Mathematical
Inequalities
& Applications
Volume 19, Number 2 (2016), 685–696 doi:10.7153/mia-19-49

# ESTIMATES OF ESSENTIAL NORM OF THE LI–STEVIĆ INTEGRAL TYPE OPERATOR BETWEEN ZYGMUND TYPE SPACES

AMIR H. SANATPOUR

(*Communicated by S. Stević*)

*Abstract.* For an analytic selfmap $\varphi$ of the open unit disc $\mathbb{D}$ and an analytic function $g$ on $\mathbb{D}$, the *Li-Stević integral type operator* $C_\varphi^g$ is given by

$$\left(C_\varphi^g f\right)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi.$$

We give essential norm estimates of the operator between Zygmund type spaces. We also apply our approach in the case of Bloch type spaces.

## 1. Introduction and preliminaries

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$. By a *weight* function $v$ we mean a continuous, strictly positive and bounded function $v:\mathbb{D}\to\mathbb{R}_+$. Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Then, for a weight $v$, the *weighted-type space* $H_v^\infty$, consists of all functions $f\in H(\mathbb{D})$ for which

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

For a weight $v$, the associated weight $\widetilde{v}$ is defined by

$$\widetilde{v}(z) = (\sup\{|f(z)| : f\in H_v^\infty, \|f\|_v \leqslant 1\})^{-1}, \quad z\in\mathbb{D}.$$

It is known that for *standard weights* $v_\alpha$ $(0<\alpha<\infty)$ given by

$$v_\alpha(z) = (1-|z|^2)^\alpha, \quad z\in\mathbb{D},$$

associated weights and weights are the same, i.e. $\widetilde{v}_\alpha = v_\alpha$. For each $0<\alpha<\infty$, we simply denote $H_{v_\alpha}^\infty$ by $H_\alpha^\infty$.

For each $0<\alpha<\infty$, the *Bloch type space* $\mathscr{B}^\alpha$ consists of all functions $f\in H(\mathbb{D})$ with

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

---

*Mathematics subject classification* (2010): 47B38, 30H30.
*Keywords and phrases*: Essential norm estimate, Li-Stević integral type operator, Zygmund type space, Bloch type space, weighted-type space.

The Bloch type space $\mathscr{B}^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)|, \quad f\in\mathscr{B}^\alpha.$$

If $\alpha = 1$, we get the classic *Bloch space* $\mathscr{B} = \mathscr{B}^1$. For each $0<\alpha<\infty$, the *Bloch type space modulo constant functions* $\widetilde{\mathscr{B}^\alpha}$ is the closed subspace of $\mathscr{B}^\alpha$ consisting of all functions $f\in\mathscr{B}^\alpha$ for which $f(0)=0$. For each $0<\alpha<\infty$, the *little Bloch type space* $\mathscr{B}_0^\alpha$ consists of those functions $f\in\mathscr{B}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

The *Zygmund space* $\mathscr{Z}$ is the class of all functions $f\in H(\mathbb{D})\cap C(\overline{\mathbb{D}})$ with

$$\sup_{\substack{e^{i\theta}\in\partial\mathbb{D}\\ h>0}} \frac{\left|f\left(e^{i(\theta+h)}\right)+f\left(e^{i(\theta-h)}\right)-2f\left(e^{i\theta}\right)\right|}{h} < \infty.$$

Based on a Zygmund's result (see, for example, [1, Theorem 5.3]), which says that the above condition is equivalent to

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty,$$

in [5], Li and Stević introduced the notions of *Zygmund space* and *little Zygmund space* and studied them, their extensions and operators from or to them considerably (see, for example, [5, 6, 7, 8, 9, 10, 11, 18, 19, 20] and references therein).

For each $0<\alpha<\infty$ the *Zygmund type space* $\mathscr{Z}^\alpha$ consists of all functions $f\in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty.$$

The Zygmund type space $\mathscr{Z}^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathscr{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)|, \quad f\in\mathscr{Z}^\alpha.$$

Similar to the case $\widetilde{\mathscr{B}^\alpha}$, for each $0<\alpha<\infty$, the space $\widetilde{\mathscr{Z}^\alpha}$ is defined to be the closed subspace of $\mathscr{Z}^\alpha$ consisting of all functions $f\in\mathscr{Z}^\alpha$ for which $f(0)=f'(0)=0$. Also, for each $0<\alpha<\infty$, the *little Zygmund type space* $\mathscr{Z}_0^\alpha$ consists of those functions $f\in\mathscr{Z}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = 0.$$

