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WEIGHTED REGULARITY ESTIMATES IN ORLICZ SPACES FOR THE PARABOLIC SCHRÖDINGER OPERATORS

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Abstract. In this paper we study regularity estimates in weighted Orlicz spaces for the parabolic Schrödinger operator

$$P = \frac{\partial}{\partial t} - \Delta + V(x,t)$$

with non-negative potentials $V(x,t)$ satisfying certain reverse Hölder class. As a corollary we obtain the classical L^p-type regularity estimates for such operator.

1. Introduction

In this paper we consider regularity estimates in weighted Orlicz spaces for the following parabolic Schrödinger differential operator

$$P = \frac{\partial}{\partial t} - \Delta + V(z) \quad \text{ in } \mathbb{R}^{n+1}, \tag{1.1}$$

with $V \in V_\infty$ (see Definition 1), where $z = (x,t) = (x^1,\dots,x^n,t)$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Shen [33] proved the L^p boundedness with $1 < p \leqslant 2$ of the nontangential maximal function of ∇u for the L^p-Neumann problem of the elliptic Schrödinger operator

$$L = -\Delta + V(x), \tag{1.2}$$

with $V \in V_\infty$ (see Definition 1) in a domain $\Omega \subset \mathbb{R}^n$. Moreover, Shen [34] obtained the following L^p estimates for (1.2)

$$\int_{\mathbb{R}^n} \left| D^2 \left(-\Delta + V(x)\right)^{-1} f \right|^p dx \leqslant C \int_{\mathbb{R}^n} |f|^p \, dx$$

for $1 < p \leqslant q$, assuming that $V \in V_q$ for some $q \geqslant n/2$.

Gao and Jiang [18] proved the following L^p estimates of the parabolic Schrödinger operator $\frac{\partial}{\partial t} - \Delta + V(x)$

$$\int_{\mathbb{R}^n \times (0,T]} \left| D^2 \left(\frac{\partial}{\partial t} - \Delta + V(x) \right)^{-1} f \right|^p dz \leqslant C \int_{\mathbb{R}^n \times (0,T]} |f|^p \, dz \quad \text{for } 1 < p \leqslant q,$$

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where $z = (x,t)$ and $V \in V_q$ for some $q \geqslant \frac{n}{2\mu}$ with $0 < \mu < \frac{n}{n+2}$ and $n \geqslant 3$. Recently, Carbonaro, Metafune and Spina [13] proved

$$\int_{\mathbb{R}^{n+1}} \left| D^2 \left(\frac{\partial}{\partial t} - \Delta + V(z) \right)^{-1} f \right|^p dz \leqslant C \int_{\mathbb{R}^{n+1}} |f|^p \, dz \tag{1.3}$$

for the parabolic Schrödinger operator $\frac{\partial}{\partial t} - \Delta + V(z)$, assuming that $V \in V_p$ for $1 < p < \infty$.

DEFINITION 1. ([3, 34, 35]) The function $V(z)$ is said to belong to the reverse Hölder class V_q for some $1 < q \leqslant \infty$ if $V \in L^q_{loc}(\mathbb{R}^{n+1})$, $V \geqslant 0$ almost everywhere and there exists a constant C such that

$$\left(\fint_Q V^q(z) \, dz \right)^{1/q} \leqslant C \fint_Q V(z) \, dz$$

for any square cubes Q in $\mathbb{R}^{n+1}$, where

$$\fint_Q V(z) \, dz = \frac{1}{|Q|} \int_Q V(z) \, dz.$$

If $q = \infty$, then the left hand side is the essential supremum in Q, i.e.,

$$\sup_Q |V(z)| \leqslant C \fint_Q V(z) \, dz.$$

Actually, if $V \in V_\infty$, it clearly implies $V \in V_q$ for every $q > 1$.

Sobolev spaces, which are sets of functions with a certain degree of smoothness, are commonly used and studied in a wide variety of fields of mathematics, and have turned out to be one of the most powerful tools in analysis created in the 20th century. Since the 1960s, the need to use wider spaces of functions than Sobolev spaces came from various practical problems. Orlicz spaces have been studied as the generalization of Sobolev spaces since they were introduced by Orlicz [31] (see [3, 4, 14, 15, 22]). The theory of Orlicz spaces plays a crucial role in many fields of mathematics including geometric, probability, stochastic, Fourier analysis and PDE (see [32]).

We denote Φ by

$$\Phi = \left\{ \phi : [0, +\infty) \longrightarrow [0, +\infty) \;\middle|\; \phi \text{ are increasing and convex} \right\}. \tag{1.4}$$

DEFINITION 2. A function $\phi \in \Phi$ is said to be a Young function if

$$\lim_{t \to 0+} \frac{\phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\phi(t)} = 0.$$

DEFINITION 3. A Young function $\phi \in \Phi$ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for every $t > 0$,

$$\phi(2t) \leqslant K\phi(t).$$

Moreover, a Young function $\phi \in \Phi$ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that for every $t > 0$,

$$\phi(t) \leqslant \frac{\phi(at)}{2a}.$$

REMARK 1. We remark that the $\Delta_2 \cap \nabla_2$ condition makes the function grow moderately. For example, $\phi(t) = t^p$ and $\phi(t) = t^p(1 + |\log t|)$ with $p > 1$ satisfy the $\Delta_2 \cap \nabla_2$ condition.

Now we consider

$$h_\phi(\lambda) = \sup_{\rho>0} \frac{\phi(\lambda\rho)}{\phi(\rho)} \quad \text{for} \quad \lambda > 0$$

and

$$i(\phi) = \lim_{\lambda\to 0+} \frac{\log h_\phi(\lambda)}{\log\lambda} = \sup_{0<\lambda<1} \frac{\log h_\phi(\lambda)}{\log\lambda}. \tag{1.5}$$

If $\phi \in \Delta_2 \cap \nabla_2$, then from [9, 17] we know that $i(\phi) > 1$ and

$$\frac{1}{c}\min\{\lambda^{\alpha_1}, \lambda^{\alpha_2}\}\phi(\rho) \leqslant \phi(\lambda\rho) \leqslant c\max\{\lambda^{\alpha_1}, \lambda^{\alpha_2}\}\phi(\rho) \quad \text{for any} \quad \lambda, \rho > 0, \tag{1.6}$$

where the constants $\alpha_1, \alpha_2 \in (1, \infty)$, $\alpha_1 \leqslant \alpha_2$. It is worth pointing out that $i(\phi)$ is equal to the supremum of those α_1 for which (1.6) holds true with $\lambda \geqslant 1$.

