Mathematical Inequalities & Applications
Volume 19, Number 2 (2016), 597–624 doi:10.7153/mia-19-44

LIMITING ULTRASYMMETRIC SEQUENCE SPACES

PEDRO FERNÁNDEZ-MARTÍNEZ AND TERESA SIGNES

(*Communicated by B. Opic*)

Abstract. The paper gives an analytic characterization of a class of ultrasymmetric sequence spaces that are very close to ℓ_∞. We introduce discrete descriptions of limiting J and K interpolation methods, and we apply the results and the techniques developed to the study of limiting approximation spaces.

1. Introduction

The study of limiting spaces in interpolation scales and their applications to adjacent topics has been an active area of research in recent years. See [1], [6], [7], [8], [9], [10], [11], [15], [17], [18], [19] among other contributions.

The interpolation spaces for the (ℓ_1, ℓ_∞) couple form the family of rearrangement invariant (r.i.) sequence spaces, among these the class of ultrasymmetric spaces stands out. An r.i. space $\mathbf{e}$, with fundamental function φ, is always intermediate for the $(\Lambda_\varphi, M_\varphi)$ couple formed by the Lorentz and the Marcinkiewicz spaces with the same fundamental function φ. Ultrasymmetric spaces are r.i. spaces which are not only intermediate, but also interpolation spaces for the $(\Lambda_\varphi, M_\varphi)$ couple. The family of ultrasymmetric sequence spaces comprises Lebesgue spaces, Lorentz spaces, Lorentz-Zygmund spaces and other families of sequence spaces.

The analytical description of the norm available for most ultrasymmetric spaces make them a very convenient framework to deal, in a general way, with those (very frequent) problems that simultaneously appear in the above-mentioned families of spaces. Unfortunately, this characterization does not always exist for those spaces that are very close to ℓ_∞ or to ℓ_1. See [28], [29], [30], [2] and [25] for more information on ultrasymmetric spaces.

We identify the ultrasymmetric spaces generated from $(\Lambda_\varphi, M_\varphi)$ as limiting interpolation spaces of the (ℓ_1, ℓ_∞) couple for an important part of the limiting cases in which Λ_φ and M_φ are very closed to ℓ_∞. This gives an analytical description of the norm of these limiting ultrasymmetric spaces that makes them easy to handle. To be precise, we say Λ_φ and M_φ are very close to ℓ_∞ if the dilation indices of the fundamental function φ satisfy $\pi_\varphi = \rho_\varphi = 0$.

Mathematics subject classification (2010): 46B70, 46E30, 47B38.
Keywords and phrases: Sequence spaces, ultrasymmetric spaces, interpolation theory, approximation spaces.
This work was supported by the Spanish Ministerio de Economí-a y Competitividad (MTM2013-42220-P) and Fundación Séneca de la Región de Murcia 19378/PI/14.

© ELEMENT, Zagreb

In the non-limiting case, $0 < \pi_\varphi \leqslant \rho_\varphi < 1$, the dilation indices provide very useful information about the rate of growth of the fundamental function. However, when $\pi_\varphi = \rho_\varphi = 0$, this information ceases to be available. This is a major setback when giving an analytical characterization of spaces whose fundamental function grows very slowly. Here, we develop a technique that allows us to consider a large family of r.i. spaces whose fundamental functions have null dilation indices. We show that an r.i. sequence space $\mathbf{g}$ in this family is ultrasymmetric if and only if

$$\|a\|_{\mathbf{g}} \sim \left\| \left(\varphi(\lambda_n) a^*_{\lambda_n}\right) \right\|_{\mathbf{e}}. \tag{1.1}$$

Here $\mathbf{e} = \mathscr{F}(\ell_1, \ell_\infty)$ and $\mathscr{F}$ is the interpolation functor that generates $\mathbf{g}$ from the $(\Lambda_\varphi, M_\varphi)$ couple. The sequence (λ_n) in $(0,\infty)$ is related to φ from equation (2.6) and Remark 2.6 below. We present a comprehensive study of this class of spaces with a detailed description of each of the elements in the characterization.

In order to illustrate the use of the above results and techniques we study approximation spaces generated by modelling the sequence of the approximation numbers in limiting ultrasymmetric sequence spaces. This approach is motivated by research by Pustylnik in 2005, see [29], and by the papers of DeVore, Riemenschenider and Sharpley [16], or Cobos and Resina [14] where the authors of the last paper use approximation spaces generated by modelling the approximation numbers in Lorentz-Zygmund spaces. The use of these techniques is an ongoing topic, as we can see in a recent paper of 2015 by Cobos and Dominguez [6]. For this limiting class of approximation spaces we obtain a representation theorem and prove that the class is stable under iteration. Finally, we give an interpolation formula for ordered couples of approximation spaces. The paper concludes with applications of our results to operator ideals.

Ultrasymmetric spaces with slowly varying fundamental function require special consideration due to several reasons of technical nature. The process of characterizing limiting ultrasymmetric spaces goes through the identification of the Lorentz space Λ_φ as an interpolation space for the (ℓ_1, ℓ_∞) couple. The previous techniques do not cover the limiting case $\pi_\varphi = \rho_\varphi = 0$. Besides this, the present approach assures the boundedness of Calderón operator (3.4) which is an essential element for our arguments.

Among the tools we had to develop it is worth mentioning the incorporation of discrete descriptions of limiting K and J interpolation methods. This approach proves to be optimal for obtaining an equivalence theorem between K and J methods. Unlike the continuous case where a logarithmic term is needed to adjust the interpolation parameters, see [12] and [25], we obtain equivalence of both methods with the same interpolation parameters. We also show that every space in [12] or [25] can be transformed to a discrete description that explains these differences. Finally, we also obtain a reiteration theorem that eventually will lead to the characterization of limiting ultrasymmetric spaces.

The organization of the paper is as follows. Section 2 contains the definitions and some basic facts of r.i. sequence spaces. Section 3 provides discrete descriptions of limiting K and J interpolation methods and the equivalence theorem. These results will be used in Section 4 which is devoted to limiting ultrasymmetric sequence spaces and includes the analytical characterization of these spaces. Finally, Section 5 uses the results to introduce a class of limiting approximation spaces and shows some applications.

We would like to thank Professor F. Cobos for some helpful advices. We would also like to show gratitude to Alba Segurado for her careful reading of the manuscript. Finally we want to thank the referee for the comments and suggestions that have definitely improved the final version of this paper.

2. Preliminaries

We begin with some notation. For a and b positive quantities, depending perhaps on some parameters or variables, we write $a \lesssim b$ to indicate that there exists a positive constant $C > 0$ such that $a \leqslant Cb$. Similarly $a \gtrsim b$ means that there exists a constant $C > 0$ such that $a \geqslant Cb$. The expression $a \sim b$ means $a \lesssim b$ and $a \gtrsim b$.

Rearrangement invariant spaces is a well-documented class of Banach spaces, see for example [3] and [26]. However, it is worth recalling some basic facts we will use in both the continuous and the discrete case.

Let E be an r.i. function space on $(0,\infty)$ with the Lebesgue measure. The fundamental function of E, φ_E, is defined as

$$\varphi_E(t) = \|\chi_{(0,t)}\|_E, \quad t > 0.$$

The behaviour of φ_E, and thereby some properties of the space E, is described by the lower and upper dilation indices of φ_E:

$$\pi_{\varphi_E} = \lim_{t\to 0} \frac{\log \overline{m}_{\varphi_E}(t)}{\log(t)} \quad \text{and} \quad \rho_{\varphi_E} = \lim_{t\to\infty} \frac{\log \overline{m}_{\varphi_E}(t)}{\log(t)}.$$

Here $\overline{m}_\varphi(t) = \sup_{s>0} \frac{\varphi(ts)}{\varphi(s)}$. Again we refer to [3] and [26] for more detailed information about the fundamental function and the dilation indices of a function φ.

The corresponding concepts for sequences spaces are absent in most monographs. Since this paper deals with sequence spaces defined on $\mathbb{N}$ with the counting measure, we recover some basic definitions in this context. An r.i. sequence space, sometimes called symmetric space, $(\mathbf{e}, \|\cdot\|_{\mathbf{e}})$ is a Banach space whose elements are sequences over $\mathbb{N}$ that satisfy:

1. If $a \in \mathbf{e}$ and $b \leqslant a$ then $b \in \mathbf{e}$ and $\|b\|_{\mathbf{e}} \leqslant \|a\|_{\mathbf{e}}$.
2. If a and b are equimeasurable,
$$m_a(k) = \sharp\{n : |a_n| > k\} = \sharp\{n : |b_n| > k\} = m_b(k) \quad \text{for all } k \in \mathbb{N},$$
then $\|b\|_{\mathbf{e}} = \|a\|_{\mathbf{e}}$.
3. The space $\mathbf{e}$ satisfies the Fatou property: if $(a_n) \nearrow a$ in $\mathbf{e}$, then $(\|a_n\|_{\mathbf{e}}) \nearrow \|a\|_{\mathbf{e}}$.

Given a sequence $(a_n)_{n\in\mathbb{N}}$, the decreasing rearrangement of $(a_n)_{n\in\mathbb{N}}$ is the sequence $(a_n^*)_{n\in\mathbb{N}}$, where

$$a_n^* = \inf\{k : m_a(k) \leqslant n\}.$$

Sequences (a_n) and (a_n^*) are equimeasurable, so $\|(a_n)\|_{\mathbf{e}} = \|(a_n^*)\|_{\mathbf{e}}$.

Let $(e_j)_{j\in\mathbb{N}}$ be the canonical sequence

$$e_j(n) = \begin{cases} 0 & \text{if } n \neq j \\ 1 & \text{for } n = j. \end{cases}$$

The fundamental function of the space $(\mathbf{e}, \|\cdot\|_{\mathbf{e}})$ is

$$\varphi_{\mathbf{e}}(0) = 0, \quad \text{and}$$
$$\varphi_{\mathbf{e}}(n) = \|e_1 + e_2 + \cdots + e_n\|_{\mathbf{e}}, \quad n \in \mathbb{N}.$$

Note that the sequence $\big(\varphi_{\mathbf{e}}(n)\big)_{n\in\mathbb{N}}$ is quasi-concave, that is, increases while the sequence $\big(\frac{\varphi_{\mathbf{e}}(n)}{n}\big)_{n\in\mathbb{N}}$ decreases.

Occasionally, we will use weighted sequence spaces. If $\mathbf{e}$ is a sequence space and $\delta : \mathbb{N} \longrightarrow (0,\infty)$ a weight sequence, we denote by $\mathbf{e}(\delta)$ the space of all those sequences (a_n) for which

$$\|(a_n)\|_{\mathbf{e}(\delta)} = \|(\delta(n)a_n)\|_{\mathbf{e}} < \infty.$$

Next we introduce the Lorentz and the Marcinkiewicz spaces.

DEFINITION 2.1. Let φ be a quasi-concave function on $\mathbb{N}\cup\{0\}$. The Lorentz space Λ_φ consists of all those scalar sequences $(a_n)_{n\in\mathbb{N}}$ for which the norm

$$\|a\|_{\Lambda_\varphi} = \sum_{n=1}^{\infty} a_n^* w_n = \|(a_n^* w_n)\|_{\ell_1} < \infty.$$

Here, $w_n = \varphi(n) - \varphi(n-1)$ for $n \geqslant 1$.

The space Λ_φ is an r.i. sequence space with fundamental function φ. The equalities

$$\|e_1 + e_2 + e_3 + \cdots + e_n\|_{\Lambda_\varphi} = \sum_{k=1}^{n} w_k = \varphi(n), \qquad n \in \mathbb{N},$$

show that φ is the fundamental function of Λ_φ. Furthermore, Λ_φ is the smallest symmetric space with fundamental function φ. Indeed, if $\mathbf{e}$ is a symmetric space with fundamental function φ and the Fatou property, for any $a \in \mathbf{e}$

$$\|a\|_{\mathbf{e}} = \|a^*\|_{\mathbf{e}} = \lim_n \|(a_k^*)_{k=1}^n\|_{\mathbf{e}}.$$

Now, for a fixed $n \in \mathbb{N}$ set $a_k^* = 0$ if $k > n$ and $F_k = \{e_1, e_2, \ldots, e_k\}$ for $1 \leqslant k \leqslant n$. The norm of the finite sequence $(a_k^*)_{k=1}^n$ satisfies the inequalities

$$\begin{aligned}\|(a_k^*)_{k=1}^n\|_{\mathbf{e}} &= \|\sum_{k=1}^n (a_k^* - a_{k+1}^*)\chi_{F_k}\|_{\mathbf{e}} \\ &\leqslant \sum_{k=1}^n (a_k^* - a_{k+1}^*)\varphi(k) \\ &= \sum_{k=1}^n a_k^*(\varphi(k) - \varphi(k-1)) \\ &= \sum_{k=1}^n a_k^* w_k.\end{aligned}$$

This shows that $\|a\|_{\mathbf{e}} \leqslant \|a\|_{\Lambda_\varphi}$ and establishes the embedding $\Lambda_\varphi \hookrightarrow \mathbf{e}$.

