sum_{i=1}^{n-1}(2i-1)(n-i)\varphi'\left(\frac{2i-1}{2n-1}\right)\\
&\geqslant \sum_{i=1}^{n-1}\frac{4(2i-1)(n-i)}{(2n+1)(2n-1)^2(2n-3)}\left(\varphi\left(\frac{4(n-i)}{(2n-3)(2n-1)}\right)+\varphi\left(\frac{2(2i-1)}{(2n-3)(2n-1)}\right)\right)\\
&\geqslant \frac{4n(n-1)}{(2n+1)(2n-1)(2n-3)}\varphi\left(\frac{1}{2n-1}\right)\geqslant 0.
\end{aligned}$$

In our next example we give a similar application of Theorem 4.

EXAMPLE 4. In this example we use Theorem 4 to see that also in this case all the lower bounds obtained by the 1-quasiconvexity of f are better that those obtained by its superquadracity, for the sequence $a_i=i,\ i=0,1,\ldots n$. In this case $A_n(f)$ becomes

$$C_n(f)=\frac{1}{n-1}\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right),$$

and because $a_{i+1}-a_i=1,\ i=0,1,\dots$, we get instead of inequality the equality

$$C_{n+1}(f)-C_n(f)=\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{i}{n}f\left(\frac{i+1}{n+1}\right)+\frac{n-i}{n}f\left(\frac{i}{n+1}\right)-f\left(\frac{i}{n}\right)\right),$$

and we obtain the following

Let φ and f be as in Lemma 2. Then, since f is 1-quasiconvex we find, because $\alpha_i=\frac{i}{n},\ x_i=\frac{i+1}{n+1},\ y_i=\frac{i}{n+1}$, and $\alpha_i x_i+(1-\alpha_i)y_i=\overline{x}_i=\frac{i}{n},\ i=0,\dots$, that

$$\begin{aligned}C_{n+1}(f)-C_n(f)&=\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{i}{n}f\left(\frac{i+1}{n+1}\right)+\frac{n-i}{n}f\left(\frac{i}{n+1}\right)-f\left(\frac{i}{n}\right)\right)\\&\geqslant\frac{1}{n-1}\sum_{i=1}^{n-1}\varphi'\left(\frac{i}{n}\right)\left(\frac{i(n-i)}{n^2(n+1)^2}\right)\geqslant 0,\end{aligned}$$

and since f is superquadratic (see Lemma 4) we get from Lemma 1 that

$$\begin{aligned}C_{n+1}(f)-C_n(f)&=\sum_{i=1}^{n-1}\frac{i}{n}f\left(\frac{i+1}{n+1}\right)+\frac{n-i}{n}f\left(\frac{i}{n+1}\right)\\&\geqslant\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{i(n-i)}{n^2(n+1)}\varphi\left(\frac{n-i}{n(n+1)}\right)+\frac{i(n-i)}{n^2(n+1)}\varphi\left(\frac{i}{n(n+1)}\right)\right)\\&=\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{2i(n-i)}{n^2(n+1)}\varphi\left(\frac{i}{n(n+1)}\right)\geqslant 0.\end{aligned}$$

This means that we have that

$$\begin{aligned}C_{n+1}(f)-C_n(f)&\geqslant\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{i(n-i)}{n^2(n+1)^2}\varphi'\left(\frac{i}{n}\right)\\&\geqslant\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{2i(n-i)}{n^2(n+1)}\varphi\left(\frac{i}{n(n+1)}\right)\geqslant 0.\end{aligned}$$

Further, if φ' is also convex, for instance when $f(x)=x^p,\ x\geqslant 0,\ p\geqslant 3$, then

$$\begin{aligned}C_{n+1}(f)-C_n(f)&\geqslant\frac{1}{n-1}\sum_{i=1}^{n-1}\varphi'\left(\frac{i}{n}\right)\left(\frac{i(n-i)}{n^2(n+1)^2}\right)\\&\geqslant\sum_{i=1}^{n-1}\frac{i(n-i)}{(n-1)n^2(n+1)^2}\varphi'\left(\frac{\sum_{i=1}^{n-1}\frac{i^2(n-i)}{n}}{\sum_{i=1}^{n-1}i(n-i)}\right)\\&=\frac{1}{6n(n+1)}\varphi'\left(\frac{1}{2}\right)\geqslant 0,\end{aligned}$$

and since φ is convex we get that

$$
\begin{aligned}
C_{n+1}(f)-C_n(f) &\geqslant \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{2i(n-i)}{n^2(n+1)}\varphi\left(\frac{i}{n(n+1)}\right)\\
&\geqslant 2\sum_{i=1}^{n-1}\frac{i(n-i)}{(n-1)n^2(n+1)}\varphi\left(\frac{\sum_{i=1}^{n-1}\frac{i^2(n-i)}{n}}{n(n+1)\sum_{i=1}^{n-1}i(n-i)}\right)\\
&=\frac{1}{3n}\varphi\left(\frac{1}{2(n+1)}\right)\geqslant 0.
\end{aligned}
$$