We shall give some definitions and properties on the weighted Lebesgue spaces (see [8, 10, 20, 23, 27, 28, 35, 36]).

DEFINITION 4. A_p for some $p > 1$ is the class of the Muckenhoupt weights: $w \in A_p$ if $w \in L^1_{loc}(\mathbb{R}^{n+1})$, $w > 0$ almost everywhere and there exists a constant C such that for all square cubes Q in $\mathbb{R}^{n+1}$,

$$\left(\fint_Q w(z)\,dz\right)\left(\fint_Q w(z)^{\frac{-1}{p-1}}\,dz\right)^{p-1} \leqslant C.$$

Moreover, we denote

$$A_\infty = \bigcup_{1<p<\infty} A_p \quad \text{and} \quad w(Q) = \int_Q w(z)\,dz,$$

where $Q \subset \mathbb{R}^{n+1}$. Furthermore, the corresponding weighted Lebesgue space $L^p_w(Q)$ consists of all functions h which satisfy

$$\|h\|_{L^p_w(Q)} =: \left(\int_Q |h|^p w(z)\,dz\right)^{1/p} < \infty.$$

DEFINITION 5. Let $\phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\phi)}$. Then the weighted Orlicz class $K_w^\phi(Q)$ is the set of all measurable functions $g : Q \to \mathbb{R}$ satisfying

$$\int_Q \phi(|g|)w(z)dz < \infty.$$

The weighted Orlicz space $L_w^\phi(Q)$ is the linear hull of $K_w^\phi(Q)$.

We use the Hardy-Littlewood maximal function which controls the local behavior of a function.

DEFINITION 6. Let v be a locally integrable function. The Hardy-Littlewood maximal function $\mathscr{M}v(z)$ is defined as

$$\mathscr{M}v(z) = \sup \fint_Q |v(y,s)|dyds,$$

where the sup is taken over all square cubes Q in $\mathbb{R}^{n+1}$ containing $z = (x,t)$.

It is well known that the maximal functions satisfy strong p-p estimate for any $1 < p < \infty$ and weak 1-1 estimate (see [35]).

LEMMA 1. (see [8, 10, 23, 27, 28, 35, 36]) *Assume that $w \in A_p$ for some $p > 1$. Then we have*

1. $A_{p_1} \subset A_p$ *for any* $1 < p_1 \leqslant p < \infty$.
2. $w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}g(z) > \mu\right\}\right) \leqslant C\mu^{-p} \int_{\mathbb{R}^{n+1}} |g(z)|^p w(z)dz$ *for any* $\mu > 0$.
3. $$\frac{1}{C_1}\left(\frac{|Q_1|}{|Q_2|}\right)^p \leqslant \frac{w(Q_1)}{w(Q_2)} \leqslant C_1\left(\frac{|Q_1|}{|Q_2|}\right)^\sigma$$
   for any square cubes $Q_1 \subset Q_2 \subset \mathbb{R}^{n+1}$, where $\sigma > 0$ and $C_1 > 1$.

LEMMA 2. *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\phi)}$.*

1. *There exists a small positive constant $\varepsilon_0 < 1$ and a constant $C > 1$ such that*
   $$\left(\fint_Q w(z)^{1+\varepsilon_0} dz\right)^{\frac{1}{1+\varepsilon_0}} \leqslant C \fint_Q w(z)dz$$
   for any square cube $Q \subset \mathbb{R}^{n+1}$.
2. *There exists a positive constant $p_2 \in (1, i(\phi))$ such that*
   $$w \in A_{p_2}.$$

3. There exists a positive constant $q>1$ such that

$$L_w^{\phi}(Q)\subset L^q(Q)\subset L^1(Q)\quad \textit{for any square cube}\quad Q\subset\mathbb{R}^{n+1}.$$

Proof. The conclusion (1) can follow from Theorem 3.5 in Chapter 9 of [36]. Since $w\in A_{i(\phi)}$, from Definition 4 we have

$$\begin{aligned}
&\left(\fint_Q w(z)dz\right)\left(\fint_Q w(z)^{\frac{-1}{i(\phi)-1}}dz\right)^{i(\phi)-1}\\
&=\left(\fint_Q \left(w(z)^{\frac{-1}{i(\phi)-1}}\right)^{1-i(\phi)}dz\right)\left(\fint_Q w(z)^{\frac{-1}{i(\phi)-1}}dz\right)^{i(\phi)-1}\\
&=\left[\left(\fint_Q \left(w(z)^{\frac{-1}{i(\phi)-1}}\right)^{-\frac{1}{\frac{i(\phi)}{i(\phi)-1}-1}}dz\right)^{\frac{i(\phi)}{i(\phi)-1}-1}\left(\fint_Q w(z)^{\frac{-1}{i(\phi)-1}}dz\right)\right]^{i(\phi)-1}\\
&\leqslant C
\end{aligned}\tag{1.7}$$

for any square cube Q in $\mathbb{R}^{n+1}$, which implies that $w(z)^{\frac{-1}{i(\phi)-1}}\in A_{\frac{i(\phi)}{i(\phi)-1}}$. Therefore, from the conclusion (1) we have

$$\left(\fint_Q w(z)^{-\frac{1+\varepsilon_0'}{i(\phi)-1}}dz\right)^{\frac{1}{1+\varepsilon_0'}}\leqslant C\fint_Q w(z)^{\frac{-1}{i(\phi)-1}}dz \tag{1.8}$$

for some $\varepsilon_0'\in(0,1)$. Let

$$p_2=1+\frac{i(\phi)-1}{1+\varepsilon_0'}\in(1,i(\phi)).$$

Then from (1.8) and the fact that $w\in A_{i(\phi)}$ we find that

$$\begin{aligned}
&\left(\fint_Q w(z)dz\right)\left(\fint_Q w(z)^{\frac{-1}{p_2-1}}dz\right)^{p_2-1}\\
&=\left(\fint_Q w(z)dz\right)\left(\fint_Q w(z)^{-\frac{1+\varepsilon_0'}{i(\phi)-1}}dz\right)^{\frac{i(\phi)-1}{1+\varepsilon_0'}}\\
&\leqslant C\left(\fint_Q w(z)dz\right)\left(\fint_Q w(z)^{-\frac{1}{i(\phi)-1}}dz\right)^{i(\phi)-1}\leqslant C,
\end{aligned}\tag{1.9}$$