DEFINITION 2.2. Let φ be a quasi-concave function on $\mathbb{N}\cup\{0\}$. The Marcinkiewicz space M_φ consists of all those scalar sequences $(a_n)_{n\in\mathbb{N}}$ for which

$$\|a\|_{M_\varphi} = \sup_{n\in\mathbb{N}} \varphi(n) a_n^{**} < \infty$$

where $a_n^{**} = \frac{1}{n}\sum_{k=1}^n a_n^*$.

It is not difficult to prove that M_φ is the largest r.i. sequence space with fundamental function φ. See [3] and [26] for more information.

The rate of growth of the fundamental function φ is an important piece of information to progress in the study of r.i. spaces. See, for example, [26, p. 57]. We will use the dilation indices of such functions to describe their behaviour. The following definition recalls these concepts.

DEFINITION 2.3. Let $\varphi : \mathbb{N}\cup\{0\} \longrightarrow (0,\infty)$, and consider the functions

$$\underline{m}_\varphi(n) = \inf_{m\in\mathbb{N}} \frac{\varphi(nm)}{\varphi(m)}, \qquad \overline{m}_\varphi(n) = \sup_{m\in\mathbb{N}} \frac{\varphi(nm)}{\varphi(m)}.$$

The lower and upper index of the function φ are

$$\pi_\varphi = \lim_{n\to\infty} \frac{\log_2 \underline{m}_\varphi(n)}{\log_2(n)}, \qquad \rho_\varphi = \lim_{n\to\infty} \frac{\log_2 \overline{m}_\varphi(n)}{\log_2(n)}.$$

In this paper we are concerned with those r.i. sequence spaces whose fundamental function, φ, grows very slowly. In terms of the dilation indices $\pi_\varphi = \rho_\varphi = 0$. In these situations, the dilation indices do not describe the rate of growth of the function. This prevents us from applying known techniques to the study of these spaces. We develop a method to overcome this obstacle. Unfortunately, it does not apply to all functions φ with $\pi_\varphi = \rho_\varphi = 0$. We restrict ourselves to a set of functions $\mathscr{P} = \cup_{N\in\mathbb{N}}\mathscr{P}_N$, which is the infinite union of smaller classes of slowly varying functions that we define next. The description of the classes $\mathscr{P}_N$ requires the reiterated logarithms

$$\begin{cases} \ell(t) = L_1(t) = 1 + |\log_2 t|, & t > 0 \\ L_{n+1}(t) = \ell(L_n(t)), & t > 0 \text{ and } n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Let $N \in \mathbb{N}$ be fixed, we define the function

$$L(t) = L_1(t)L_2(t)\cdots L_N(t), \quad t > 0. \tag{2.2}$$

DEFINITION 2.4. We say that a positive function $\varphi : \mathbb{N} \cup \{0\} \longrightarrow [0,\infty)$ belongs to the family $\mathscr{P}_N$, $N \in \mathbb{N}$, if it satisfies the following properties:

a) φ is quasi-concave.

b) $\varphi(0) = 0$.

c)
$$w_n = \varphi(n) - \varphi(n-1) \sim \frac{\varphi(n)}{nL(n)}, \quad \text{for} \quad n \in \mathbb{N}. \tag{2.3}$$

REMARK 2.5. [1] Let $N \in \mathscr{P}_N$ be fixed. It is not difficult to check that a function $\varphi \in \mathscr{P}_N$ satisfies that

$$\varphi(n^2) \sim \varphi(n) \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

Moreover, there exist a function $\Phi : (0,\infty) \longrightarrow (0,\infty)$, with dilation indices $0 < \pi_\Phi \leqslant \rho_\Phi < \infty$, such that

$$\varphi(n) = \Phi\big(L_N(n)\big) \quad \text{for } n \in \mathbb{N}.$$

It must be said that in the last section of this paper we will use the extension of φ defined as

$$\varphi(t) = \Phi\big(L_N(t)\big) \quad \text{for } t > 1. \tag{2.5}$$

It is also convenient to have a partition of the interval $(1,\infty)$ which suits the functions of the class $\mathscr{P}_N$. To define such a partition, we use the sequence of functions

$$L_1^{-1}(s) = 2^{s-1}, \; s > 1$$
$$L_n^{-1}(s) = L_{n-1}^{-1}(2^{s-1}), \; s > 1 \text{ and } n \geqslant 2 \text{ in } \mathbb{N}.$$

These functions are inverse to functions L_n, $n \in \mathbb{N}$, on the interval $(1,\infty)$. Given a fixed $N \in \mathbb{N}$, the sequence $(\lambda_n)_{n \geqslant 0}$, where

$$\lambda_n = L_N^{-1}(2^n), \quad n \geqslant 0, \tag{2.6}$$

defines a partition on $[1,\infty)$ with the property that for any $\varphi \in \mathscr{P}_N$

$$\varphi(\lambda_n) \sim \varphi(t) \sim \varphi(\lambda_{n+1}) \quad \text{for } t \in [\lambda_n, \lambda_{n+1}], \tag{2.7}$$

with the equivalence constants being independent of $n \in \mathbb{N}$. This follows from (2.4).

[1] We thank the referee for suggesting the above definition and this remark.

REMARK 2.6. The sequence (λ_n) plays an important role in this paper. It is worth mentioning that, for any given $N \in \mathbb{N}$, $(\lambda_n)_{n\geqslant 0}$ is uniquely defined and consists only of natural numbers. Throughout the paper the function φ determines N by means of the relation $\varphi \in \mathscr{P}_N$.

We refer to [25] for more information on functions in the family $\mathscr{P} = \cup_{N=1}^{\infty}\mathscr{P}_N$. Other properties of the class $\mathscr{P}_N$ and the sequence $(\lambda_n)_n \geqslant 0$ are collected in the following lemma.

LEMMA 2.7. *Let $\varphi \in \mathscr{P}_N$, then for any $n \in \mathbb{N}$ the following inequalities hold:*

$$\sum_{k=0}^{n-1} \varphi(\lambda_k) \lesssim \varphi(\lambda_n), \qquad \sum_{k=n+1}^{\infty} \frac{\varphi(\lambda_k)}{\lambda_k} \lesssim \frac{\varphi(\lambda_n)}{\lambda_n L(\lambda_n)}, \tag{2.8}$$

$$\sum_{k=n}^{\infty} \frac{1}{\varphi(\lambda_k)} \lesssim \frac{1}{\varphi(\lambda_n)}, \qquad \sum_{k=0}^{n-1} \frac{\lambda_k}{\varphi(\lambda_k)} \lesssim \frac{\lambda_n}{\varphi(\lambda_n)L(\lambda_n)}. \tag{2.9}$$

Proof. The partition $(\lambda_k)_{k\geqslant 0}$ satisfies that $\int_{\lambda_k}^{\lambda_{k+1}} \frac{dt}{tL(t)} = \log 2$, for all $k \geqslant 0$. So, by Lemma 2.7 (i) from [25],

$$\varphi(\lambda_n) \gtrsim \int_0^{\lambda_n} \varphi(s)\frac{ds}{sL(s)} \geqslant \sum_{k=0}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} \varphi(s)\frac{ds}{sL(s)} \sim \sum_{k=0}^{n-1} \varphi(\lambda_k).$$

This proves the first inequality of (2.8). For the second inequality, notice that the function $\varphi(t)/tL(t)$, $t > 0$, has negative dilation indices. Hence, by using Corollary 4 of [26, p. 57],

$$\begin{aligned} \frac{\varphi(\lambda_n)}{\lambda_n L(\lambda_n)} &\sim \int_{\lambda_n}^{\infty} \frac{\varphi(s)}{s}\frac{ds}{sL(s)} = \sum_{k=n}^{\infty} \int_{\lambda_k}^{\lambda_{k+1}} \frac{\varphi(s)}{s}\frac{ds}{sL(s)} \\ &\gtrsim \sum_{k=n}^{\infty} \frac{\varphi(\lambda_{k+1})}{\lambda_{k+1}} = \sum_{k=n+1}^{\infty} \frac{\varphi(\lambda_k)}{\lambda_k}. \end{aligned}$$

Inequalities of equation (2.9) can likewise be proved. □

REMARK 2.8. It should be stressed that N may be any natural number, however it has to be chosen and fixed. This allows to define the function L, see (2.2), which depends on N but does not contain it explicitly.

3. Limiting K and J discrete spaces

In the next section we will characterize limiting ultrasymmetric sequence spaces by explicitly describing their norms. This will be done through precise identification of the interpolation methods that generate them from the (ℓ_1, ℓ_∞) couple. Before this we need to introduce discrete limiting K and J interpolation spaces. We begin by describing the sequence spaces we use as interpolation parameters. Let $\mathbf{e}$ be an r.i. sequence space, generated from the (ℓ_1, ℓ_∞) couple by the interpolation functor $\mathscr{F}$, that is to say,

$$\mathbf{e} = \mathscr{F}(\ell_1, \ell_\infty).$$

See [3] Theorem 3.2.12. We use $\widehat{\mathbf{e}}$ to denote the space $\mathscr{F}(\widehat{\ell_1}, \ell_\infty)$ that results from applying the same interpolation functor $\mathscr{F}$ that generates $\mathbf{e}$ to the $(\widehat{\ell_1}, \ell_\infty)$ couple, where $\widehat{\ell_1}$ is the space of all sequences $(a_n)_{n\in\mathbb{N}}$ such that

$$\|(a_n)_{n\in\mathbb{N}}\|_{\widehat{\ell_1}} = \sum_{k=1}^{\infty} \frac{|a_n|}{nL(n)} < \infty.$$

It is worth mentioning that the space $\widehat{\mathbf{e}}$ depends on N as the function L does. See Remark 2.8 above. Next lemma relates the norms of the spaces $\mathbf{e}$ and $\widehat{\mathbf{e}}$.

LEMMA 3.1. *Let $\varphi \in \mathscr{P}_N$ and let $\mathbf{e}$ be an r.i. sequence space. Then, for any positive decreasing sequence $(a_n)_{n\in\mathbb{N}}$*

$$\left\|\left(\varphi(n)a_n\right)_{n\geqslant 1}\right\|_{\widehat{\mathbf{e}}} \sim \left\|\left(\varphi(\lambda_n)a_{\lambda_n}\right)_{n\geqslant 0}\right\|_{\mathbf{e}} \tag{3.1}$$

with equivalence constant independent of $(a_n)_{n\in\mathbb{N}}$ and $\mathbf{e}$.

Proof. Let $a = (a_n)_{n\in\mathbb{N}}$ be a positive decreasing sequence. Consider the linear operator that maps a sequence $b = (b_n)_{n\in\mathbb{N}}$ into the sequence $((Tb)_n)_{n\in\mathbb{N}}$ where

$$(Tb)_n = b_{\lambda_k} \quad \text{if } \lambda_k \leqslant n < \lambda_{k+1}, \text{ for } k \geqslant 0.$$

Then, by (2.7)

$$\varphi(n)a_n \leqslant \varphi(\lambda_{k+1})a_{\lambda_k} \lesssim \varphi(\lambda_k)a_{\lambda_k} \leqslant (T\varphi(n)a_n)_n,$$

for all $n \in \mathbb{N}$ and $\lambda_k \leqslant n < \lambda_{k+1}$. Hence, the lattice property of the space $\widehat{\mathbf{e}}$ yields

$$\|(\varphi(n)a_n)_{n\geqslant 1}\|_{\widehat{\mathbf{e}}} \lesssim \|(T\varphi(n)a_n)_{n\in\mathbb{N}}\|_{\widehat{\mathbf{e}}}.$$

On the other hand, for the weight

$$\delta(n) = \begin{cases} 1 & \text{if } n = \lambda_k,\ k \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and any sequence $b = (b_n)_{n\in\mathbb{N}}$, we obtain the equality

$$\|b\|_{\mathbf{e}(\delta)} = \|(b_{\lambda_n})_{n\geqslant 0}\|_{\mathbf{e}}.$$

Thus, to prove $\|(\varphi(n)a_n)_{n\in\mathbb{N}}\|_{\widehat{\mathbf{e}}} \lesssim \left\|\left(\varphi(\lambda_n)a_{\lambda_n}\right)_{n\geqslant 0}\right\|_{\mathbf{e}}$ it suffices to show that the operator

$$T : \mathbf{e}(\delta) \longrightarrow \widehat{\mathbf{e}}$$

is bounded. We first check that T is bounded when $\mathbf{e} = \ell_1$. Let $b = (b_n)_{n\in\mathbb{N}} \in \ell_1(\delta)$,

$$\|Tb\|_{\widehat{\ell_1}} = \sum_{k\geqslant 0} \sum_{\lambda_k \leqslant n < \lambda_{k+1}} |b_{\lambda_k}| \frac{1}{nL(n)} \sim \sum_{k\geqslant 0} |b_{\lambda_k}| = \|b\|_{\ell_1(\delta)}.$$

The operator $T : \ell_\infty(\delta) \longrightarrow \ell(\infty)$ is clearly bounded and, thus, by the interpolation properties of the spaces $\mathbf{e}(\delta)$ and $\widehat{\mathbf{e}}$, the operator

$$T : \mathbf{e}(\delta) \longrightarrow \widehat{\mathbf{e}}$$

is bounded.

