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CONCAVITY OF THE ERROR FUNCTION WITH RESPECT TO HÖLDER MEANS

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Abstract. In this paper, we present a necessary and sufficient condition for the concavity of the error function with respect to Hölder means.

1. Introduction

For $x \in \mathbb{R}$, the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The error function has numerous applications in statistics, probability theory, and partial differential equations. Recently, the error function have been the subject of intensive research. In particular, many remarkable inequalities for the error function can be found in the literature [1, 2, 3, 8, 9, 13, 15, 18].

For $p \in \mathbb{R}$, the pth Hölder mean $H_p(x,y)$ of two positive numbers x and y is defined by

$$H_p(x,y) = \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{xy}, & p = 0. \end{cases}$$

It is well known that $H_p(x,y)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $x, y > 0$ with $x \neq y$. The main properties of the Hölder mean were given in [12].

A real-valued function $f : I \to \mathbb{R}$ is said to be $H_{p,q}$-convex (concave) on I if it satisfies

$$f(H_p(x,y)) \leqslant (\geqslant) H_q(f(x), f(y))$$

for all $x, y \in I$, and strictly $H_{p,q}$-convex (concave) if the inequality is strict except for $x = y$.

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Recently, the $H_{p,q}$-convexity (concavity) has attracted the attention of many researcher (see [6, 7, 11, 14, 19, 20, 21, 22]). Baricz [6] discussed the convexity of the Gaussian hypergeometric series and general power series with respect to Hölder means, and proved that the complete elliptic integrals of the first kind is strictly $H_{p,p}$-convex on $(0,1)$ for $p \in (0,2]$. In [14, 20], the authors presented the necessary and sufficient conditions such that the complete elliptic integrals of the second kind is $H_{p,q}$-convex and the complete elliptic integrals of the first kind is $H_{p,q}$-concave. Zhou, Qiu and Wang [22] discussed the $H_{p,p}$ concavity of the generalized elliptic integral $\mathscr{K}_a(r)$ for $a \in (0,1/2]$. Some convexity and concavity properties for generalized trigonometric functions with respected to Hölder mean were established by Bhayo and Vuorinen in [10]. More results involving the convexity and concavity with respected to other bivariat means can be found in the literature [5, 16, 17].

In this paper, we shall establish a necessary and sufficient condition for the concavity of the error function with respect to Hölder means. Our main result is the following Theorem 1.1.

THEOREM 1.1. *The Error function* $\mathrm{erf}(t)$ *is strictly* $H_{p,q}$*-concave on* $(0,+\infty)$ *if and only if*

$$(p,q) \in \{(p,q)|p \geqslant L(q)\},$$

where

$$L(q) = \sup_{t\in(0,+\infty)} \left\{[2(q-1)te^{-t^2}]/[\sqrt{\pi}\mathrm{erf}(t)] + 1 - 2t^2\right\}$$

is a continuous function with $L(q) = q$ *for* $q \geqslant -2$ *and* $L(q) > q$ *for* $q < -2$*. There are no real values of* p *and* q *for which* $\mathrm{erf}(t)$ *is* $H_{p,q}$*-convex on* $(0,+\infty)$.

REMARK 1.1. By use of the mathematical software, we draw the picture (Fig. 1) of the curve $p = L(q)$ in the pq-plane as illustrated above.

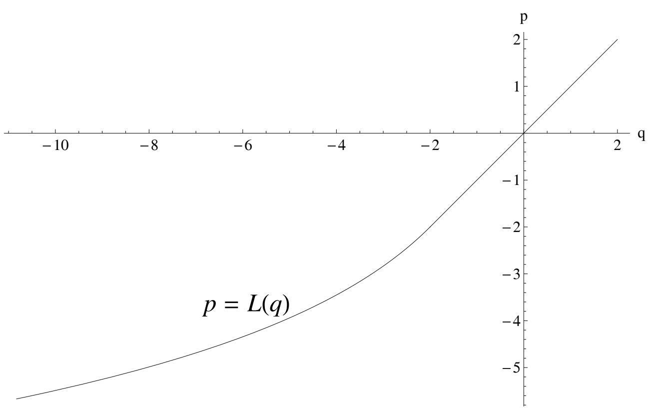

Figure 1: *Curve* $p = L(q)$

2. Lemmas

LEMMA 2.1. [4, Theorem 1.25] *For* $-\infty < a < b < \infty$, *let* $f,g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$, *and be differentiable on* (a,b), *let* $g'(x)\neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

LEMMA 2.2. *Then function* $\rho(t)=\sqrt{\pi}(2t^2-1)\mathrm{erf}(t)+2te^{-t^2}$ *is strictly increasing from* $(0,+\infty)$ *onto* $(0,+\infty)$.