which implies that $w\in A_{p_2}$. Thus, the conclusion (2) is true. Recalling that $i(\phi)>p_2>1$ and the fact that $i(\phi)=\sup\alpha_1>1$, where the supremum is taken over those α_1 for which (1.6) holds true with $\lambda\geqslant 1$, we can choose a proper constant

$$\alpha_1^0\in(p_2,i(\phi))\quad\text{satisfying (1.6).}\tag{1.10}$$

Let

$$q = \frac{\alpha_1^0}{p_2} \in (1, \alpha_1^0). \tag{1.11}$$

Then, from Hölder's inequality and (1.9) we have

$$\begin{aligned}
\int_Q |f|^q \, dz &= \int_Q |f|^q \, w(z)^{\frac{1}{p_2}} w(z)^{-\frac{1}{p_2}} dz \\
&\leqslant \left(\int_Q |f|^{\alpha_1^0} w(z) dz \right)^{\frac{1}{p_2}} \left(\int_Q w(z)^{-\frac{1}{p_2 - 1}} dz \right)^{1 - \frac{1}{p_2}} \\
&\leqslant \left(\int_Q (1 + |f|)^{\alpha_1^0} w(z) dz \right)^{\frac{1}{p_2}} \left(\frac{|Q|}{w(Q)} \right)^{\frac{1}{p_2}} \\
&\leqslant C \left(\int_Q (1 + |f|)^{\alpha_1^0} w(z) dz \right)^{\frac{1}{p_2}},
\end{aligned}$$

since $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ almost everywhere. Furthermore, if $f \in L^\phi_w(Q)$, then from (1.6) and (1.10) we find that

$$\begin{aligned}
\int_Q |f|^q \, dz &\leqslant C \left(\int_Q \phi\,(1 + |f|)\, w(z) dz \right)^{\frac{1}{p_2}} \\
&\leqslant C \left(1 + \int_Q \phi\,(|f|)\, w(z) dz \right)^{\frac{1}{p_2}} \leqslant C.
\end{aligned}$$

This finishes our proof. □

LEMMA 3. (see [9, 21]) *Let $\phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i(\phi)}$ and $g \in L^\phi_w(\mathbb{R}^{n+1})$. Then we have*

$$\int_{\mathbb{R}^{n+1}} \phi(|g|) w(z) dz = \int_0^\infty w\left(\{z \in \mathbb{R}^{n+1} : |g| > \lambda\}\right) d\left[\phi(\lambda)\right]$$

and

$$\int_{\mathbb{R}^{n+1}} \phi\,(|g|)\, w(z) dz \leqslant \int_{\mathbb{R}^{n+1}} \phi\,(\mathscr{M}\,(|g|))\, w(z) dz \leqslant C \int_{\mathbb{R}^{n+1}} \phi\,(|g|)\, w(z) dz,$$

where $C = C(n, \phi, w)$.

Now let us state the main results of this work: Theorem 1 and Theorem 2. We shall give the direct proofs of the main results via the maximal function approach which was employed by [2, 8, 11, 24, 27, 28].

THEOREM 1. *Assume that $\phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i(\phi)}$ and $f \in L^\phi_w(\mathbb{R}^{n+1})$. If u is the solution of the heat equation*

$$u_t(z) - \Delta u(z) = f(z) \quad \text{in } \mathbb{R}^{n+1},$$

then we have

$$\int_{\mathbb{R}^{n+1}} \phi(|u_t|)\, w(z)dz + \int_{\mathbb{R}^{n+1}} \phi\left(\left|D^2 u\right|\right) w(z)dz \leqslant C \int_{\mathbb{R}^{n+1}} \phi(|f|)\, w(z)dz.$$

REMARK 2. We remark that the condition $\phi \in \Delta_2 \cap \nabla_2$ is optimal for the type of regularity results (see [37]). Especially when $w(z) \equiv 1$ and $\phi(t) = t^p$ with $p > 1$, the above estimate is reduced to the classical L^p estimate.

Recently, Bramanti, Brandolini, Harboure's & Viviani [5] proved that

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leqslant C\left(\|f\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right),$$

if $u \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$-a_{ij}u_{x_i x_j} + Vu = f, \tag{1.12}$$

where $V \in V_q$ with $1 < p \leqslant q$ and $q \geqslant n/2$. Furthermore, Zhang [38] extended the result in [5] in the setting of the general Orlicz spaces.

THEOREM 2. *Assume that* $\phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i(\phi)}$, $V \in V_\infty$ *and* $f \in L_w^\phi(\mathbb{R}^{n+1})$. *If* $u \in C_0^\infty(\mathbb{R}^{n+1})$ *is the solution of the following parabolic Schrödinger equation*

$$u_t(z) - \Delta u(z) + V(z)u(z) = f(z) \qquad in\ \mathbb{R}^{n+1}, \tag{1.13}$$

then we have

$$\int_{\mathbb{R}^{n+1}} \phi(|Vu|)\, w(z) + \phi(|u_t|)\, w(z) + \phi\left(\left|D^2 u\right|\right) w(z)dz \leqslant C \int_{\mathbb{R}^{n+1}} \phi(|f|)\, w(z)dz.$$

Our approach is much influenced by [8, 9, 10, 25, 27, 28]. The authors [8, 9, 27, 28] obtained the local/global weighted gradient estimates for second-order elliptic and parabolic equations in the bounded domain, where they used harmonic analysis tools such as the maximal function operator which is first developed by Caffarelli and Peral [11]. Moreover, Byun and Ryu [10] proved the global weighted gradient estimates for nonlinear elliptic equations of p-Laplacian type, in which they used the harmonic analysis free approach based on the covering/iteration argument (see [1, 6, 7, 29]).

2. Proofs of the main results

In this section we shall finish the proofs of the main results: Theorem 1 and Theorem 2.

2.1. Proof of Theorem 1

In this subsection we shall prove Theorem 1. We first give the following Calderón-Zygmund decomposition, which is much influenced by [25].