Moreover, when φ is defined on $x \geqslant 0$ we obtain by Lemma 3 using that $\frac{1}{2}\varphi'\left(\frac{1}{2}\right) \geqslant \varphi\left(\frac{1}{2}\right)$ and $2(n+1)\varphi\left(\frac{1}{2(n+1)}\right) \leqslant 2\varphi\left(\frac{1}{2}\right)$ that

$$
\frac{1}{6n(n+1)}\varphi'\left(\frac{1}{2}\right) \geqslant \frac{1}{3n}\varphi\left(\frac{1}{2(n+1)}\right) \geqslant 0.
$$

Therefore, again, the bound obtained by using the 1-quasiconvexity of f is better than the bound we get using its superquadracity, that is,

$$
C_{n+1}(f)-C_n(f) \geqslant \frac{1}{6n(n+1)}\varphi'\left(\frac{1}{2}\right) \geqslant \frac{1}{3n}\varphi\left(\frac{1}{2(n+1)}\right) \geqslant 0.
$$

We also remark that by taking in A_n the sequence $\{a_i\}$ where $a_i = 2i+1,\ i = 1,\ldots$ and $a_0 = 0$ the bounds of $\frac{1}{2n+1}\sum_{i=1}^{n} f\left(\frac{2i+1}{2n+3}\right) - \frac{1}{2n-1}\sum_{i=1}^{n-1} f\left(\frac{2i+1}{2n+1}\right)$ obtained by using the 1-quasiconvexity of f is better that the bounds obtained by using supequadracity (c.f. Example 3).

Finally, we present our announced refinements of (1.11)

EXAMPLE 5. The functions $f(x) = x^p,\ p \geqslant 2,\ \ x \geqslant 0,$ are the basic cases of 1-quasiconvex functions as well as superquadratic functions. Therefore we obtain from Example 2 that the ratio $\left(\frac{(n+1)\sum_{i=1}^{n-1} i^p}{n\sum_{i=1}^{n} i^p}\right)^{\frac{1}{p}}$ is not only bounded below by $\frac{n-1}{n}$ but by strictly better lower bounds when $p \geqslant 2$ instead of $p \geqslant 1$. Indeed, from the 1-quasiconvexity of $f(x) = x^p,\ x \geqslant 0,\ \ p \geqslant 2$ we find from Example 2 that when $n \geqslant 2$ the first inequality in (3.1) holds

$$
\left(\frac{(n+1)\sum_{i=1}^{n-1} i^p}{n\sum_{i=1}^{n} i^p}\right)^{\frac{1}{p}} \geqslant \frac{n-1}{n}(1+\Delta_1)^{\frac{1}{p}} \geqslant \frac{n-1}{n}(1+\Delta_2)^{\frac{1}{p}} \geqslant 0, \tag{3.1}
$$

where Δ_1 is defined by (3.2).

The second inequality in (3.1) holds since $0 < \frac{i}{n} \leqslant 1,\ i = 1,\ldots,n,\ \sum_{i=1}^{n}\left(\frac{i}{n}\right)^p \leqslant \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2$, when $p \geqslant 2$ and therefore

$$
\Delta_1 = \frac{(p-1)(n+1)}{2^{p-2}6n(n-1)\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p} \geqslant \Delta_2 = \frac{p-1}{2^{p-2}(n-1)(2n+1)} \geqslant 0 \tag{3.2}
$$

is satisfied. The first inequality in (3.1) follows from Example 2. Therefore (3.1)holds as a result of (3.2) and Example 2 and the proof of the example is complete.

Similarly from Example 4 we obtain that:

EXAMPLE 6. For $p \geqslant 2$ when $n \geqslant 2$

$$\left(\frac{(n-1)\sum_{i=1}^{n} i^p}{n\sum_{i=1}^{n-1} i^p}\right)^{\frac{1}{p}} \geqslant \frac{n+1}{n}(1+\Delta_1)^{\frac{1}{p}} \geqslant \frac{n+1}{n}(1+\Delta_2)^{\frac{1}{p}} \geqslant 0, \tag{3.3}$$

holds where

$$\Delta_1 = \frac{(p-1)(n-1)}{2^{p-2}6n(n+1)\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^p} \geqslant \Delta_2 = \frac{p-1}{2^{p-2}(n+1)(2n-1)} \geqslant 0.$$

From the superquadracity of $f(x) = x^p \;\; p \geqslant 2,\;\; x \geqslant 0,$ we obtain similar inequalities to (3.1) and (3.3), but because of Theorems 3 and 4 we know that (3.1) and (3.3) are better inequalities than those derived by superquadracity.

Examples 5 and 6 can be generalized by replacing $f(x) = x^p,\; x \geqslant 0,\; p \geqslant 2$ by a general 1-quasiconvex function $f(x) = x\varphi(x)$ where φ is a convex increasing function satisfying $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi'(x)$.

FINAL REMARK. Another reason that motivated us to deal with the lower bounds of the differences of averages is that to use Lemma 5 in order to compare a bound obtained by the 1-quasiconvexity of a function with its bound obtained by the superquadracity we need that the conditions

$$0 \leqslant x_i \leqslant 2\overline{x}, \quad 0 < b \leqslant \infty, \quad 0 \leqslant \alpha_i \leqslant 1, \quad 1 = 1,\ldots,m, \quad \sum_{i=1}^{m}\alpha_i = 1, \quad \overline{x} = \sum_{i=1}^{m}\alpha_i x_i$$

hold. In this paper we see that this condition is very natural and automatically holds when dealing with bounds of differences of averages as seen in Theorems 1, 2, 3, 4 and in Examples 2, 3 and 4.

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