Proof. The result follows easily from $\rho(0^+)=0$, $\rho(+\infty)=+\infty$ and $\rho'(t)=4\sqrt{\pi}t\mathrm{erf}(t)>0$. □

LEMMA 2.3. *The function*

$$f(t)=\frac{2\pi te^{t^2}\mathrm{erf}(t)^2}{\sqrt{\pi}(2t^2-1)\mathrm{erf}(t)+2te^{-t^2}}$$

is strictly increasing from $(0,+\infty)$ *onto* $(3,+\infty)$.

Proof. Let $f_1(t)=2\pi te^{t^2}\mathrm{erf}(t)^2$ and $f_2(t)=\sqrt{\pi}(2t^2-1)\mathrm{erf}(t)+2te^{-t^2}$. Then simple calculations lead to

$$f(t)=\frac{f_1(t)}{f_2(t)}, \tag{2.1}$$

$$f_1(0)=f_2(0)=0, \tag{2.2}$$

$$f_1'(t)=8\sqrt{\pi}t\mathrm{erf}(t)+2\pi e^{t^2}(2t^2+1)\mathrm{erf}(t)^2,$$

$$f_2'(t)=4\sqrt{\pi}t\mathrm{erf}(t),$$

$$\frac{f_1'(t)}{f_2'(t)}=2+\frac{g_1(t)}{g_2(t)}, \tag{2.3}$$

where $g_1(t)=\sqrt{\pi}e^{t^2}(2t^2+1)\mathrm{erf}(t)$ and $g_2(t)=2t$. Moreover,

$$g_1(0)=g_2(0)=0, \tag{2.4}$$

$$\frac{g_1'(t)}{g_2'(t)}=1+2t^2+\sqrt{\pi}t(2t^2+3)e^{t^2}\mathrm{erf}(t). \tag{2.5}$$

Equation (2.5) implies that $g_1'(t)/g_2'(t)$ is strictly increasing on $(0,+\infty)$. Making use of l'Hôpital's rule we have

$$\lim_{t\to 0^+}f(t)=\lim_{t\to 0^+}\frac{f_1'(t)}{f_2'(t)}=2+\lim_{t\to 0^+}\frac{g_1'(t)}{g_2'(t)}=3. \tag{2.6}$$

Therefore, Lemma 2.3 follows easily from (2.1)–(2.4), (2.6), $f(+\infty)=+\infty$, the monotonicity of $g_1'(t)/g_2'(t)$ and Lemma 2.1. □

LEMMA 2.4. *Let* $q \in \mathbb{R}$,

$$\phi(t) = (q-1)\frac{2te^{-t^2}}{\sqrt{\pi}\mathrm{erf}(t)} + 1 - 2t^2$$

and $L(q) = \sup_{t \in (0,+\infty)} \phi(t)$. *Then the following statements hold true:*

(1) If $q \geqslant -2$, *then* $\phi(t)$ *is strictly decreasing from* $(0,+\infty)$ *onto* $(-\infty, q)$;

(2) If $q < -2$, *then there exists* $\delta_0 \in (0,+\infty)$ *such that* $\phi(t)$ *is strictly increasing on* $(0,\delta_0)$ *and strictly decreasing on* $(\delta_0,+\infty)$. *In particular,* $L(q) > q$ *and the range of* $\phi(t)$ *is* $(-\infty, L(q)]$.

Proof. By simple computations, one has

$$\lim_{t \to 0^+} \phi(t) = q, \tag{2.7}$$

$$\lim_{t \to +\infty} \phi(t) = -\infty, \tag{2.8}$$

$$\begin{aligned} \phi'(t) &= 2(q-1)\frac{\sqrt{\pi}(1-2t^2)e^{-t^2}\mathrm{erf}(t) - 2te^{-2t^2}}{\pi \mathrm{erf}(t)^2} - 4t \\ &= \frac{2\rho(t)}{\pi e^{t^2}\mathrm{erf}(t)^2}[1 - q - f(t)], \end{aligned} \tag{2.9}$$

where $\rho(t)$ and $f(t)$ are defined as in Lemmas 2.2 and 2.3, respectively.