LEMMA 4. *Let D be a square cube in $\mathbb{R}^{n+1}$ and $A,B \subset D$ be measurable sets. Assume that $0 < w(A) < \mu w(D)$ for $0 < \mu < 1$. Then there exists a sequence of disjoint square cubes $\{Q_k\}_{k\in\mathbb{N}}$ satisfying*

1. $w\left(A \setminus \bigcup_{k\in\mathbb{N}} Q_k\right) = 0$,
2. $w(A \cap Q_k) > \mu w(Q_k)$,
3. $w\left(A \cap \widetilde{Q_k}\right) \leqslant \mu w\left(\widetilde{Q_k}\right)$ *if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.*

Furthermore, if for any Q_k, its predecessor $\widetilde{Q_k}$ satisfies

$$w\left(B \cap \widetilde{Q_k}\right) > \alpha w\left(\widetilde{Q_k}\right) \quad \text{for } 0 < \alpha < 1, \tag{2.14}$$

then we have

$$w(A) \leqslant \frac{\mu}{\alpha} w(B).$$

Proof. **1.** We first divide D into 2^{n+1} (denote by $\left\{Q_1^{j_1}\right\}_{j_1=1}^{2^n}$) disjoint square cubes (daughters) with the same size. Choose those square cubes satisfying $w\left(A \cap Q_1^{j_1}\right) > \mu w\left(Q_1^{j_1}\right)$ and continue to divide every remaining square cube $Q_1^{j_1}$ into 2^{n+1} (denote by $\left\{Q_2^{j_1,j_2}\right\}_{j_2=1}^{2^n}$) disjoint square cubes with the same size. Therefore, we obtain a sequence of disjoint square cubes $\{Q_k\}_{k\in\mathbb{N}}$ which satisfy (2)–(3) by repeating the process above. If $z \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, then there is a sequence of square cubes P_i containing z with the diameters of P_i converging to 0 and

$$w(A \cap P_i) \leqslant \mu w(P_i).$$

That is to say,

$$\int_{A\cap P_i} w(z)\, dz \leqslant \mu \int_{P_i} w(z)\, dz.$$

From the elementary measure theory and the fact that $w(z) > 0$ almost everywhere we can conclude that $z \in D \setminus A$ for almost every $z \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, which implies that

$$\left|A \setminus \{Q_k\}_{k\in\mathbb{N}}\right| = 0.$$

Thus, from Lemma 1 (3) we conclude that (1) is true.

2. Let $\widetilde{Q_k}$ be the predecessor (father) of Q_k. Now we choose a disjoint predecessor subsequence $\left\{\widetilde{Q_{k_j}}\right\}$ (still denote by $\left\{\widetilde{Q_k}\right\}$) such that

$$\bigcup_{k\in\mathbb{N}} Q_k \subset \bigcup_{k\in\mathbb{N}} \widetilde{Q_k}.$$

Thus, from (1), (3) and the hypothesis (2.14) we deduce that

$$w(A) = \sum_k w\left(A \cap \widetilde{Q_k}\right) \leqslant \mu \sum_k w\left(\widetilde{Q_k}\right) < \frac{\mu}{\alpha} \sum_k w\left(B \cap \widetilde{Q_k}\right) \leqslant \frac{\mu}{\alpha} w(B),$$

which finishes our proof. □

Next, we shall prove the following important result.

LEMMA 5. (cf. Lemma 9) *Assume that* $\phi \in \Delta_2 \cap \nabla_2$ *and* $w \in A_{i(\phi)}$ *for* $i(\phi) > \alpha_1^0 > q > 1$, *where* α_1^0, q *are defined in (1.10)–(1.11). For any* $\mu \in (0,1)$ *there exist two constants* $M_2 = M_2(n) > 1$ *and* $\delta = \delta(n, \sigma, \mu) \in (0,1)$ *such that if*

$$w\left(\left\{z \in \widetilde{Q} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}\left(|f|^q\right)(z) \leqslant \delta^q\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.15}$$

where $\widetilde{Q}$ *is a so-called predecessor (father) of the square cube* Q *with* $\left|\widetilde{Q}\right| = |2Q|$, *then we have*

$$w\left(\left\{z \in Q : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \geqslant M_2^q\right\}\right) \leqslant \mu w(Q).$$

Proof. From Lemma 1 (3) and (2.15) we find that

$$\begin{aligned}
&\frac{\left|\left\{z \in \widetilde{Q} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}\left(|f|^q\right)(z) \leqslant \delta^q\right\}\right|}{\left|\widetilde{Q}\right|} \\
&\geqslant \left[\frac{w\left(\left\{z \in \widetilde{Q} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}\left(|f|^q\right)(z) \leqslant \delta^q\right\}\right)}{C_1 w\left(\widetilde{Q}\right)}\right]^{\frac{1}{\sigma}} \\
&\geqslant (2C_1)^{-\frac{1}{\sigma}} \in (0,1),
\end{aligned}$$

since $C_1 > 1$ and $\sigma > 0$. That is to say,

$$\left|\left\{z \in \widetilde{Q} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}\left(|f|^q\right)(z) \leqslant \delta^q\right\}\right| \geqslant (2C_1)^{-\frac{1}{\sigma}} \left|\widetilde{Q}\right|.$$

Therefore, there exists $z_0 \in \widetilde{Q}$ satisfying

$$\mathscr{M}\left(\left|D^2 u\right|^q\right)(z_0) \leqslant 1 \quad \text{and} \quad \mathscr{M}\left(|f|^q\right)(z_0) \leqslant \delta^q.$$

Since $z_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$⨍_{4Q} \left|D^2 u\right|^q dz \leqslant 1 \quad \text{and} \quad ⨍_{4Q} |f|^q dz \leqslant \delta^q. \tag{2.16}$$

Let $\overline{f}$ be the zero extention of f from $4Q$ to $\mathbb{R}^{n+1}$ and v_1 be the solution of

$$(v_1)_t - \Delta v_1 = \overline{f} \qquad \text{in } \mathbb{R}^{n+1}.$$

Then recalling the elementary L^p-type estimates, we have

$$\int_{\mathbb{R}^{n+1}} \left|D^2 v_1\right|^q dz \leqslant C \int_{\mathbb{R}^{n+1}} |\bar{f}|^q dz,$$

which implies that

$$\int_{4Q} \left|D^2 v_1\right|^q dz \leqslant \int_{\mathbb{R}^{n+1}} \left|D^2 v_1\right|^q dz \leqslant C \int_{\mathbb{R}^{n+1}} |\bar{f}|^q dz = C \int_{4Q} |f|^q dz.$$

Therefore, from (2.16) we conclude that

$$⨍_{4Q} \left|D^2 v_1\right|^q dz \leqslant C ⨍_{4Q} |f|^q dz \leqslant C\delta^q. \tag{2.17}$$