For the reverse inequality, consider the sequence transformation

$$R\big((b_n)_{n\in\mathbb{N}}\big) = \left(\sum_{\lambda_{n-1} \leqslant k < \lambda_n} \frac{b_k}{kL(k)} \right)_{n\in\mathbb{N}}.$$

Since $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence and φ is an increasing function satisfying that $\varphi(\lambda_n) \sim \varphi(\lambda_{n+1})$, we obtain the inequalities

$$\varphi(\lambda_n) a_{\lambda_n} \lesssim \varphi(\lambda_n) a_{\lambda_n} \sum_{\lambda_{n-1} \leqslant k < \lambda_n} \frac{1}{kL(k)} \leqslant \sum_{\lambda_{n-1} \leqslant k < \lambda_n} \frac{\varphi(k) a_k}{kL(k)}, \quad \text{for } n \geqslant 1.$$

Hence,

$$\left\| \big(\varphi(\lambda_n) a_{\lambda_n}\big)_{n\geqslant 0} \right\|_{\mathbf{e}} \leqslant \left\| \big(R(\varphi(n) a_n)\big)_{n\in\mathbb{N}} \right\|_{\mathbf{e}}.$$

Thus, to finish the proof it suffices to show that the operator

$$R : \widehat{\mathbf{e}} \longrightarrow \mathbf{e}$$

is bounded. A straightforward computation shows that $R : \widehat{\ell_1} \longrightarrow \ell_1$ is an isometry, while the boundedness of $R : \ell_\infty \longrightarrow \ell_\infty$ follows from the inequality

$$\sum_{n=\lambda_{k-1}}^{\lambda_k - 1} \frac{1}{nL(n)} \leqslant \int_{\lambda_{k-2}}^{\lambda_k} \frac{dt}{tL(t)} = 2\log 2.$$

This completes the proof. $\square$

Before defining the interpolation methods let us recall some basic definitions. Let $\overline{A} = (A_0, A_1)$ be a Banach couple, that is to say, A_0 and A_1 are Banach spaces that are continuously embedded in the same Hausdorff topological vector space. For $t > 0$, Peetre's K-functional of the element $a \in A_0 + A_1$ is

$$\begin{aligned} K(t,a) &= K(t,a;A_0,A_1) \\ &= \inf\Big\{ \|a_0\|_0 + t\|a_1\|_1;\ a = a_0 + a_1,\ a_i \in A_i,\ i = 0,1 \Big\}, \end{aligned}$$

while the J-functional for an element $a \in A_0 \cap A_1$ is

$$J(t,a) = J(t,a;A_0,A_1) = \max\big\{ \|a\|_{A_0}, t\|a\|_{A_1} \big\}, \quad t > 0.$$

We are now in a position to give the discrete definition of the interpolation spaces we use. We work with ordered couples ($A_0 \hookrightarrow A_1$), although the definitions and the subsequent results can be extended with no difficulty to the general case.

DEFINITION 3.2. Let $\overline{A} = (A_0, A_1)$ be an ordered Banach couple, $A_0 \hookrightarrow A_1$. Given $\mathbf{e}$, an r.i. sequence space and φ a function in $\mathscr{P}_N$, the space $\overline{A}^K_{1,\varphi,\widehat{\mathbf{e}}}$ consists of all those elements of A_1 for which

$$\|a\|^K_{1,\varphi,\widehat{\mathbf{e}}} = \left\| \left(\frac{\varphi(n)}{n} K(n,a) \right)_{n \in \mathbb{N}} \right\|_{\widehat{\mathbf{e}}} < \infty.$$

The space $\overline{A}^K_{1,\varphi,\widehat{\mathbf{e}}}$ is an intermediate space for the (A_0, A_1) couple which also has the interpolation property. See [21] for a proof of the continuous case. Next we introduce J interpolation spaces.

DEFINITION 3.3. Let $\overline{A} = (A_0, A_1)$ be an ordered Banach couple, $A_0 \hookrightarrow A_1$. Given $\mathbf{e}$, an r.i. sequence space, and a function $\varphi \in \mathscr{P}_N$, we define the space $\overline{A}^J_{1,\varphi,\widehat{\mathbf{e}}}$ as the set of all those elements $a \in A_1$ which can be represented as

$$a = \sum_{n \geqslant 0} u_n \qquad \text{with } u_n \in A_0 \cap A_1 \tag{3.2}$$

and

$$\left\| \left(\frac{\varphi(\lambda_n)}{\lambda_n} J(\lambda_n, u_n) \right)_{n \geqslant 0} \right\|_{\mathbf{e}} < \infty. \tag{3.3}$$

We equip this space with the norm

$$\|a\|^J_{1,\varphi,\widehat{\mathbf{e}}} = \inf \left\{ \left\| \left(\frac{\varphi(\lambda_n)}{\lambda_n} J(\lambda_n, u_n) \right)_{n \geqslant 0} \right\|_{\mathbf{e}} \right\},$$

where the infimum is taken over all representations of a satisfying (3.2) and (3.3).

The main result of this section shows that limiting K and J discrete interpolation spaces, as defined, coincide with equivalence of norms if we use the same interpolation parameters. The equivalence of J and K methods requires the boundedness of the Calderón operator

$$S\big((a_n)\big) = \Big(\sum_{s \geqslant 1} \min\{1, \tfrac{\lambda_n}{\lambda_s}\} |a_s| \Big)_{n \geqslant 1}$$

when it acts between the sequence spaces that model the norm of our interpolation space.

PROPOSITION 3.4. *The operator*

$$S : \mathbf{e}\Big(\frac{\varphi(\lambda_n)}{\lambda_n} \Big) \longrightarrow \mathbf{e}\Big(\frac{\varphi(\lambda_n)}{\lambda_n} \Big) \tag{3.4}$$

is bounded for any r.i. sequence space $\mathbf{e}$ *and any* $\varphi \in \mathscr{P}_N$.

Proof. We begin by showing that S is bounded when $\mathbf{e}=\ell_1$. Let $(a_n)_{n\geqslant 0}\in \ell_1\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)$, we use Lemma 2.7 to establish the inequalities

$$\begin{aligned}
\|S(a_n)\|_{\ell_1\left(\frac{\varphi(\lambda_n)}{\lambda_n}\right)} &= \sum_{n\geqslant 0}\frac{\varphi(\lambda_n)}{\lambda_n}\Big|\sum_{s\geqslant 0}\min\{1,\tfrac{\lambda_n}{\lambda_s}\}a_s\Big| \\
&\leqslant \sum_{n\geqslant 0}\frac{\varphi(\lambda_n)}{\lambda_n}\Big(\sum_{s=0}^{n-1}|a_s|+\sum_{s\geqslant n}\frac{\lambda_n}{\lambda_s}|a_s|\Big) \\
&= \sum_{s=0}^{\infty}\Big(\sum_{n>s}\frac{\varphi(\lambda_n)}{\lambda_n}\Big)|a_s| + \sum_{s=0}^{\infty}\Big(\sum_{n\leqslant s}\varphi(\lambda_n)\Big)\frac{|a_s|}{\lambda_s} \\
&\lesssim \sum_{s=0}^{\infty}\frac{\varphi(\lambda_s)}{\lambda_s L(\lambda_s)}|a_s| + \sum_{s=0}^{\infty}\frac{\varphi(\lambda_s)}{\lambda_s}|a_s| \\
&\lesssim \|(a_s)_{s\geqslant 0}\|_{\ell_1\left(\frac{\varphi(\lambda_s)}{\lambda_s}\right)}.
\end{aligned}$$

This proves that $S:\ell_1\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)\longrightarrow \ell_1\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)$ is bounded. Now we study the case $\mathbf{e}=\ell_\infty$. Let $(a_n)_{n\geqslant 0}\in \ell_\infty\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)$,

$$\begin{aligned}
&\|S(a_n)_{n\geqslant 0}\|_{\ell_\infty\left(\frac{\varphi(\lambda_n)}{\lambda_n}\right)} \\
&\quad\leqslant \sup_{n\in\mathbb{N}}\frac{\varphi(\lambda_n)}{\lambda_n}\sum_{s\geqslant 0}\min\{1,\tfrac{\lambda_n}{\lambda_s}\}|a_s| \\
&\quad\lesssim \|(a_n)_{n\geqslant 0}\|_{\ell_\infty\left(\frac{\varphi(\lambda_n)}{\lambda_n}\right)}\sup_{n\in\mathbb{N}}\Big\{\frac{\varphi(\lambda_n)}{\lambda_n}\Big(\sum_{s=0}^{n-1}\frac{\lambda_s}{\varphi(\lambda_s)}+\sum_{s\geqslant n}\frac{\lambda_n}{\varphi(\lambda_s)}\Big)\Big\} \\
&\quad\lesssim \|(a_n)_{n\geqslant 0}\|_{\ell_\infty\left(\frac{\varphi(\lambda_n)}{\lambda_n}\right)}.
\end{aligned}$$

The last inequality follows from Lemma 2.7. Since $\mathbf{e}$ is an exact interpolation space for the (ℓ_1,ℓ_∞) couple, we conclude that

$$S:\mathbf{e}\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)\longrightarrow \mathbf{e}\Big(\frac{\varphi(\lambda_n)}{\lambda_n}\Big)$$

is a bounded linear operator. □

Having proved the boundedness of the Calderón operator (3.4), we are in a position to establish the equivalence for limiting J and K discrete methods.

THEOREM 3.5. *Let $\overline{A}=(A_0,A_1)$ be an ordered Banach couple, $A_0\hookrightarrow A_1$. For any $\varphi\in\mathscr{P}_N$ and any r.i. sequence space $\mathbf{e}$,*

$$(A_0,A_1)^J_{1,\varphi,\widehat{\mathbf{e}}}=(A_0,A_1)^K_{1,\varphi,\widehat{\mathbf{e}}}.$$

Proof. Let $0 \neq a \in (A_0, A_1)^J_{1,\varphi,\widehat{\mathbf{e}}}$ and choose $a = \sum_{n \geqslant 0} a_n$ be a J-representation of a such that

$$\Big\| \Big(\frac{\varphi(\lambda_n)}{\lambda_n} J(\lambda_n, a_n) \Big)_{n \geqslant 0} \Big\|_{\mathbf{e}} \leqslant 2\|a\|^J_{1,\varphi,\widehat{\mathbf{e}}}.$$

Then, for any $m \geqslant 0$ we have

$$K(\lambda_m, a) \leqslant \sum_{n \geqslant 0} \min\big(1, \frac{\lambda_m}{\lambda_n}\big) J(\lambda_n, a_n).$$

Since the right-hand side of the previous inequality is precisely the m-th coordinate of the sequence $S\big(J(\lambda_n, a_n)\big)$, Lemma 3.1 and Proposition 3.4 yield

$$\begin{aligned}
\|a\|_{(A_0,A_1)^K_{1,\varphi,\widehat{\mathbf{e}}}} &\sim \|\big(K(\lambda_n, a)\big)\|_{\mathbf{e}(\frac{\varphi(\lambda_n)}{\lambda_n})} \\
&\leqslant \big\|S\big(J(\lambda_n, a_n)\big)\big\|_{\mathbf{e}(\frac{\varphi(\lambda_n)}{\lambda_n})} \\
&\lesssim \|S\| \big\|\big(J(\lambda_n, a_n)\big)\big\|_{\mathbf{e}(\frac{\varphi(\lambda_n)}{\lambda_n})} \\
&\lesssim \|S\| \|a\|^J_{1,\varphi,\widehat{\mathbf{e}}}.
\end{aligned}$$

This proves that $(A_0, A_1)^J_{1,\varphi,\widehat{\mathbf{e}}} \hookrightarrow (A_0, A_1)^K_{1,\varphi,\widehat{\mathbf{e}}}$.