Therefore, Lemma 2.4 follows from (2.7)–(2.9) and Lemmas 2.2 and 2.3. □

LEMMA 2.5. *Let* $p, q \in \mathbb{R}$ *and*

$$\varphi(t) = \frac{\mathrm{erf}(t)^{q-1}e^{-t^2}}{t^{p-1}}.$$

Then the following statements hold true:

(1) If $q \geqslant -2$, *then* $\varphi(t)$ *is strictly decreasing on* $(0,+\infty)$ *if and only if* $p \geqslant q$. *In particular, if* $p < q$, *then there exists* $\delta_1 \in (0,+\infty)$ *such that* $\varphi(t)$ *is strictly increasing on* $(0,\delta_1)$ *and strictly decreasing on* $(\delta_1,+\infty)$,

(2) If $q < -2$, *then* $\varphi(t)$ *is strictly decreasing on* $(0,+\infty)$ *if and only if* $p \geqslant L(q)$, *where* $L(q)$ *is define as in Lemma 2.4. In particular, if* $p < L(q)$, *then there exists* $\delta_2 \in (\delta_0,+\infty)$ *such that* $\varphi(t)$ *is strictly increasing on* (δ_0,δ_2) *and strictly decreasing on* $(\delta_2,+\infty)$.

Proof. By logarithmic differentiation, one has

$$\frac{\varphi'(t)}{\varphi(t)} = (q-1)\frac{2e^{-t^2}}{\sqrt{\pi}\mathrm{erf}(t)} + \frac{1}{t} - 2t - \frac{p}{t} = \frac{1}{t}[\phi(t) - p], \tag{2.10}$$

where $\phi(t)$ is defined as in Lemma 2.4.

We divide the proof into four cases.

Case I $q \geqslant -2$ and $p \geqslant q$. Then it follows from (2.10) and Lemma 2.4(1) that $\varphi'(t) < 0$ on $(0, +\infty)$. Hence $\varphi(t)$ is strictly decreasing on $(0, +\infty)$.

Case II $q \geqslant -2$ and $p < q$. Then (2.10) and Lemma 2.4(1) lead to the conclusion that there exists $\delta_1 \in (0, +\infty)$ such that $\varphi'(t) > 0$ for $t \in (0, \delta_1)$ and $\varphi'(t) < 0$ for $t \in (\delta_1, +\infty)$. Thus $\varphi(t)$ is strictly increasing on $(0, \delta_1)$ and strictly decreasing on $(\delta_1, +\infty)$.

Case III $q < -2$ and $p \geqslant L(q)$. Then it follows from (2.10) and Lemma 2.4(2) that $\varphi'(t) < 0$ on $(0, +\infty)$. Hence $\varphi(t)$ is strictly decreasing on $(0, +\infty)$.

Case IV $q < -2$ and $p < L(q)$. Then (2.10) and Lemma 2.4(2) lead to the conclusion that there exists $\delta_2 \in (\delta_0, +\infty)$ such that $\varphi(t)$ is strictly increasing on (δ_0, δ_2) and strictly decreasing on $(\delta_2, +\infty)$. □

3. Proof of Theorem 1.1

Proof. The proof of Theorem 1.1 can be divided into two cases.

Case 1 $q \neq 0$. Without loss of generality, we assume that $x \leqslant y$. Define

$$F(x,y) = \mathrm{erf}(H_p(x,y))^q - \frac{\mathrm{erf}(x)^q + \mathrm{erf}(y)^q}{2}, \quad (x,y) \in (0,\infty) \times (0,\infty). \tag{3.1}$$

Let $u = H_p(x,y)$, then $\partial u / \partial x = (x/u)^{p-1}/2$. If $x < y$, then $x < u < y$. Taking the differentiation of (3.1) with respect to x, we have

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{q}{\sqrt{\pi}} \mathrm{erf}(u)^{q-1} e^{-u^2} \left(\frac{x}{u}\right)^{p-1} - \frac{q}{\sqrt{\pi}} \mathrm{erf}(x)^{q-1} e^{-x^2} \\ &= \frac{q}{\sqrt{\pi}} x^{p-1} \left[\frac{\mathrm{erf}(u)^{q-1} e^{-u^2}}{u^{p-1}} - \frac{\mathrm{erf}(x)^{q-1} e^{-x^2}}{x^{p-1}} \right].\end{aligned} \tag{3.2}$$

Next, we divide the proof of Case 1 into four subcases.

Subcase 1.1 $q \geqslant -2$ and $p \geqslant q$. It follows from (3.2) and Lemma 2.5(1) that $\partial F / \partial x > 0$ if $q < 0$, and $\partial F / \partial x < 0$ if $q > 0$. Hence $F(x,y) < F(y,y) = 0$ if $q < 0$ and $F(x,y) > F(y,y) = 0$ if $q > 0$. This in conjunction with (3.1) implies that

$$\mathrm{erf}(H_p(x,y)) \geqslant H_q(\mathrm{erf}(x), \mathrm{erf}(y))$$

holds for $x, y > 0$ with equality if and only if $x = y$.