Set $h_1 = u - v_1$. From the definition of $\bar{f}$, we find that h_1 satisfies the heat equation

$$(h_1)_t - \Delta h_1 = 0 \quad \text{ in } 4Q.$$

It is easy to check that $D^2 h_1$ still satisfies the heat equation in $4Q$. Moreover, it follows from the local bounded estimates (see [16], Theorem 9 in §2.3) that

$$\sup_{3Q} |D^2 h_1| \leqslant M_1,$$

where $M_1 > 1$ only depends on n. The proof is totally similar to the proof of Lemma 9. Here we omit the details. □

COROLLARY 1. (cf. Corollary 3) *Assume that $\mu \in (0,1)$ with $C_1\mu^\sigma < 1$ and w, δ, q, M_2 satisfy the same conditions as those in Lemma 5. For any $\lambda > 0$ we have*

$$\begin{aligned} & w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) \geqslant \lambda^q M_2^q\right\}\right) \\ & \leqslant 2C_1\mu^\sigma \left[w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) > \lambda^q\right\}\right)\right. \\ & \qquad \left. + w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|f|^q\right)(z) > \lambda^q \delta^q\right\}\right)\right]. \end{aligned}$$

Furthermore, we can obtain the following result from Corollary 1.

COROLLARY 2. *Assume that $\mu \in (0,1)$ with $C_1\mu^\sigma < 1$ and w, δ, q, M_2 satisfy the same conditions as those in Lemma 5. For any $\lambda > 0$ we have*

$$\begin{aligned} & w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(\left|D^2 u\right|\right)(z) \geqslant \lambda M_2\right\}\right) \\ & \leqslant 2C_1\mu^\sigma \left[w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(\left|D^2 u\right|^q\right)(z) > \lambda^q\right\}\right)\right. \\ & \qquad \left. + w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|f|^q\right)(z) > \lambda^q \delta^q\right\}\right)\right]. \end{aligned}$$

Proof. Let z be a point in $\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|D^2u|^q\right)(z) < \lambda^q M_2^q\right\}$. Assume that Q' is any square cube with $z \in Q'$. Then from Hölder's inequality we have

$$\fint_{Q'} |D^2u|\,dz \leqslant \left(\fint_{Q'} |D^2u|^q\,dz\right)^{\frac{1}{q}} < \lambda M_2, \tag{2.18}$$

which implies that $\mathscr{M}\left(|D^2u|\right)(z) < \lambda M_2$. Therefore, we have

$$\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|D^2u|^q\right)(z) < \lambda^q M_2^q\right\} \subset \left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|D^2u|\right)(z) < \lambda M_2\right\}, \tag{2.19}$$

which implies that

$$\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|D^2u|\right)(z) \geqslant \lambda M_2\right\} \subset \left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|D^2u|^q\right)(z) \geqslant \lambda^q M_2^q\right\}.$$

Thus, we can finish the proof. □

Moreover, we need the following result.

LEMMA 6. *Assume that $\phi \in \triangle_2 \cap \nabla_2$ and $w \in A_{i(\phi)}$ for $i(\phi) > \alpha_1^0 > q > 1$, where α_1^0, q are defined in (1.10)–(1.11). Then we have*

$$\int_0^\infty w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|f|^q\right)(x) > \lambda^q\right\}\right) d\left[\phi(M_2\lambda)\right] \leqslant C \int_{\mathbb{R}^{n+1}} \phi(|f|)w(x)dz.$$

Proof. Let

$$f_1(z) = \begin{cases} f(z), & if \quad |f(z)| > \frac{\lambda}{2}, \\ 0, & if \quad |f(z)| \leqslant \frac{\lambda}{2}, \end{cases}$$

and $f_2(z) = f(z) - f_1(z)$. Then it is easy to see that

$$\mathscr{M}\left(|f|^q\right)(z) \leqslant 2^{q-1}\left(\mathscr{M}\left(|f_1|^q\right)(z) + \mathscr{M}\left(|f_2|^q\right)(z)\right) \leqslant 2^{q-1}\left(\mathscr{M}\left(|f_1|^q\right)(z) + \left(\frac{\lambda}{2}\right)^q\right).$$

Therefore, from Lemma 1 (2) we have

$$\begin{aligned} & w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|f|^q\right)(z) > \lambda^q\right\}\right) \\ & \leqslant w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}\left(|f_1|^q\right)(z) > \left(\frac{\lambda}{2}\right)^q\right\}\right) \\ & \leqslant C\lambda^{-q} \int_{\mathbb{R}^{n+1}} |f_1|^q w(z)dz = C\lambda^{-q} \int_{\left\{z \in \mathbb{R}^{n+1} : |f| > \frac{\lambda}{2}\right\}} |f|^q w(z)dz. \end{aligned}$$

Interchanging the order of integration and integrating by parts, we deduce that

$$\begin{aligned}
&\int_0^\infty w\left(\left\{z\in\mathbb{R}^{n+1}:\mathscr{M}\left(|f|^q\right)>\lambda^q\right\}\right)d\left[\phi(M_2\lambda)\right]\\
&\leqslant C\int_0^\infty \lambda^{-q}\int_{\left\{z\in\mathbb{R}^{n+1}:|f|>\frac{\lambda}{2}\right\}}|f|^q w(z)dzd\left[\phi(M_2\lambda)\right]\\
&=C\int_0^\infty \lambda^{-q}\int_{\mathbb{R}^{n+1}}\chi_{\left\{z\in\mathbb{R}^{n+1}:|f|>\frac{\lambda}{2}\right\}}|f|^q w(z)dzd\left[\phi(M_2\lambda)\right]\\
&=C\int_{\mathbb{R}^{n+1}}|f|^q w(z)\left\{\int_0^{2|f|}\lambda^{-q}d\left[\phi(M_2\lambda)\right]\right\}dz\\
&=\int_{\mathbb{R}^{n+1}}|f|^q w(z)\left\{\frac{\phi(2M_2|f|)}{(2|f|)^q}+q\int_0^{2|f|}\frac{\phi(M_2\lambda)}{\lambda^{q+1}}d\lambda\right\}dz.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^\infty w\left(\left\{z\in\mathbb{R}^{n+1}:\mathscr{M}\left(|f|^q\right)>\lambda^q\right\}\right)d\left[\phi(M_2\lambda)\right]\\
&\leqslant C\int_{\mathbb{R}^{n+1}}\phi(|f|)w(z)dz+C\int_{\mathbb{R}^{n+1}}\phi\left(2M_2|f|\right)|f|^{q-\alpha_1^0}w(z)\left\{\int_0^{2|f|}\frac{1}{\lambda^{q+1-\alpha_1^0}}\,d\lambda\right\}dz\\
&\leqslant C\int_{\mathbb{R}^{n+1}}\phi(|f|)w(z)dz,
\end{aligned}$$

since $\alpha_1^0>q$ and

$$\phi\left(2M_2|f|\right)=\phi\left(M_2\lambda\cdot\frac{2|f|}{\lambda}\right)\geqslant\frac{1}{c}\left(\frac{2|f|}{\lambda}\right)^{\alpha_1^0}\phi\left(M_2\lambda\right)\quad\text{for}\quad 0\leqslant\lambda\leqslant 2|f|$$

in view of (1.6) and (1.10). Thus we complete the proof. □

Now we are ready to prove the main result: Theorem 1.