To establish the reverse inclusion, let $a \in (A_0, A_1)^K_{1,\varphi,\widehat{\mathbf{e}}}$. Using the properties of the K-functional, choose decompositions

$$a = a_{n,0} + a_{n,1}, \quad a_{i,n} \in A_i \text{ for } i = 0, 1$$

such that $\|a_{n,0}\|_{A_0} + \lambda_n \|a_{n,1}\|_{A_1} \leqslant 2K(\lambda_n, a)$, for $n \in \mathbb{N} \cup \{0\}$. Set $u_0 = a_{0,0}$ and let

$$u_n = a_{0,n} - a_{0,n-1} = a_{1,n-1} - a_{1,n}, \qquad \text{for } n \geqslant 1.$$

The sequence $(u_n)_{n \geqslant 0}$ is in $A_0 \cap A_1$ and satisfies that $a = \sum_{n \geqslant 0} u_n$ in $A_0 + A_1$, which follows from

$$\Big\| a - \sum_{n=0}^{M} u_n \Big\|_{A_0 + A_1} = \|a_{1,M}\|_{A_0 + A_1} \leqslant 2 \frac{K(\lambda_M, a)}{\lambda_M} \to 0 \text{ as } M \to \infty.$$

See [25, Prop. 3.3]. This provides with a representation of the element a of the form (3.2) which we can use to estimate the norm $\|a\|^J_{1,\varphi,\widehat{\mathbf{e}}}$. In fact, we estimate the terms of the sequence $\big(J(\lambda_n, u_n)\big)_{n \geqslant 0}$. For $n = 0$, $J(\lambda_0, u_0) \leqslant K(\lambda_0, a)$, and for $n \geqslant 1$

$$\begin{aligned}
\frac{J(\lambda_n, u_n)}{\lambda_n} &\lesssim \max\Big\{ \frac{1}{\lambda_n} \|a_{0,n} - a_{0,n-1}\|_{A_0}, \|a_{1,n-1} - a_{1,n}\|_{A_1} \Big\} \\
&\leqslant \frac{1}{\lambda_n} \|a_{0,n}\|_{A_0} + \|a_{1,n}\|_{A_1} + \frac{1}{\lambda_{n-1}} \|a_{0,n-1}\|_{A_0} + \|a_{1,n-1}\|_{A_1} \\
&\lesssim \frac{K(\lambda_{n-1}, a)}{\lambda_{n-1}}.
\end{aligned}$$

Now, using that $\varphi(\lambda_n) \sim \varphi(\lambda_{n-1})$ for all $n \geqslant 1$, we derive

$$\Big\| \Big(\frac{\varphi(\lambda_n)}{\lambda_n} J(\lambda_n, u_n) \Big) \Big\|_{\mathbf{e}} \lesssim \Big\| \Big(\frac{\varphi(\lambda_n)}{\lambda_n} K(\lambda_n, a) \Big) \Big\|_{\mathbf{e}},$$

which establishes the embedding $(A_0, A_1)^K_{1,\varphi,\widehat{\mathbf{e}}} \hookrightarrow (A_0, A_1)^J_{1,\varphi,\widehat{\mathbf{e}}}$.

This finishes the proof. □

REMARK 3.6. The interpolation methods introduced in [12], [21] or [25] by means of continuous description, are related with the limiting methods we have just defined. In fact, for any Banach couple $\overline{A} = (A_0, A_1)$, any $\varphi \in \mathscr{P}_N$ and any r.i. function space E, the space $\overline{A}^K_{1,\varphi,\widehat{E}}$ of [21] or [23] coincides with the present space $\overline{A}^K_{1,\varphi,\widehat{\mathfrak{e}}}$ where $\widehat{\mathfrak{e}}$ stands for the discretization of the space $\widehat{E}$, where

$$\|(a_\mu)\|_{\widehat{\mathfrak{e}}} = \|\sum_\mu a_\mu \chi_{(\lambda_\mu, \lambda_{\mu+1})}\|_{\widehat{E}}. \tag{3.5}$$

The case for the J-spaces that appear in [12], [21] or [25] is slightly different. Indeed, it is not difficult to check that, with the above notation, for any $\varphi \in \mathscr{P}_N$ and any r.i. function space E

$$\overline{A}^J_{1,\varphi,\widehat{E}} = \overline{A}^J_{1,\frac{\varphi}{L},\widehat{\mathfrak{e}}}$$

with equivalence of norms. This fits perfectly with the fact that the equivalence results in [12], [21] or [25] have to add the logarithmic term L to obtain the equivalence result between J and K spaces.

We can now state the following reiteration theorem.

THEOREM 3.7. *Let $\mathscr{F}$ be an interpolation functor and $\overline{A} = (A_0, A_1)$ an ordered Banach couple, $A_0 \hookrightarrow A_1$. Then, for any $\varphi \in \mathscr{P}_N$ and any r.i. sequence space* $\mathbf{e}$

$$\mathscr{F}\Big((A_0, A_1)^K_{1,\varphi,\widehat{\ell_1}}, (A_0, A_1)^K_{1,\varphi,\ell_\infty} \Big) = (A_0, A_1)^K_{1,\varphi,\widehat{\mathbf{e}}}, \tag{3.6}$$

where $\mathbf{e} = \mathscr{F}(\ell_1, \ell_\infty)$.

The proof of the theorem uses Theorem 3.5 and then follows the usual techniques which we shall not reproduce here. See [25] for the continuous version of this theorem.

4. Limiting ultrasymmetric sequence spaces

Ultrasymmetric sequence spaces are the interpolation spaces for the couples $(\Lambda_\varphi, M_\varphi)$ of the Lorentz and the Marcinkiewicz spaces with the same fundamental functions. In this section we give an analytical description of these spaces in many of the limiting cases $\pi_\varphi = \rho_\varphi = 0$, or more precisely for $\varphi \in \mathscr{P}$. We begin with a result of boundedness for the Hardy operator.

PROPOSITION 4.1. *Let $\varphi : \mathbb{N} \longrightarrow (0,\infty)$ be a function with upper index $\rho_\varphi < 1$. Then, for any r.i. sequence space* $\mathbf{e}$, *the Hardy operator*

$$(\mathscr{H}a)_n = \frac{1}{n}\sum_{k=1}^{n} a_k$$

is bounded on the space $\widehat{\mathbf{e}}\big(\varphi(n)\big)$. In particular,

$$\big\|\big(\varphi(n)a_n^{**}\big)\big\|_{\widehat{\mathbf{e}}} \sim \big\|\big(\varphi(n)a_n^{*}\big)\big\|_{\widehat{\mathbf{e}}} \tag{4.1}$$

and

$$\big\|\big(\varphi(\lambda_n)a_{\lambda_n}^{**}\big)\big\|_{\mathbf{e}} \sim \big\|\big(\varphi(\lambda_n)a_{\lambda_n}^{*}\big)\big\|_{\mathbf{e}}. \tag{4.2}$$

Proof. We claim that the operators

$$\mathscr{H} : \widehat{\ell_1}\big(\varphi(n)\big) \longrightarrow \widehat{\ell_1}\big(\varphi(n)\big) \tag{4.3}$$

$$\mathscr{H} : \ell_\infty\big(\varphi(n)\big) \longrightarrow \ell_\infty\big(\varphi(n)\big) \tag{4.4}$$

are bounded. In order to study the first one, let $a = (a_n)_{n\in\mathbb{N}} \in \widehat{\ell_1}(\varphi)$, then

$$\begin{aligned}\|\mathscr{H}a\|_{\widehat{\ell_1}(\varphi)} &= \sum_{n=1}^{\infty} \frac{\varphi(n)}{n}\Big|\sum_{k=1}^{n} a_k\Big|\frac{1}{nL(n)} \\ &\leqslant \sum_{k=1}^{\infty} |a_k|\Big(\sum_{n=k}^{\infty}\frac{\varphi(n)}{n^2L(n)}\Big) \leqslant \sum_{k=1}^{\infty}|a_k|\frac{\varphi(k)}{kL(k)} = \|a\|_{\widehat{\ell_1}(\varphi)},\end{aligned}$$

since
$$\sum_{n=k}^{\infty}\frac{\varphi(n)}{n^2L(n)} \leqslant \frac{\varphi(k)}{k^2L(k)} + \int_k^{\infty}\frac{\varphi(t)}{tL(t)}\frac{dt}{t} \sim \frac{\varphi(k)}{k^2L(k)} + \frac{\varphi(k)}{kL(k)} \lesssim \frac{\varphi(k)}{kL(k)}$$
for $k \in \mathbb{N}$. See [26, p. 57]. This establishes the boundedness of (4.3).

To check that the operator (4.4) is bounded, let $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty(\varphi(n))$, then

$$\begin{aligned}\|\mathscr{H}a\|_{\ell_\infty(\varphi)} &= \sup_{n\in\mathbb{N}} \tfrac{\varphi(n)}{n}\Big|\sum_{k=1}^{n} a_k\Big| \\ &\leqslant \|a\|_{\ell_\infty(\varphi)} \sup_{n\in\mathbb{N}} \tfrac{\varphi(n)}{n}\Big(\sum_{k=1}^{n}\tfrac{1}{\varphi(k)}\Big) \\ &\leqslant \|a\|_{\ell_\infty(\varphi(n))}.\end{aligned}$$

Notice that $\sum_{k=1}^{n}\frac{1}{\varphi(k)} \leqslant \int_0^n \frac{1}{\varphi(t)}dt \sim \frac{n}{\varphi(n)}$, see again [26, p. 57]. The usual interpolation argument proves that $\mathscr{H}$ is bounded on $\widehat{\mathbf{e}}\big(\varphi(n)\big)$.

Finally, observing that $a_n^* \leqslant a_n^{**} = (\mathscr{H}a^*)_n$ for all $n \in \mathbb{N}$, we obtain equivalence (4.1). Equivalence (4.2) follows now from Lemma 3.1. □

The next proposition characterizes Lorentz and Marcinkiewicz sequence spaces as K-interpolation spaces for the (ℓ_1,ℓ_∞) couple.

PROPOSITION 4.2. *Let $\varphi \in \mathscr{P}_N$, then*

$$\Lambda_\varphi = (\ell_1, \ell_\infty)^K_{1,\varphi,\widehat{\ell}_1} \qquad \textit{and} \qquad M_\varphi = (\ell_1, \ell_\infty)^K_{1,\varphi,\ell_\infty}.$$

Proof. The proof of both equalities is a consequence of the previous results and the fact that $K(n,a;\ell_1,\ell_\infty) = na_n^{**}$, for all $n \in \mathbb{N}$. Indeed, by (2.3)

$$\|a\|_{\Lambda_\varphi} = \|(w_n a_n^*)\|_{\ell_1} \sim \left\| \left(\frac{\varphi(n)}{nL(n)} a_n^* \right) \right\|_{\ell_1}$$

$$\sim \left\| \left(\frac{\varphi(n)}{n} K(n,a;\ell_1,\ell_\infty) \right) \right\|_{\widehat{\ell}_1} = \|a\|^K_{1,\varphi,\widehat{\ell}_1}.$$

Similarly,

$$\|a\|_{M_\varphi} = \sup_n \varphi(n) a_n^{**} = \left\| \left(\frac{\varphi(n)}{n} K(n,a;\ell_1,\ell_\infty) \right) \right\|_{\ell_\infty} = \|a\|^K_{1,\varphi,\ell_\infty}. \quad \square$$

We are now in position to establish the main result of this section.

THEOREM 4.3. *Let $\mathbf{g}$ be an r.i. sequence space whose fundamental function φ belongs to $\mathscr{P}_N$, then*

$$\mathbf{g} \textit{ is ultrasymmetric if and only if } \mathbf{g} = (\ell_1, \ell_\infty)^K_{1,\varphi,\widehat{\mathbf{e}}}$$

for some r.i. sequence space $\mathbf{e}$. Moreover, if $\mathscr{F}$ is the interpolation method that generates $\mathbf{g}$ from the $(\Lambda_\varphi, M_\varphi)$ couple then $\mathbf{e} = \mathscr{F}(\ell_1, \ell_\infty)$.

Proof. Let $\mathscr{F}$ be the interpolation method that generates the ultrasymmetric space as $\mathbf{g} = \mathscr{F}(\Lambda_\varphi, M_\varphi)$. Then, by Proposition 4.2 and Theorem 3.7 we get

$$\mathbf{g} = \mathscr{F}(\Lambda_\varphi, M_\varphi) = \mathscr{F}\left((\ell_1, \ell_\infty)^K_{1,\varphi,\widehat{\ell}_1}, (\ell_1, \ell_\infty)^K_{1,\varphi,\ell_\infty} \right) = (\ell_1, \ell_\infty)^K_{1,\varphi,\widehat{\mathbf{e}}},$$

where $\widehat{\mathbf{e}} = \mathscr{F}(\widehat{\ell_1}, \ell_\infty)$. $\square$

We can now obtain formula (1.1) that characterizes ultrasymmetric sequence spaces whose fundamental function lies in $\mathscr{P}$.