Therefore, $\mathrm{erf}(t)$ is strictly $H_{p,q}$-concave on $(0, +\infty)$ for $(p,q) \in \{(p,q) | p \geqslant -2, p \geqslant q, q \neq 0\}$.

Subcase 1.2 $q \geqslant -2$ and $p < q$. Equation (3.2) and Lemma 2.5(1) implies that $\partial F / \partial x < 0$ if $q < 0$, and $\partial F / \partial x > 0$ if $q > 0$ for $0 < x < y < \delta_1$; while $\partial F / \partial x > 0$ if $q < 0$, and $\partial F / \partial x < 0$ if $q > 0$ for $\delta_1 < x < y < +\infty$. This in conjunction with (3.1) yields that

$$\mathrm{erf}(H_p(x,y)) < H_q(\mathrm{erf}(x), \mathrm{erf}(y))$$

holds for $0 < x < y < \delta_1$ and

$$\operatorname{erf}(H_p(x,y)) > H_q(\operatorname{erf}(x), \operatorname{erf}(y))$$

holds for $\delta_1 < x < y < +\infty$.

Therefore, $\operatorname{erf}(t)$ is neither $H_{p,q}$-convex nor $H_{p,q}$-concave on $(0,+\infty)$ for $(p,q) \in \{(p,q) | q \geqslant -2, p < q, q \neq 0\}$.

Subcase 1.3 $q < -2$ and $p \geqslant L(q)$. With the similar argument in Subcase 1.1, equations (3.1) and (3.2) together with Lemma 2.5(2) lead to the conclusion that $\operatorname{erf}(t)$ is strictly $H_{p,q}$-concave on $(0,+\infty)$ for $(p,q) \in \{(p,q) | q < -2, p \geqslant L(q)\}$.

Subcase 1.4 $q < -2$ and $p < L(q)$. Making use of the similar argument in Subcase 1.2, it follows from (3.1) and (3.2) together with Lemma 2.5(2) that

$$\operatorname{erf}(H_p(x,y)) < H_q(\operatorname{erf}(x), \operatorname{erf}(y))$$

holds for $\delta_0 < x < y < \delta_2$, and

$$\operatorname{erf}(H_p(x,y)) > H_q(\operatorname{erf}(x), \operatorname{erf}(y))$$

holds for $\delta_2 < x < y < +\infty$.

Therefore, $\operatorname{erf}(t)$ is neither $H_{p,q}$-convex nor $H_{p,q}$-concave on $(0,+\infty)$ for $(p,q) \in \{(p,q) | q < -2, p \geqslant L(q)\}$.

Case 2 $q = 0$. Without loss of generality, we assume that $x \leqslant y$. Define

$$G(x,y) = \frac{\operatorname{erf}(H_p(x,y))^2}{\operatorname{erf}(x)\operatorname{erf}(y)}. \tag{3.3}$$

Let $u = H_p(x,y)$, then $\partial u/\partial x = (x/u)^{p-1}/2$. Logarithmic differentiation of (3.3) gives

$$\frac{1}{G(x,y)} \frac{\partial G}{\partial x} = \frac{2}{\sqrt{\pi}} x^{p-1} \left[\frac{e^{-u^2}}{u^{p-1}\operatorname{erf}(u)} - \frac{e^{-x^2}}{x^{p-1}\operatorname{erf}(x)} \right]. \tag{3.4}$$

Next, we divide the proof of Case 2 into two subcases.

Subcase 2.1 $p \geqslant 0$. It follows from (3.4) and Lemma 2.5(1) that $\partial G/\partial x < 0$ and thereby $G(x,y) > G(y,y) = 1$. From (3.3) we have

$$\operatorname{erf}(H_p(x,y)) \geqslant H_q(\operatorname{erf}(x), \operatorname{erf}(y))$$

holds for $x, y > 0$ with equality if and only if $x = y$.

Therefore, $\operatorname{erf}(t)$ is strictly $H_{p,q}$-concave on $(0,+\infty)$ for $(p,q) \in \{(p,q) | p \geqslant q = 0\}$.

Subcase 2.2 $p < 0$. Then by equations (3.3) and (3.4) together with Lemma 2.5(1) we can conclude that $\operatorname{erf}(t)$ is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on $(0,+\infty)$ analogously. $\square$

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