Proof. From the fact that $|D^2u|(z)\leqslant\mathscr{M}\left(|D^2u|\right)(z)$, Lemma 3, Corollary 2 and Lemma 6 we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}}\phi\left(\left|D^2u\right|\right)w(z)dz\\
&\leqslant\int_{\mathbb{R}^{n+1}}\phi\left(\mathscr{M}\left(\left|D^2u\right|\right)\right)w(z)dz\\
&=\int_0^\infty w\left(\left\{z\in\mathbb{R}^{n+1}:\mathscr{M}\left(\left|D^2u\right|\right)(z)>M_2\lambda\right\}\right)d\left[\phi(M_2\lambda)\right]\\
&\leqslant 2C_1\mu^\sigma\int_0^\infty w\left(\left\{z\in\mathbb{R}^{n+1}:\mathscr{M}\left(\left|D^2u\right|^q\right)(z)>\lambda^q\right\}\right)d\left[\phi(M_2\lambda)\right]\\
&\quad+2C_1\mu^\sigma\int_0^\infty w\left(\left\{z\in\mathbb{R}^{n+1}:\mathscr{M}\left(|f|^q\right)(z)>\delta^q\lambda^q\right\}\right)d\left[\phi(M_2\lambda)\right],
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^{n+1}}\phi\left(|D^2u|\right)w(z)dz\leqslant C_3\mu^\sigma\int_{\mathbb{R}^{n+1}}\phi\left(|D^2u|\right)w(z)dz+C_4\int_{\mathbb{R}^{n+1}}\phi\left(|f|\right)w(z)dz,$$

for any $\mu \in (0,1)$ with $C_1\mu^\sigma < 1$, where $C_3 = C_3(\phi,w,n)$ and $C_4 = C_4(n,\phi,\mu,w)$. In a standard way via an approximation argument as the one in [1, 37] we can assume that the integral $\int_{\mathbb{R}^{n+1}} \phi\left(|D^2u|\right) w(z)dz$ is finite. Now selecting μ small enough to ensure that $C_3\mu^\sigma \leqslant 1/2$, we have

$$\int_{\mathbb{R}^{n+1}} \phi\left(|D^2u|\right) w(z)dz \leqslant C \int_{\mathbb{R}^{n+1}} \phi\left(|f|\right) w(z)dz,$$

for some constant $C = C(n,\phi,w) > 0$. This completes our proof. □

2.2. Proof of Theorem 2

In this subsection we shall finish the proof of Theorem 2. Now let us first recall the following result.

LEMMA 7. ([35], page 195) *Assume that $V \in V_\infty$. There exist $1 \leqslant t < \infty$ and $C > 0$ such that*

$$⨍_Q g\,dz \leqslant \left(\frac{C}{V(Q)} \int_Q V g^t\,dz\right)^{\frac{1}{t}}$$

holds for any nonnegative function g and all square cubes Q, where

$$V(Q) = \int_Q V\,dz.$$

Let h be the solution of the following homogeneous equation

$$h_t(z) - \Delta h(z) + V(z)h(z) = 0, \quad z = (x,t) \in 2Q \subset \mathbb{R}^{n+1}. \tag{2.20}$$

Then we have the following local bounded property.

LEMMA 8. *Assume that $V \in V_\infty$. If $h(z)$ satisfies (2.20) in $2Q$, then*

$$\sup_Q |h| \leqslant \frac{C}{V(2Q)} \int_{2Q} V|h|\,dz.$$

Proof. In view of the fact that $u \in C_0^\infty(\mathbb{R}^{n+1})$ we may as well assume that

$$\operatorname{supp} u \subset Q_{r_0}$$

for some $r_0 > 0$. Moreover, since $V \in V_\infty$ and $u \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies $u_t(z) - \Delta u(z) + V(z)u(z) = f(z)$, we can suppose that

$$V(z) \equiv 0 \quad \text{in } \mathbb{R}^{n+1} \setminus Q_{r_0} \quad \text{and} \quad |V(z)| \leqslant C \quad \text{in } \mathbb{R}^{n+1}.$$

Using the elementary local bounded property of the second-order parabolic equation (see [26], Theorem 7.21), we have

$$\sup_Q |h| \leqslant C\left(⨍_{2Q} |h|^r\,dz\right)^{\frac{1}{r}}$$

for any $r > 0$, which implies that

$$\sup_Q |h| \leqslant C \left(\fint_{2Q} |h|^{\frac{1}{t}} \, dz \right)^t \leqslant \frac{C}{V(2Q)} \int_{2Q} V|h| \, dz$$

in view of Lemma 7 with $r = \frac{1}{t}$. Thus, we finish the proof. □

Next, we shall prove the following important result.