COROLLARY 4.4. *Let $\mathbf{g}$ be an r.i. sequence space with fundamental function $\varphi \in \mathscr{P}_N$, for some $N \in \mathbb{N}$. Then, $\mathbf{g}$ is ultrasymmetric if and only if*

$$\|a\|_{\mathbf{g}} \sim \left\| \left(\varphi(\lambda_n) a^*_{\lambda_n} \right) \right\|_{\mathbf{e}} \tag{4.5}$$

where $\mathscr{F}$ is the interpolation functor that satisfies $\mathbf{g} = \mathscr{F}(\Lambda_\varphi, M_\varphi)$ and $\mathbf{e} = \mathscr{F}(\ell_1, \ell_\infty)$.

Proof. The previous theorem proves from (4.2) that an r.i. sequence space $\mathbf{g}$ is ultrasymmetric if and only if

$$\|a\|_{\mathbf{g}} = \|(\varphi(n) a_n^{**})_{n\in\mathbb{N}}\|_{\widehat{\mathbf{e}}} \sim \|(\varphi(n) a_n^*)_{n\in\mathbb{N}}\|_{\widehat{\mathbf{e}}}.$$

We now recover formula (4.5) from Lemma 3.1. $\square$

Subsequently, we will denote the ultrasymmetric space $\mathbf{g}$, whose norm depends only on φ and $\widehat{e}$, as $\ell_{\varphi,\widehat{e}}$.

5. Approximation spaces

In this section we present a class of approximation spaces obtained by modelling the sequence of the approximation numbers in a limiting ultrasymmetric space. Let us begin by introducing the concept of approximation family (some authors use "approximation scheme").

DEFINITION 5.1. Let A be a quasi-Banach space. By an approximation family in A we mean a sequence of subsets of A, $\{G_n\}_{n\geqslant 0}$, satisfying the following conditions:

1. $G_0 = \{0\}$ and $G_n \subset G_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$.
2. $\lambda G_n \subseteq G_n$ for any scalar λ and $n \in \mathbb{N}$.
3. $G_n + G_m \subset G_{m+n}$ for $m,n \in \mathbb{N}$.

The error of approximation of an element $f \in A$ by elements of G_{n-1} is given by

$$E_n(f) = E_n(f,A) = \inf\{\|f-g\|_A;\ g \in G_{n-1}\}.$$

Clearly, $(E_n(T))_n$ is a decreasing sequence and $E_1(T) = \|T\|_A$.

The classical approximation space A_p^α, $\alpha > 0$ and $0 < p \leqslant \infty$, is the set of all those elements $f \in A$ for which

$$\|f\|_{A_p^\alpha} = \|n^\alpha E_n(f,A)\|_{\widetilde{\ell}_p} < \infty. \tag{5.1}$$

Here $\widetilde{\ell}_p$ stands for the weighted Lebesgue sequence space $\ell_p(\frac{1}{n})$, $0 < p \leqslant \infty$. These spaces were defined by Butzer and Scherer in [5]. See also [4], [27], [13] and the references therein. Later on, Cobos and Resina introduce new spaces by modelling the approximation numbers in Lorentz-Zygmund spaces rather than in L_p-spaces, see [14]. In that paper they use, with a different notation, the approximation spaces $A_{\ell_q}^{\ell^\gamma}$ defined through the quasi-norm

$$\|f\|_{A_{\ell_q}^{\ell^\gamma}} = \|\ell(n)^\gamma E_n(f)\|_{\widetilde{\ell}_q},$$

where ℓ is the logarithmic function defined in (2.1). They were able to make the exponent $\alpha = 0$ in (5.1) at the expense of introducing a logarithmic power. See also [20]. In 2005 Pustylnik generalizes these results to a larger class of approximation spaces by modelling the sequence of approximation numbers in a wider class of sequence spaces, including Lebesgue ℓ_p spaces, Lorentz-Zygmund $\ell^{p,r}(\log\ell)^\alpha$ spaces and many others. More precisely, given $\ell_{\varphi,\mathbf{e}}$ a (non-limiting) ultrasymmetric sequence space, Pustylnik considers the approximation space $A_{\mathbf{e}}^\varphi$ formed by all those elements of A for which the quasi-norm

$$\|f\|_{A_{\mathbf{e}}^\varphi} = \|E_n(f)\|_{\ell_{\varphi,\mathbf{e}}}$$

is finite. See [30]. In a subsequent paper, [29], he uses a case of limiting ultrasymmetric spaces that allows the function φ to range in a wider class of functions. In this section we extend these results to a significantly larger family of limiting ultrasymmetric spaces by applying the techniques developed in the previous sections. This approach will provide a better insight of all the processes involved in using limiting ultrasymmetric spaces to define quasi-norms of approximation spaces. Let us begin with the definition of limiting ultrasymmetric approximation spaces. Before that, it is worth recalling that N will be a fixed natural number.

DEFINITION 5.2. Let $\varphi \in \mathscr{P}_N$, and let $\mathbf{e}$ be an r.i. sequence space. We define the limiting ultrasymmetric approximation space $A^{\varphi}_{\widehat{\mathbf{e}}}$ as the set of all those elements $f \in A$ for which

$$\|f\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} = \|(E_n(f))\|_{\ell_{\varphi,\widehat{\mathbf{e}}}} = \big\|\big(\varphi(n)E_n(f)\big)\big\|_{\widehat{\mathbf{e}}} < \infty.$$

5.1. The representation theorem

In this subsection we prove a representation theorem for approximation spaces $A^{\varphi}_{\widehat{\mathbf{e}}}$. The theorem generalizes known representation theorems. We begin with an auxiliary result.

LEMMA 5.3. *Let $\varphi \in \mathscr{P}_N$ and let a linear operator T be defined as*

$$T(a_n) = \Big(\sum_{k \geqslant n} a_k\Big)_{n\in\mathbb{N}}.$$

Then T is bounded in $\mathbf{e}\big(\varphi(\lambda_n)\big)$ for any r.i. sequence space $\mathbf{e}$.

Proof. We show the operator is bounded for $\mathbf{e} = \ell_1$ and for $\mathbf{e} = \ell_\infty$. The result follows from the fact that $\mathbf{e}$ is an exact interpolation space for the couple (ℓ_1, ℓ_∞). Choose $(a_n) \in \ell_1(\varphi(\lambda_n))$, then

$$\begin{aligned}\|T(a_n)\|_{\ell_1\left(\varphi(\lambda_n)\right)} &= \Big\|\Big(\varphi(\lambda_n)\sum_{k\geqslant n} a_k\Big)\Big\|_{\ell_1} \leqslant \sum_{n\geqslant 1}\varphi(\lambda_n)\sum_{k\geqslant n}|a_k| \\ &= \sum_{k=1}^{\infty}|a_k|\sum_{n\leqslant k}\varphi(\lambda_n) \lesssim \sum_{k=1}^{\infty}\varphi(\lambda_k)|a_k| \\ &= \|(a_n)\|_{\ell_1\left(\varphi(\lambda_n)\right)}.\end{aligned}$$

The last inequality follows from Lemma 2.7.

Now, let $(a_n) \in \ell_\infty(\varphi(\lambda_n))$, then

$$\begin{aligned}\|T(a_n)\|_{\ell_\infty\left(\varphi(\lambda_n)\right)} &= \sup_{n\in\mathbb{N}}\varphi(\lambda_n)\Big|\sum_{k\geqslant n} a_k\Big| \\ &\leqslant \|(a_n)\|_{\ell_\infty\left(\varphi(\lambda_n)\right)}\sup_{n\in\mathbb{N}}\Big\{\varphi(\lambda_n)\sum_{k\geqslant n}\frac{1}{\varphi(\lambda_k)}\Big\} \\ &\sim \|(a_n)\|_{\ell_\infty\left(\varphi(\lambda_n)\right)}.\end{aligned}$$

This concludes the proof. □

We are now in a position to establish the following representation theorem.

THEOREM 5.4. *Let $\varphi \in \mathscr{P}_N$ and let $\mathbf{e}$ be an r.i. sequence space. The following are equivalent:*

1. $f \in A^{\varphi}_{\widehat{\mathbf{e}}}$.

2. *There exists a sequence $(f_{\lambda_n})_{n\geqslant 0}$, with $f_{\lambda_n} \in G_{\lambda_n}$, such that*

$$f = \sum_{n=0}^{\infty} f_{\lambda_n} \in A \quad \textit{and} \quad \Big(\varphi(\lambda_n)\|f_{\lambda_n}\|_A\Big) \in \mathbf{e}.$$

Moreover

$$\|f\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \sim \inf\Big\{\Big\|\Big(\varphi(\lambda_n)\|f_{\lambda_n}\|_A\Big)\Big\|_{\mathbf{e}}\Big\}$$

where the infimum runs over all possible representations of f as above.

Proof. (1) $\Rightarrow$ (2) Let $f \in A^{\varphi}_{\widehat{\mathbf{e}}}$. For each $n \in \mathbb{N}\cup\{0\}$ choose $g_{\lambda_n} \in G_{\lambda_n}$ such that

$$\|f - g_{\lambda_n}\| \leqslant 2E_{1+\lambda_n}(f) \lesssim E_{\lambda_n}(f).$$

Put $f_{\lambda_{n+1}} = g_{\lambda_n} - g_{\lambda_{n-1}}$ for $n \geqslant 1$, $f_{\lambda_1} = g_{\lambda_0}$ and $f_{\lambda_0} = 0$. For this choice, $f_{\lambda_n} \in G_{\lambda_n}$ for $n \in \mathbb{N}$. Indeed, for $n \geqslant 2$, $f_{\lambda_n} \in G_{\lambda_n}$ since $G_{\lambda_n} + G_{\lambda_{n-1}} \subseteq G_{\lambda_{n+1}}$, while for $n = 0,1$ we have that

$$f_{\lambda_0} \in G_{\lambda_0} \quad \text{and} \quad f_{\lambda_1} \in G_{\lambda_0} \subset G_{\lambda_1}.$$

Thus,

$$\Big\|f - \sum_{k=0}^{n+1} f_{\lambda_k}\Big\|_A = \|f - g_{\lambda_n}\|_A \lesssim E_{\lambda_n}(f) \to 0 \quad \text{as } n \to \infty.$$

This proves that $f = \sum_{n=0}^{\infty} f_{\lambda_n}$ in A, with $f_{\lambda_n} \in G_{\lambda_n}$. In addition,

$$\|f_{\lambda_n}\|_A \leqslant \|f - g_{\lambda_{n-1}}\|_A + \|f - g_{\lambda_{n-2}}\|_A \leqslant 4E_{1+\lambda_{n-2}}(f) \lesssim E_{\lambda_{n-2}}(f), \quad \text{for } n \geqslant 2,$$

and therefore, by rearranging the sequence, we obtain

$$\Big\|\Big(\varphi(\lambda_n)\|f_{\lambda_n}\|_A\Big)_{n\geqslant 2}\Big\|_{\mathbf{e}} \lesssim \Big\|\Big(\varphi(\lambda_n)E_{\lambda_n}(f)\Big)_{n\geqslant 0}\Big\|_{\mathbf{e}}.$$

Besides, for $n = 0,1$ we have that

$$\varphi(\lambda_1)\|f_{\lambda_1}\|_A \,\|e_2\|_{\mathbf{e}} \leqslant \Big\|\Big(\varphi(\lambda_n)E_{\lambda_n}(f)\Big)_{n\geqslant 0}\Big\|_{\mathbf{e}}$$
$$\varphi(\lambda_0)\|f_{\lambda_0}\|_A \,\|e_1\|_{\mathbf{e}} \leqslant \Big\|\Big(\varphi(\lambda_n)E_{\lambda_n}(f)\Big)_{n\geqslant 0}\Big\|_{\mathbf{e}}.$$

Now, a simple use of the triangular inequality yields that

$$\Big\|\Big(\varphi(\lambda_n)\|f_{\lambda_n}\|_A\Big)_{n\geqslant 0}\Big\|_{\mathbf{e}} \lesssim \Big\|\Big(\varphi(\lambda_n)E_{\lambda_n}(f)\Big)_{n\geqslant 0}\Big\|_{\mathbf{e}}.$$

(2) $\Rightarrow$ (1) In order to prove the reverse inequality, recall that $G_{\lambda_0} + G_{\lambda_1} + \cdots + G_{\lambda_{n-1}} \subseteq G_{\lambda_n}$, therefore

$$E_{\lambda_{n+1}}(f) \leqslant \Big\| f - \sum_{k=0}^{n-1} f_{\lambda_k} \Big\|_A = \Big\| \sum_{k=n}^{\infty} f_{\lambda_k} \Big\|_A \leqslant \sum_{k=n}^{\infty} \|f_{\lambda_k}\|_A.$$