LEMMA 9. *Assume that $\phi \in \triangle_2 \cap \nabla_2$, $w \in A_{i(\phi)}$ and $V \in V_\infty$. For any $\mu \in (0,1)$ there exist two constants $N_2 = N_2(n) > 1$ and $\delta = \delta(n, \sigma, \mu) \in (0,1)$ such that if*

$$w\left(\left\{z \in \widetilde{Q} : \mathscr{M}(V|u|)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}(|f|)(z) \leqslant \delta\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.21}$$

where $\widetilde{Q}$ is a so-called predecessor (father) of the square cube Q with $\left|\widetilde{Q}\right| = |2Q|$, then we have

$$w(\{z \in Q : \mathscr{M}(V|u|)(z) \geqslant N_2\}) \leqslant \mu w(Q).$$

Proof. **1.** We first find that

$$\left|\left\{z \in \widetilde{Q} : \mathscr{M}(V|u|)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}(|f|)(z) \leqslant \delta\right\}\right| \geqslant (2C_1)^{-\frac{1}{\sigma}} \left|\widetilde{Q}\right|,$$

since

$$\begin{aligned} &\frac{\left|\left\{z \in \widetilde{Q} : \mathscr{M}(V|u|)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}(|f|)(z) \leqslant \delta\right\}\right|}{\left|\widetilde{Q}\right|} \\ &\geqslant \left[\frac{w\left(\left\{z \in \widetilde{Q} : \mathscr{M}(V|u|)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q} : \mathscr{M}(|f|)(z) \leqslant \delta\right\}\right)}{C_1 w\left(\widetilde{Q}\right)}\right]^{\frac{1}{\sigma}} \\ &\geqslant (2C_1)^{-\frac{1}{\sigma}} \in (0,1), \end{aligned}$$

in view of Lemma 1 (3) and (2.21). Therefore, there exists $z_0 \in \widetilde{Q}$ such that

$$\mathscr{M}(V|u|)(z_0) \leqslant 1 \quad \text{and} \quad \mathscr{M}(|f|)(z_0) \leqslant \delta. \tag{2.22}$$

Since $z_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$\fint_{4Q} |Vu| \, dz \leqslant 1 \quad \text{and} \quad \fint_{4Q} |f| \, dz \leqslant \delta. \tag{2.23}$$

Let v be the solution of

$$v_t - \Delta v + V(z) v = \bar{f}, \quad z \in \mathbb{R}^{n+1},$$

where $\bar{f}$ is the zero extention of f from $4Q$ to $\mathbb{R}^{n+1}$. Then recalling the well-known L^1 maximal inequality (see [13], Lemma 3.1), we have

$$\int_{\mathbb{R}^{n+1}} V|v|\,dz \leqslant \int_{\mathbb{R}^{n+1}} |\bar{f}|\,dz,$$

which implies that

$$\int_{4Q} V|v|\,dz \leqslant \int_{\mathbb{R}^{n+1}} V|v|\,dz \leqslant \int_{\mathbb{R}^{n+1}} |\bar{f}|\,dz = \int_{4Q} |f|\,dz.$$

Therefore, from (2.23) we conclude that

$$⨍_{4Q} V|v|\,dz \leqslant ⨍_{4Q} |f|\,dz \leqslant \delta. \tag{2.24}$$

Set $h = u - v$. From the definition of $\bar{f}$, we find that h satisfies

$$h_t - \Delta h + V(z)h = 0 \qquad \text{in } 4Q.$$

Moreover, it follows from (2.23) and (2.24) that

$$⨍_{4Q} V|h|\,dz \leqslant ⨍_{4Q} V|v|\,dz + ⨍_{4Q} V|u|\,dz < 2.$$

Then from the above inequality and Lemma 8 we find that

$$\sup_{3Q} V|h| \leqslant C \sup_{4Q} V \,\left[V(4Q)\right]^{-1} \int_{4Q} V|h|\,dz \leqslant C \sup_{4Q} V \left(⨍_{4Q} V\,dz\right)^{-1},$$

which implies that

$$\sup_{3Q} V|h| \leqslant N_1, \tag{2.25}$$

since $V \in V_\infty$, where $N_1 > 1$ only depends on n.

2. Now we shall prove that

$$\{z \in Q : \mathscr{M}(V|u|)(z) > N_2\} \subset \{z \in Q : \mathscr{M}(|Vv|)(z) > N_1\}, \tag{2.26}$$

where $N_2 =: \max\{2N_1, 9^{n+1}\}$. To prove this, we fix

$$z \in \{z \in Q : \mathscr{M}(V|v|)(z) \leqslant N_1\}.$$

Case 1: $z \in Q_1 \subset 3Q$. Then we have

$$⨍_{Q_1} V|u|dz \leqslant ⨍_{Q_1} V|v|dz + N_1 \leqslant 2N_1,$$

since (2.25) and

$$V|u| \leqslant V|v| + V|h| \leqslant V|v| + N_1 \quad \text{for any} \quad z \in 3Q.$$

Case 2: $z \in Q_1 \not\subset 3Q$. Then we have $z \in Q \subset 3Q_1$ and $3Q \subset 9Q_1$. Since $z_0 \in \widetilde{Q} \subset 3Q \subset 9Q_1$ and $\mathscr{M}(V|u|)(z_0) \leqslant 1$, from (2.22) we find that

$$\fint_{Q_1} V|u|dy \leqslant 9^{n+1} \fint_{9Q_1} V|u|dy \leqslant 9^{n+1}.$$

Thus, combining Case 1 and Case 2, we conclude that

$$\mathscr{M}(V|u|)(z) \leqslant N_2,$$

which implies that the desired result (2.26) is true. Finally, from (2.24), (2.26) and the weak (1,1) estimate of the maximal functions we have

$$\begin{aligned} |\{z \in Q : \mathscr{M}(V|u|)(z) > N_2\}| &\leqslant |\{z \in Q : \mathscr{M}(|Vv|)(z) > N_1\}| \\ &\leqslant C \int_Q |Vv|\,dz \leqslant C \int_{4Q} |Vv|\,dz \leqslant C\delta\,|4Q| \leqslant C\delta\,|Q|. \end{aligned}$$

Then Lemma 1 (3) implies that

$$w(\{z \in Q : \mathscr{M}(V|u|)(z) \geqslant N_2\}) \leqslant C\delta^{\sigma} w(Q) \leqslant \mu w(Q)$$

by choosing δ small enough satisfying the last inequality. Thus we complete the proof. □

Furthermore, we can obtain the following result.

COROLLARY 3. *Assume that* $\mu \in (0,1)$ *with* $C_1\mu^{\sigma} < 1$ *and* w, δ, N_2 *satisfy the same conditions as those in Lemma 9. For any* $\lambda > 0$ *we have*

$$\begin{aligned} & w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(V|u|)(z) \geqslant \lambda N_2\right\}\right) \\ & \leqslant 2C_1\mu^{\sigma}\left[w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(V|u|)(z) > \lambda\right\}\right) + w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(|f|)(z) > \lambda\delta\right\}\right)\right]. \end{aligned}$$

Proof. Without loss of generality, we may as well assume that $\lambda = 1$. Let

$$\mathbb{R}^{n+1} = \bigcup_{i=1}^{\infty} \overline{D_i},$$

where $\{D_i\}$ is a sequence of disjoint square cubes. Moreover, from weak 1-1 estimate and the well-known L^1 maximal inequality (see [13], Lemma 3.1) we conclude that

$$\left|\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(V|u|)(z) \geqslant N_2\right\}\right| \leqslant \frac{C}{N_2}\|Vu\|_{L^1(\mathbb{R}^{n+1})} \leqslant \frac{C}{N_2}\|f\|_{L^1(\mathbb{R}^{n+1})}.$$

We may as well assume that $f \in C_0^{\infty}(\mathbb{R}^{n+1})$ via an elementary approximation argument. So, we can obtain

$$|\{z \in D_i : \mathscr{M}(V|u|)(z) \geqslant N_2\}| \leqslant \mu\,|D_i|$$

by selecting $|D_i|$ large enough for $i \in \mathbb{N}$. Furthermore, from Lemma 1 (3) we have

$$w(\{z \in D_i : \mathscr{M}(V|u|)(z) \geqslant N_2\}) \leqslant C_1\mu^{\sigma} w(D_i).$$

We denote

$$A = \{z \in D_i : \mathscr{M}(V|u|)(z) \geqslant N_2\}$$

and

$$B = \{z \in D_i : \mathscr{M}(V|u|)(z) > 1\} \cup \{z \in D_i : \mathscr{M}(|f|)(z) > \delta\}.$$

Then $A, B \subset D_i$ and $w(A) \leqslant C_1 \mu^\sigma w(D_i)$ with $C_1 \mu^\sigma < 1$. Therefore, it follows from Lemma 4 that there exists a sequence of disjoint square cubes $\{Q_k\}$ satisfying

1. $w(A \setminus \bigcup_{k \in \mathbb{N}} Q_k) = 0$,
2. $w(A \cap Q_k) > C_1 \mu^\sigma w(Q_k)$,
3. $w\left(A \cap \widetilde{Q_k}\right) \leqslant C_1 \mu^\sigma w\left(\widetilde{Q_k}\right)$ if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.

If $w\left(\widetilde{Q_k} \cap B\right) \leqslant \frac{1}{2} w\left(\widetilde{Q_k}\right)$, where $\widetilde{Q_k}$ is the predecessor of Q_k, then we obtain

$$\begin{aligned} & w\left(\left\{z \in \widetilde{Q_k} : \mathscr{M}(V|u|)(z) \leqslant 1\right\} \cap \left\{z \in \widetilde{Q_k} : \mathscr{M}(|f|)(z) \leqslant \delta\right\}\right) \\ & > w\left(\widetilde{Q_k} \cap \overline{B}\right) > \frac{1}{2} w\left(\widetilde{Q_k}\right). \end{aligned}$$

Furthermore, it follows from Lemma 9 that

$$w(A \cap Q_k) \leqslant w(\{z \in Q_k : \mathscr{M}(V|u|)(z) \geqslant N_2\}) \leqslant C_1 \mu^\sigma w(Q_k).$$

So, we get a contradiction with (2) and then know that $w\left(\widetilde{Q_k} \cap B\right) > \frac{1}{2} w\left(\widetilde{Q_k}\right)$. Finally, we can use Lemma 4 again to get that

$$w(A) \leqslant 2C_1 \mu^\sigma w(B),$$

which implies that

$$\begin{aligned} & w(\{z \in D_i : \mathscr{M}(V|u|)(z) \geqslant N_2\}) \\ & \leqslant 2C_1 \mu^\sigma \Big[w(\{z \in D_i : \mathscr{M}(V|u|)(z) > 1\}) + w(\{z \in D_i : \mathscr{M}(|f|)(z) > \delta\}) \Big]. \end{aligned}$$

Thus, we obtain the desired estimate by the summation. This finishes our proof. □

Now we are ready to prove the main result: Theorem 2.

Proof. From Lemma 3 and Corollary 3 we have

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}} \phi(|\mathscr{M}(V|u|)|) w(z) dz \\ & = \int_0^\infty w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(V|u|)(z) > N_2 \lambda\right\}\right) d[\phi(N_2 \lambda)] \\ & \leqslant 2C_1 \mu^\sigma \int_0^\infty w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(V|u|)(z) > \lambda\right\}\right) d[\phi(N_2 \lambda)] \\ & \quad + 2C_1 \mu^\sigma \int_0^\infty w\left(\left\{z \in \mathbb{R}^{n+1} : \mathscr{M}(|f|)(z) > \lambda \delta\right\}\right) d[\phi(N_2 \lambda)] \\ & \leqslant C_5 \mu^\sigma \int_{\mathbb{R}^{n+1}} \phi(|\mathscr{M}(V|u|)|) w(z) dz + C_6 \int_{\mathbb{R}^{n+1}} \phi(|\mathscr{M}(|f|)|) w(z) dz \end{aligned}$$

for any $\mu \in (0,1)$ with $C_1\mu^{\sigma} < 1$, where $C_5 = C_5(n,w,\phi)$ and $C_6 = C_6(n,w,\phi,\mu,\sigma)$. Without loss of generality we may as well assume that $f \in C_0^{\infty}(\mathbb{R}^{n+1})$. Then choosing a suitable μ such that $C_5\mu^{\sigma} < 1$, we obtain

$$\int_{\mathbb{R}^{n+1}} \phi\left(|\mathscr{M}\left(V|u|\right)|\right) w(z)dz \leqslant C\int_{\mathbb{R}^{n+1}} \phi\left(|\mathscr{M}\left(|f|\right)|\right) w(z)dz \leqslant C\int_{\mathbb{R}^{n+1}} \phi\left(|f|\right) w(z)dz$$

in view of Lemma 3. From the fact that $V|u|(z) \leqslant \mathscr{M}\left(V|u|\right)(z)$, we can obtain

$$\int_{\mathbb{R}^{n+1}} \phi\left(V|u|\right) w(z)dz \leqslant C\int_{\mathbb{R}^{n+1}} |f|^p w(z)dz.$$

Thus from Theorem 1 and (1.13) we deduce that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \phi\left(|D^2u|\right) w(z)dz &\leqslant C\left(\int_{\mathbb{R}^{n+1}} \phi\left(V|u|\right) w(z)dz + \int_{\mathbb{R}^{n+1}} |f|^p w(z)dz\right)\\ &\leqslant C\int_{\mathbb{R}^{n+1}} |f|^p w(z)dz,\end{aligned}$$

which completes the proof. □

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