Hence, $\varphi(\lambda_{n+1}) E_{\lambda_{n+1}(f)} \leqslant \varphi(\lambda_n) \sum_{k=n}^{\infty} \|f_{\lambda_k}\|_A$, for $n \geqslant 1$, and

$$\Big\| \Big(\varphi(\lambda_n) E_{\lambda_n}(f) \Big)_{n\geqslant 2} \Big\|_{\mathbf{e}} \leqslant \Big\| \Big(\varphi(\lambda_n) \sum_{k=n}^{\infty} \|f_{\lambda_k}\|_A \Big)_{n\geqslant 1} \Big\|_{\mathbf{e}}.$$

For $n = 1$ we have that $E_{\lambda_1}(f) \leqslant \|f - g_{\lambda_0}\| \leqslant \sum_{n\geqslant 1} \|f_{\lambda_k}\|_A$ and so

$$\varphi(\lambda_1) E_{\lambda_1}(f) \leqslant \varphi(\lambda_1) \sum_{n\geqslant 1} \|f_{\lambda_k}\|_A,$$

while for $n = 0$, $\varphi(\lambda_0) E_{\lambda_0}(f) \leqslant \varphi(\lambda_1) \sum_{n\geqslant 0} \|f_{\lambda_k}\|_A$. Thus, arranging the sequence properly and using the triangular inequality, we obtain

$$\Big\| \Big(\varphi(\lambda_n) E_{\lambda_n}(f) \Big)_{n\geqslant 0} \Big\|_{\mathbf{e}} \leqslant \Big\| \Big(\varphi(\lambda_n) \sum_{k=n}^{\infty} \|f_{\lambda_k}\|_A \Big)_{n\geqslant 0} \Big\|_{\mathbf{e}}.$$

This, together with Lemma 5.3, yields $f \in A^{\varphi}_{\widehat{\mathbf{e}}}$ and

$$\|f\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \lesssim \inf \Big\{ \Big\| \Big(\varphi(\lambda_n) \|f_{\lambda_n}\|_A \Big) \Big\|_{\mathbf{e}} \Big\}. \quad \square$$

5.2. The iteration theorem

Let $\varphi \in \mathscr{P}_N$ and $\mathbf{e}$ be an r.i. sequence space. The approximation family for A, $\{G_n\}_{n\geqslant 0}$, is also an approximation family for the space $A^{\varphi}_{\widehat{\mathbf{e}}}$. Therefore, given any $\psi \in \mathscr{P}_N$ and any r.i. sequence space $\mathbf{f}$, we may consider the limiting ultrasymmetric approximation space for $A^{\varphi}_{\widehat{\mathbf{e}}}$

$$(A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}}.$$

We prove that this space can be obtained as an approximation space for A. Before we proceed to this result, it is desirable to remark that in order to avoid ambiguity we denote the approximation errors of an element f by $E_n(f,A)$ or $E_n(f,A^{\varphi}_{\widehat{\mathbf{e}}})$ depending on whether we refer to the approximation family $\{G_n\}_{n\geqslant 0}$ with respect to A or $A^{\varphi}_{\widehat{\mathbf{e}}}$. We will need the following Jackson type inequality; for $f \in A^{\varphi}_{\widehat{\mathbf{e}}}$

$$\varphi(\lambda_n) E_{\lambda_n}(f,A) \lesssim \|f\|_{A^{\varphi}_{\widehat{\mathbf{e}}}}, \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{5.2}$$

We will also use the Bernstein type inequality for the elements $f \in G_{\lambda_n}$:

$$\|f\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \lesssim \varphi(\lambda_n) \|f\|_A, \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{5.3}$$

Inequalities (5.2) and (5.3) follow from the representation theorem by means of similar arguments to those in [20].

REMARK 5.5. The approximation numbers of both approximation families can be compared. Indeed, they satisfy the inequality

$$\varphi(\lambda_n)E_{\lambda_{n+1}}(f,A) \lesssim E_{\lambda_n}(f,A^{\varphi}_{\widehat{\mathbf{e}}}).$$

Next we prove that the approximation process is stable under iteration.

THEOREM 5.6. *Let φ and ψ be slowly varying functions in $\mathscr{P}_N$, and $\mathbf{e}$, $\mathbf{f}$ be r.i. sequence spaces. Then,*

$$(A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}} = A^{\varphi\psi}_{\widehat{\mathbf{f}}}$$

with equivalence of norms.

Proof. Let $f \in (A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}}$. From Remark 5.5 and the embedding $A^{\varphi}_{\widehat{\mathbf{e}}} \hookrightarrow A$, we obtain the inequalities

$$\begin{aligned}\|f\|^{\varphi\psi}_{A_{\widehat{\mathbf{f}}}} &\sim |\varphi(\lambda_0)\psi(\lambda_0)E_{\lambda_0}(f,A)|\ \|e_1\|_{\mathbf{f}} + \Big\|\Big(\varphi(\lambda_{n+1})\psi(\lambda_{n+1})E_{\lambda_{n+1}}(f,A)\Big)_{n\geqslant 0}\Big\|_{\mathbf{f}}\\ &\lesssim |\psi(\lambda_0)E_{\lambda_0}(f,A^{\varphi}_{\widehat{\mathbf{e}}})|\ \|e_1\|_{\mathbf{f}} + \Big\|\Big(\varphi(\lambda_{n+1})\psi(\lambda_{n+1})E_{\lambda_{n+1}}(f,A^{\varphi}_{\widehat{\mathbf{e}}})\Big)_{n\geqslant 0}\Big\|_{\mathbf{f}}\\ &\lesssim \|f\|_{(A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}}}\end{aligned}$$

In order to prove the reverse embedding, let $f \in A^{\varphi\psi}_{\widehat{\mathbf{f}}}$. Following Theorem 5.4, we choose an arbitrary representation $f = \sum_{n=0}^{\infty} f_{\lambda_n} \in A$ with $f_{\lambda_n} \in G_{\lambda_n}$ and

$$\Big\|\Big(\varphi(\lambda_n)\psi(\lambda_n)\|f_{\lambda_n}\|_A\Big)_{n\geqslant 0}\Big\|_{\mathbf{f}} \lesssim \|f\|_{A^{\varphi\psi}_{\widehat{\mathbf{f}}}} < \infty.$$

Besides, by Bernstein Inequality (5.3)

$$\|f_{\lambda_n}\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \lesssim \varphi(\lambda_n)\|f_{\lambda_n}\|_A \lesssim \frac{1}{\psi(\lambda_n)}\|f_{\lambda_n}\|_{A^{\varphi}_{\widehat{\mathbf{f}}}}.$$

Hence,

$$\Big\|f - \sum_{k=0}^{n} f_{\lambda_k}\Big\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \leqslant \sum_{k=n+1}^{\infty} \|f_{\lambda_k}\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \lesssim \|f\|_{A^{\varphi\psi}_{\widehat{\mathbf{f}}}} \cdot \sum_{k=n+1}^{\infty} \frac{1}{\psi(\lambda_n)}$$

and from Lemma 2.7, we obtain that

$$\Big\|f - \sum_{k=0}^{n} f_{\lambda_k}\Big\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} \lesssim \frac{1}{\psi(\lambda_{n+1})}\|f\|_{A^{\varphi\psi}_{\widehat{\mathbf{f}}}} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $f = \sum_{n=0}^{\infty} f_{\lambda_n}$ with convergence in $A^{\varphi}_{\widehat{\mathbf{e}}}$ and

$$\Big\|\Big(\psi(\lambda_n)\|f_{\lambda_n}\|_{A^{\varphi}_{\widehat{\mathbf{e}}}}\Big)_{n\geqslant 0}\Big\|_{\mathbf{f}} \lesssim \Big\|\Big(\varphi(\lambda_n)\psi(\lambda_n)\|f_{\lambda_n}\|_A\Big)_{n\geqslant 0}\Big\|_{\mathbf{f}} < \infty.$$

This proves that $f \in (A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}}$ with $\|f\|_{(A^{\varphi}_{\widehat{\mathbf{e}}})^{\psi}_{\widehat{\mathbf{f}}}} \lesssim \|f\|_{A^{\varphi\psi}_{\widehat{\mathbf{f}}}}$ and concludes the proof. □

5.3. Interpolation of approximation spaces

Let $\varphi \in \mathscr{P}_N$ and let $\mathbf{e}$ be an r.i. sequence space. Since $A^{\varphi}_{\widehat{\mathbf{e}}} \hookrightarrow A$, the pair $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ is an ordered interpolation couple. Next, we identify the space generated by applying interpolation methods defined by a functional parameter and an r.i. function space to the $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ couple. Let us briefly recall the definition of these interpolation methods.

DEFINITION 5.7. Let $\overline{A} = (A_0, A_1)$ be a Banach couple. Consider an r.i. function space F on $(0,\infty)$ with the Lebesgue measure and a function parameter $\psi : (0,\infty) \longrightarrow (0,\infty)$ with dilation indices satisfying

$$-1 < \pi_{\psi} \leqslant \rho_{\psi} < 0. \tag{5.4}$$

By $(A_0,A_1)^K_{\psi,F}$ we denote the space of all elements f in $A_0 + A_1$ for which

$$\|f\|_{(A_0,A_1)^K_{\psi,F}} = \|\psi(t)K(t,f)\|_{\widetilde{F}} < \infty.$$

Here $\widetilde{F} = \mathscr{F}(L_1(\frac{dt}{t}), L_\infty)$ and $\mathscr{F}$ is the interpolation functor that generates F from the (L_1, L_∞) couple, see [3].

We will show that the interpolation space $\left(A^{\varphi}_{\widehat{\mathbf{e}}}, A\right)^K_{\psi,F}$ is an approximation space for A. We begin with an estimate for the K-functional of the $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ couple.

PROPOSITION 5.8. *For all* $f \in A^{\varphi}_{\widehat{\mathbf{e}}}$ *and* $n \in \mathbb{N}$

$$\varphi(\lambda_n)E_{\lambda_n}(f) \lesssim K\Big(\varphi(\lambda_n), f; A^{\varphi}_{\widehat{\mathbf{e}}}, A\Big) \lesssim \sum_{k=0}^{n} \varphi(\lambda_k)E_{\lambda_k}(f). \tag{5.5}$$

Proof. Let $h \in A^{\varphi}_{\widehat{\mathbf{e}}}$, from Jackson inequality (5.2) we get that

$$\begin{aligned} E_{\lambda_n}(f) &\leqslant \|f-h\|_A + E_{\lambda_n}(h,A) \\ &\leqslant \|f-h\|_A + \frac{1}{\varphi(\lambda_n)}\|h\|_{A^{\varphi}_{\widehat{\mathbf{e}}}}. \end{aligned}$$

Now, taking infimum over all $h \in A^{\varphi}_{\widehat{\mathbf{e}}}$ we obtain

$$\varphi(\lambda_n)E_{\lambda_n}(f) \leqslant K\Big(\varphi(\lambda_n), f; A^{\varphi}_{\widehat{\mathbf{e}}}, A\Big).$$

To prove the second inequality of (5.5), choose functions $g_n \in G_{\lambda_n - 1}$ such that $\|f - g_n\|_A \leqslant 2E_{\lambda_n}(f)$ for $n \geqslant 0$, and put

$$f_n = g_{n-1} - g_{n-2} \qquad \text{for } n \geqslant 2.$$

Clearly, $g_n = \sum_{k=2}^{n+1} f_k$, and for $k \geqslant 2$

$$\|f_k\|_A \leqslant \|f - g_{k-1}\|_A + \|f - g_{k-2}\|_A \leqslant 4E_{\lambda_{k-2}}(f).$$

Hence, using Bernstein inequality (5.3),

$$
\begin{aligned}
K\Big(\varphi(\lambda_n), f; A^{\varphi}_{\widehat{\mathbf{e}}}, A\Big) &\leqslant \|g_n\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} + \varphi(\lambda_n)\|f-g_n\|_A \\
&\lesssim \sum_{k=2}^{n+1} \|f_k\|_{A^{\varphi}_{\widehat{\mathbf{e}}}} + \varphi(\lambda_n)E_{\lambda_n}(f) \\
&\lesssim \sum_{k=2}^{n+1} \varphi(\lambda_k)\|f_k\|_A + \varphi(\lambda_n)E_{\lambda_n}(f) \\
&\lesssim \sum_{k=2}^{n+1} \varphi(\lambda_k)E_{\lambda_{k-2}}(f) + \varphi(\lambda_n)E_{\lambda_n}(f) \\
&\lesssim \sum_{k=0}^{n} \varphi(\lambda_k)E_{\lambda_k}(f).
\end{aligned}
$$

This concludes the proof. □

Some considerations are required before establishing the interpolation formula for the $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ couple. First we recall that $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ is an ordered couple, $A^{\varphi}_{\widehat{\mathbf{e}}} \hookrightarrow A$. Therefore, to interpolate by the real K-method it suffices to integrate on the interval $(1,\infty)$. More precisely, for any ψ as in (5.4) and any r.i. space F

$$
\|f\|_{(A^{\varphi}_{\widehat{\mathbf{e}}}, A)^K_{\psi,F}} \sim \|\psi(t)K(t,f)\|_{\widetilde{F}(1,\infty)}.
$$

A second fact worth mentioning is that we may assume without loss of generality that $\varphi(1) = 1$. This and the hypothesis $\varphi \in \mathscr{P}_N$ yield the function

$$
\varphi : (1,\infty) \longrightarrow (1,\infty),
$$

with $\varphi(t) = \Phi(L_N(t))$ as in (2.5), defines a measure preserving transformation (up to equivalences) in the sense of [3, Def. 7.1, p. 80] between the measure spaces $\big((1,\infty), \frac{dt}{tL(t)}\big)$ and $\big((1,\infty), \frac{dt}{t}\big)$. Therefore, any measurable function f on $\big((1,\infty), \frac{dt}{t}\big)$ is equimeasurable with the function $f\circ\varphi$ on $\big((1,\infty), \frac{dt}{tL(t)}\big)$. This, in particular, implies that for any rearrangement invariant space F and any measurable function f the equivalence of norms

$$
\|f\circ\varphi\|_{\widehat{F}(1,\infty)} \sim \|f\|_{\widetilde{F}(1,\infty)} \tag{5.6}
$$

holds. It is also convenient to comment that since $\varphi(\lambda_n) \sim \varphi(\lambda_{n+1})$ and the function ψ satisfies (5.4) (ψ decreases, while $t\psi(t)$, $t > 0$, increases), then

$$
\psi(\varphi(\lambda_n)) \sim \psi(\varphi(\lambda_{n+1})) \tag{5.7}
$$

for all $n \in \mathbb{N}$.

The proof of the interpolation formula requires the discrete Hardy type inequality we collect in Lemma 5.10. First, we need to establish some estimates.

LEMMA 5.9. *Let $\varphi \in \mathscr{P}_N$, ψ as in (5.4) and $(\lambda_n)_{n\geqslant 0}$ defined by equation (2.6). Then for $n \in \mathbb{N}$*

$$\sum_{k\geqslant n} \psi(\varphi(\lambda_k)) \sim \psi(\varphi(\lambda_n)), \tag{5.8}$$

$$\sum_{k=0}^{n} \frac{1}{\psi(\varphi(\lambda_k))} \sim \frac{1}{\psi(\varphi(\lambda_n))}. \tag{5.9}$$

Proof. To prove (5.8), recall that $\psi\big(\varphi(t)\big) = \psi \circ \Phi(L_N(t))$, $t > 0$, and $\psi \circ \Phi$ has negative dilation indices ($\rho_\Psi < 0$). Then, by (5.7),

$$\begin{aligned} \psi(\varphi(\lambda_n)) &\sim \int_{\lambda_n}^{\infty} (\psi\circ\varphi)(t)\frac{dt}{tL(t)} \\ &= \sum_{k=n}^{\infty} \int_{\lambda_k}^{\lambda_{k+1}} (\psi\circ\varphi)(t)\frac{t}{tL(t)} \\ &\sim \sum_{k=n}^{\infty} \psi(\varphi(\lambda_k)). \end{aligned}$$

Equivalence (5.9) can be proved with similar arguments. □

LEMMA 5.10. *Let $\varphi \in \mathscr{P}_N$ and ψ as in (5.4), then for any r.i. sequence space $\mathbf{f}$ we have*

$$\left\|\left(\psi(\varphi(\lambda_n)) \sum_{k=0}^{n} \varphi(\lambda_k)a_k\right)\right\|_{\mathbf{f}} \lesssim \left\|\Big(\psi(\varphi(\lambda_n))\varphi(\lambda_n)a_n\Big)\right\|_{\mathbf{f}}.$$

Proof. We prove the inequality for $\mathbf{f} = \ell_1$ and for $\mathbf{f} = \ell_\infty$. Consider the sequence transformation

$$T\Big((a_n)_{n\geqslant 0}\Big) = \left(\sum_{k=0}^{n} \varphi(\lambda_k)a_k\right)_{n\geqslant 0}.$$

We claim the operator $T : \ell_1\big(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\big) \longrightarrow \ell_1\big(\psi(\varphi(\lambda_n))\big)$ is bounded. Let $(a_n)_{n\geqslant 0} \in \ell_1\big(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\big)$. Then, by Lemma 5.9,

$$\begin{aligned} \left\|\left(\sum_{k=0}^{n} \varphi(\lambda_k)a_k\right)\right\|_{\ell_1\left(\psi(\varphi(\lambda_n))\right)} &= \sum_{n=0}^{\infty} \psi(\varphi(\lambda_n)) \sum_{k=0}^{n} \varphi(\lambda_k)a_k \\ &= \sum_{k=0}^{\infty} \varphi(\lambda_k)a_k \sum_{n\leqslant k} \psi(\varphi(\lambda_n)) \\ &\sim \sum_{k=0}^{\infty} \psi(\varphi(\lambda_k))\varphi(\lambda_k)a_k. \end{aligned}$$

For $(a_n)_{n\geqslant 0} \in \ell_\infty(\psi(\varphi(\lambda_n)))$ Lemma 5.9 yields that

$$\begin{aligned} \|T(a_n)\|_{\ell_\infty(\psi(\varphi(\lambda_n)))} &= \sup_{n\geqslant 0} \psi(\varphi(\lambda_n)) \sum_{k=0}^{n} \varphi(\lambda_k) a_k \\ &\leqslant \sup_{n\geqslant 0} \|(a_n)\|_{\ell_\infty\left(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\right)} \sum_{k=0}^{n} \frac{\psi(\varphi(\lambda_n))}{\psi(\varphi(\lambda_k))} \\ &\sim \|(a_n)\|_{\ell_\infty\left(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\right)}. \end{aligned}$$

This shows the operator $T : \ell_\infty\big(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\big) \longrightarrow \ell_\infty\big(\psi(\varphi(\lambda_n))\big)$ is also bounded.

Now, since $\mathbf{f}$ is an exact interpolation space for the (ℓ_1, ℓ_∞) couple, we conclude that the operator $T : \mathbf{f}\big(\psi(\varphi(\lambda_n))\varphi(\lambda_n)\big) \longrightarrow \mathbf{f}\big(\psi(\varphi(\lambda_n))\big)$ is bounded, which completes the proof. $\square$

Next we establish an interpolation formula for the $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$ couple.

THEOREM 5.11. *Let A be a Banach space, $\mathbf{e}$ an r.i. space and $\varphi \in \mathscr{P}_N$. Consider the ordered couple $(A^{\varphi}_{\widehat{\mathbf{e}}}, A)$. Then, for any r.i. space F and any function ψ as in* (5.4)

$$\big(A^{\varphi}_{\widehat{\mathbf{e}}}, A\big)^{K}_{\psi,F} = A^{\Phi}_{\widehat{\mathfrak{f}}},$$

where $\mathfrak{f}$ stands for the discretization of F, see (3.5)*, and $\Phi(t) = \psi(\varphi(t))\varphi(t)$, for $t > 0$.*

Proof. Let $f \in A^{\Phi}_{\widehat{\mathfrak{f}}}$, then by (5.5) and (5.6) we get that

$$\begin{aligned} \|f\|_{A^{\Phi}_{\widehat{\mathfrak{f}}}} &= \Big\| \Big(\psi(\varphi(\lambda_n))\varphi(\lambda_n) E_{\lambda_n}(f, A) \Big)_{n\geqslant 0} \Big\|_{\mathfrak{f}} \\ &\leqslant \Big\| \Big(\psi(\varphi(\lambda_n)) K\big(\varphi(\lambda_n), f\big) \Big)_{n\geqslant 0} \Big\|_{\mathfrak{f}} \\ &\sim \big\| \psi(\varphi(t)) K(\varphi(t), f) \big\|_{\widehat{F}(1,\infty)} \\ &\sim \big\| \psi(t) K(t, f) \big\|_{\widetilde{F}(1,\infty)}. \end{aligned}$$

For the reverse inequality we use equations (5.5), (5.6), (5.7), Lemma 3.1 and (3.5) to obtain that for any $f \in \big(A^{\varphi}_{\widehat{\mathbf{e}}}, A\big)^{K}_{\psi,F}$

$$\begin{aligned} \|f\|_{\left(A^{\varphi}_{\widehat{\mathbf{e}}}, A\right)^{K}_{\psi,F}} &= \big\| \psi(t) K(t, f) \big\|_{\widetilde{F}(1,\infty)} \\ &\sim \Big\| \psi(\varphi(t)) K\big(\varphi(t), f\big) \Big\|_{\widehat{F}(1,\infty)} \\ &= \Big\| \sum_{n\geqslant 0} \psi(\varphi(t)) K\big(\varphi(t), f\big) \chi_{(\lambda_n, \lambda_{n+1})} \Big\|_{\widehat{F}(1,\infty)} \\ &\sim \Big\| \sum_{n\geqslant 0} \psi(\varphi(\lambda_n)) K\big(\varphi(\lambda_n), f\big) \chi_{(\lambda_n, \lambda_{n+1})} \Big\|_{\widehat{F}(1,\infty)} \end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(\psi(\varphi(\lambda_n)) K(\varphi(\lambda_n), f) \right)_{n \geqslant 0} \right\|_{\mathfrak{f}} \\
&\lesssim \left\| \left(\psi(\varphi(\lambda_n)) \sum_{k=0}^{n} \varphi(\lambda_k) E_{\lambda_k}(f, A) \right)_{n \geqslant 0} \right\|_{\mathfrak{f}} \\
&\lesssim \left\| \left(\psi(\varphi(\lambda_n)) \varphi(\lambda_n) E_{\lambda_n}(f, A) \right)_{n \geqslant 0} \right\|_{\mathfrak{f}} \\
&= \|f\|_{A^{\Phi}_{\widehat{\mathfrak{f}}}}.
\end{aligned}
$$

This concludes the proof. □

5.4. Applications

Let X, Y be Banach spaces, and let $\mathscr{L}(X,Y)$ be the Banach space of all bounded linear operators from X to Y. Put

$$G_n = \{R \in \mathscr{L}(X,Y) \text{ with } \operatorname{rank}(\mathrm{R}) \leqslant \mathrm{n}\}.$$

The family $\{G_n\}_{n \geqslant 0}$ is an approximation scheme for $\mathscr{L}(X,Y)$. In this case, the approximation errors for an operator T coincide with the approximation numbers of T,

$$E_n(T) = a_n(T) = \inf\left\{ \|T - R\|_{\mathscr{L}(X,Y)}; \text{ with } \operatorname{rank}(\mathrm{R}) \leqslant \mathrm{n} \right\}.$$

Given a function $\varphi \in \mathscr{P}_N$, for some $N \in \mathbb{N}$, and an r.i. sequence space $\mathbf{f}$, the approximation space $\left(\mathscr{L}(X,Y)\right)^{\varphi}_{\widehat{\mathbf{f}}}$ coincides with the operator ideal $\mathscr{L}(X,Y)_{\varphi, \widehat{\mathbf{f}}}$ defined as

$$\mathscr{L}_{\varphi, \widehat{\mathbf{f}}} = \left\{ T \in \mathscr{L}(X,Y); \text{ with } \|\varphi(n) a_n(T)\|_{\widehat{\mathbf{f}}} < \infty \right\}.$$

We begin with an example that shows that our representation theorem, Thm. 5.4, includes that of Cobos and Resina in [14] as a particular case.

EXAMPLE 5.12. Choose $\varphi(n) = \ell(n)^{\gamma + 1/q}$, $n \in \mathbb{N}$, and $\mathbf{e} = \ell_q$. The limiting ultrasymmetric approximation space $\left(\mathscr{L}(X,Y)\right)^{\varphi}_{\widehat{\mathbf{e}}}$ coincides with the operator ideal $\mathscr{L}_{\ell^{\gamma+1/q}, \widehat{\ell_q}}(X,Y)$, $\mathscr{L}_{\infty, q, \gamma}$ in the notation of [14]. In order to apply Theorem 5.4, recall that in the present setting $N = 1$, and therefore $\lambda_n = \frac{1}{2} 2^{2^n}$, then

$$T \in \left(\mathscr{L}(X,Y)\right)^{\ell(n)^{\gamma+1/q}}_{\widehat{\ell_q}} \Leftrightarrow T = \sum_{n=0}^{\infty} T_{\lambda_n}, \text{ with } T_{\lambda_n} \in G_{\lambda_n} \text{ and}$$
$$\left\| \left(\ell(\lambda_n)^{\gamma+1/q} \|T_{\lambda_n}\|_{\mathscr{L}(X,Y)} \right) \right\|_{\ell_q} < \infty.$$

That is to say

$$T \in \mathscr{L}_{\infty, q, \gamma} \Leftrightarrow T = \sum_{n=0}^{\infty} T_{\lambda_n}, \text{ with } \operatorname{rank}(T_{\lambda_n}) < \lambda_n \text{ and}$$
$$\left(\sum_{n=0}^{\infty} \left(2^{n(\gamma+1/q)} \|T_{\lambda_n}\|_{\mathscr{L}(X,Y)} \right)^q \right)^{1/q} < \infty,$$

which is precisely the representation theorem that Cobos and Resina established in [14].

The next two theorems establish interpolation formulas for operator ideals.

THEOREM 5.13. *Let $\varphi \in \mathscr{P}_N$ and let* $\mathbf{e}$ *be an r.i. sequence space. Then, for any* $\Psi : (0,\infty) \longrightarrow (0,\infty)$ *satisfying* (5.4) *and any r.i. function space* F,

$$\mathscr{L}_{\Phi,\widehat{\mathfrak{f}}}(X,Y) = \Big(\mathscr{L}_{\varphi,\widehat{\mathbf{e}}}(X,Y), \mathscr{L}(X,Y)\Big)^K_{\Psi,F}$$

where $\Phi(t) = \Psi(\varphi(t))\varphi(t)$, $t > 0$, *and* $\mathfrak{f}$ *is the discretization of* F.

Proof. It suffices to apply Theorem 5.11 to obtain

$$\begin{aligned}\Big(\mathscr{L}_{\varphi,\widehat{\mathbf{e}}}(X,Y), \mathscr{L}(X,Y)\Big)^K_{\Psi,F} &= \Big((\mathscr{L}(X,Y))^{\varphi}_{\widehat{\mathbf{e}}}, \mathscr{L}(X,Y)\Big)^K_{\Psi,F} \\ &= \Big(\mathscr{L}(X,Y)\Big)^{\Phi}_{\widehat{\mathfrak{f}}} \\ &= \mathscr{L}_{\Phi,\widehat{\mathfrak{f}}}(X,Y).\end{aligned}$$

This completes the proof. $\square$

We are now in a position to obtain more general interpolation formulas for couples of operator ideals. To do this we need a more precise setting.

Let $\Psi, \Psi_0, \Psi_1 : (0,\infty) \longrightarrow (0,\infty)$ be functions satisfying that

$$-1 < \pi_\Psi = \rho_\Psi < 0 \tag{5.10}$$

$$-1 < \pi_{\Psi_i} = \rho_{\Psi_i} < 0, \quad i = 0,1, \text{ and } \pi_{\Psi_0} < \pi_{\Psi_1}. \tag{5.11}$$

Under these conditions for Ψ, the interpolation method $(\cdot)^K_{\Psi,F}$ introduced in Definition 5.7 coincides with the interpolation method studied by the present authors in [22] $(\cdot)^K_{\theta,t^\theta\Psi(t),F}$, with $\theta = \pi_\Psi$. See also [24, 23]. Thus, we can use the reiteration theorem [22, Thm. 5.1] to obtain the following reiteration result.

THEOREM 5.14. *Let* (A_0,A_1) *be a Banach couple. Then, for any* F, F_0 *and* F_1 *r.i. function spaces, and* Ψ, Ψ_0 *and* Ψ_1 *as in* (5.10) *and* (5.11),

$$\Big((A_0,A_1)^K_{\Psi_0,F_0}, (A_0,A_1)^K_{\Psi_1,F_1}\Big)^K_{\Psi,F} = (A_0,A_1)_{\Psi_0^{1-\theta}\Psi_1^{\theta}\Psi(\frac{\Psi_0}{\Psi_1}),F}.$$

Here $\Big(\Psi_0^{1-\theta}\Psi_1^{\theta}\Psi(\frac{\Psi_0}{\Psi_1})\Big)(t) = \Psi_0^{1-\theta}(t)\Psi_1^{\theta}(t)\Psi\big(\frac{\Psi_0(t)}{\Psi_1(t)}\big)$, $t > 0$.

Given any $\varphi \in \mathscr{P}_N$, consider the functions

$$\Phi_0(t) = \Psi_0(\varphi(t))\varphi(t), \quad t > 0$$
$$\Phi_1(t) = \Psi_1(\varphi(t))\varphi(t), \quad t > 0.$$

Now we are in condition to state the following result.

THEOREM 5.15. *Let F, F_0 and F_1 be r.i. function spaces and let Ψ, Ψ_0 and Ψ_1 be as in* (5.10) *and* (5.11)*. Then, with the above notation,*

$$\left(\mathscr{L}_{\Phi_0,\widehat{\mathfrak{f}}_0}(X,Y),\mathscr{L}_{\Phi_1,\widehat{\mathfrak{f}}_1}(X,Y)\right)^K_{\Psi,F}=\mathscr{L}_{\Phi,\widehat{\mathfrak{f}}}(X,Y).$$

Here $\Phi(t)=\Phi_0^{1-\theta}(t)\Phi_1^{\theta}(t)\Psi\big(\frac{\Psi_0(t)}{\Psi_1(t)}\big)$, $t>0$, *and* $\mathbf{f}$, $\mathbf{f}_0$ *and* $\mathbf{f}_1$ *are the discretizations of* F, F_0 *and* F_1*, respectively.*

Proof. Use Theorems 5.13 and 5.14 to obtain that

$$\begin{aligned}
&\left(\mathscr{L}_{\Phi_0,\widehat{\mathfrak{f}}_0}(X,Y),\mathscr{L}_{\Phi_1,\widehat{\mathfrak{f}}_1}(X,Y)\right)^K_{\Psi,F}\\
&\qquad=\left(\left(\mathscr{L}_{\varphi,\widehat{e}}(X,Y),\mathscr{L}(X,Y)\right)^K_{\Psi_0,F_0},\left(\mathscr{L}_{\varphi,\widehat{e}}(X,Y),\mathscr{L}(X,Y)\right)^K_{\Psi_1,F_1}\right)^K_{\Psi,F}\\
&\qquad=\left(\mathscr{L}_{\varphi,\widehat{e}}(X,Y),\mathscr{L}(X,Y)\right)^K_{\Psi_0^{1-\theta}\Psi_1^{\theta}\Psi(\frac{\Psi_0}{\Psi_1}),F}\\
&\qquad=\mathscr{L}_{\Phi,\widehat{\mathfrak{f}}}(X,Y).
\end{aligned}$$

This concludes the proof. □

REFERENCES

[1] I. AHMED, D. E. EDMUNDS, W. D. EVANS AND G. E. KARADZHOV, *Reiteration theorems for the K-interpolation method in limiting cases*, Math. Nachr. **284** (2011), no. 4, 421–442.

[2] S. ASTASHKIN AND L. MALIGRANDA, *Ultrasymmetric Orlicz spaces.*, J. Math. Anal. Appl. **1** (2008), 273–285.

[3] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.

[4] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces. An introduction*, Springer, Berlin-Heidelberg-New York, 1976.

[5] P. L. BUTZER AND K. SCHERER, *Approximationsprozesse und interpolationsmethoden*, Bibliographisches Institut, Mannheim-Zürich, 1968.

[6] F. COBOS AND O. DOMINGUEZ, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory **189** (2015), 43–36.

[7] F. COBOS, L. M. FERNÁNDEZ CABRERA, T. KÜHN AND T. ULLRICH, *On an extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.

[8] F. COBOS AND L. M. FERNÁNDEZ-CABRERA AND A. MARTÍNEZ, *On a paper of Edmunds and Opic on limiting interpolation of compact operators between L_p spaces*, Math. Nachr. **288** (2015), no. 2–3, 167–175.

[9] F. COBOS AND L. M. FERNÁNDEZ-CABRERA AND M. MASTYLO, *Abstract limit J-spaces*, J. London Math. Soc. **82** (2010), 83–101.

[10] F. COBOS AND L. M. FERNÁNDEZ-CABRERA AND P. SILVESTRE, *Limiting J-spaces for general couples*, Z. Anal. Anwend. **32** (2013), no. 1, 3696–3722.

[11] F. COBOS AND L. M. FERNÁNDEZ-CABRERA AND P. SILVESTRE, *New limiting real interpolation methods and their connection with the methods associated to the unit square*, Math. Nachr. **286** (2013), no. 5-6, 569–578.

[12] F. COBOS AND T. KÜHN, *Equivalence of K- and J-methods for limiting real interpolation spaces*, J. Funct. Anal. **216** (2011), 3696–3722.

[13] F. COBOS AND M. MILMAN, *On a limit class of approximation spaces*, Numer. Funct. Anal. Optim. **11** (1990), no. 1–2, 11–31.

[14] F. COBOS AND I. RESINA, *Representation theorems for some operator ideals*, J. London Math. Soc. **39** (1989), no. 2, 324–334.
[15] F. COBOS AND A. SEGURADO, *Limiting real interpolation methods for arbitrary Banach couples*, Studia Math. **213** (2012), 243–273.
[16] R. A. DEVORE, S. D. RIEMENSCHNEIDER AND R. C. SHARPLEY, *Weak Interpolation in Banach Spaces*, J. Funct. Anal. (1979), no. 33, 58–94.
[17] D. E. EDMUNDS AND B. OPIC, *Limiting variants of Krasnosel'skiĭ's compact interpolation theorem*, J. Funct. Anal. **266** (2014), no. 5, 3265–3285.
[18] W. D. EVANS AND B. OPIC, *Real Interpolation with Logarithmic Functors and Reiteration*, Canad. J. Math. **52** (2000), no. 5, 920–960.
[19] W. D. EVANS, B. OPIC AND L. PICK, *Real Interpolation with Logarithmic Functors*, J. of Inequal. & Appl. **7** (2002), no. 2, 187–269.
[20] F. FEHÉR AND G. GRÄSSLER, *On an extremal scale of approximation spaces*, J. Comput. Anal. Appl. **3** (2001), no. 2, 95–108.
[21] P. FERNÁNDEZ-MARTÍNEZ, A. SEGURADO AND T. SIGNES, *Compactness results for a class of limiting interpolation methods*, Mediterranean J. Math., 2016, DOI:10.1007/s00009-015-0667-3
[22] P. FERNÁNDEZ-MARTÍNEZ AND T. SIGNES, *Real interpolation with slowly varying functions and symmetric spaces*, Quart. J. Math. **63** (2012), no. 1, 133–164.
[23] P. FERNÁNDEZ-MARTÍNEZ AND T. SIGNES, *Reiteration theorems with extreme values of parameters*, Ark. Mat. **52** (2014), no. 2, 227–256.
[24] P. FERNÁNDEZ-MARTÍNEZ AND T. SIGNES, *Limit cases of reiteration theorems*, Math. Nachr. **288** (2015), no. 1, 25–47.
[25] P. FERNÁNDEZ-MARTÍNEZ AND T. SIGNES, *A Limiting Case of Ultrasymmetric Spaces*. Preprint, `http://webs.um.es/pedrofdz/miwiki/doku.php?id=research`
[26] S. G. KREIN, JU. I. PETUNIN, AND E. M. SEMENOV, *Interpolation of Linear Operators*, Translations of Mathematical Monographs, vol. 54, American Mathematical Society, 1982.
[27] A. PIETSCH, *Approximation spaces*, J. Approx. Theory **32** (1981), no. 2, 115–134.
[28] E. PUSTYLNIK, *Ultrasymmetric Spaces*, J. London Math. Soc. **68** (2003), no. 2, 165–182.
[29] E. PUSTYLNIK, *A new class of approximation spaces*, Rend. Circ. Mat. Palermo **2** (2005), no. 76, 517–532.
[30] E. PUSTYLNIK, *Ultrasymmetric sequence spaces in approximation theory*, Collect. Math. **57** (2006), no. 3, 257–532.

(Received June 22, 2015)

Pedro Fernández-Martínez
Departamento de Matemáticas, Facultad de Matemáticas
Universidad de Murcia, Campus de Espinardo
30071 Espinardo (Murcia), Spain
e-mail: pedrofdz@um.es

Teresa Signes
Departamento de Matemáticas, Facultad de Matemáticas
Universidad de Murcia, Campus de Espinardo
30071 Espinardo (Murcia), Spain
e-mail: tmsignes@um.es