Recall that for Banach spaces $X$ and $Y$, a linear operator $T: X\to Y$ is *bounded* if it takes bounded sets to bounded sets, and $T$ is *compact* if it takes bounded sets to sets with compact closure. The space of all bounded operators and compact operators $T: X\to Y$ are denoted by $B(X,Y)$ and $K(X,Y)$, respectively. The *operator norm* of

$T \in B(X,Y)$ is denoted by $\|T\|_{X\to Y}$. The *essential norm* of $T \in B(X,Y)$, denoted by $\|T\|_{e,X\to Y}$, is defined as the distance from $T$ to $K(X,Y)$, that is

$$\|T\|_{e,X\to Y} = \operatorname{dist}(T, K(X,Y)) = \inf_{S\in K(X,Y)} \|T-S\|_{X\to Y}.$$

Clearly, an operator $T \in B(X,Y)$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. Therefore, essential norm estimates of bounded operators also give necessary and/or sufficient conditions for the compactness of such operators.

Essential norm estimates of different types of operators between certain classes of Banach spaces have been investigated by many authors. See, for example, [13, 14, 17, 21] and references therein. In this paper, we investigate the essential norms of certain types of operators, called the *Li-Stević integral type operator*, defined as follows.

Let $\psi \in H(\mathbb{D})$ and $\varphi$ be an analytic selfmap of $\mathbb{D}$. The *weighted composition operator* $\psi C_\varphi$, on $H(\mathbb{D})$, is the operator given by

$$\left(\psi C_\varphi f\right)(z) = \psi(z) f(\varphi(z)).$$

In the special case of $\psi = 1$ we get the *composition operator*

$$\left(C_\varphi f\right)(z) = f(\varphi(z)).$$

Boundedness and compactness of weighted composition operators between Bloch type spaces are studied in [16]. Also, essential norms of weighted composition operators between Bloch type spaces are given in [13, 14]. Characterizations for bounded and compact weighted composition operators between Zygmund type spaces and Bloch type spaces and essential norms of such operators are investigated in [2, 17].

For $g \in H(\mathbb{D})$ and an analytic selfmap $\varphi$ of $\mathbb{D}$, Li and Stević in [6] introduced the following operator

$$\left(C_\varphi^g f\right)(z) = \int_0^z f'(\varphi(\xi)) g(\xi)\, d\xi.$$

If $g = \varphi'$ then $C_\varphi^{\varphi'}$ is the composition operator $C_\varphi$ up to a point evaluation operator. The operator is nowadays known as *Li-Stević integral type operator* (see, e.g., [21]). An $n$-dimensional extension of the operator was given in [18]. In [6], Li and Stević investigated boundedness and compactness of the operator between Bloch type spaces and Zygmund type spaces. Boundedness and compactness of the operator between different spaces of analytic functions have been investigated by many authors. See, for example, recent papers [12, 18, 19, 22, 23] and references therein. Essential norm estimates of the operator between weighted Bergman spaces and Zygmund type spaces are investigated in [21]. In this paper, we give essential norm estimates of the operator between Zygmund type spaces and Bloch type spaces.

## 2. Main results

In this section, we first give essential norm estimates of the Li-Stević integral type operator between Zygmund type spaces. Then, we also apply our approach in the case of Bloch type spaces.

In order to get the upper estimate, for the essential norm of the operator $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$, we decompose the operator $C_\varphi^g$ using operators $D_\alpha$ and $S_\alpha$, defined as follows:

$$D_\alpha : \mathscr{Z}^\alpha \to \mathscr{B}^\alpha, \quad (D_\alpha f)(z) = f'(z),$$

$$S_\alpha : \mathscr{B}^\alpha \to \mathscr{Z}^\alpha, \quad (S_\alpha f)(z) = \int_0^z f(\xi) d\xi.$$

Note that operators $D_\alpha : \mathscr{Z}^\alpha \to \mathscr{B}^\alpha$ and $S_\alpha : \mathscr{B}^\alpha \to \mathscr{Z}^\alpha$ are bounded operators satisfying

$$\|D_\alpha f\|_{\mathscr{B}^\alpha} \leqslant \|f\|_{\mathscr{Z}^\alpha}, \quad f \in \mathscr{B}^\alpha,$$

$$\|S_\alpha f\|_{\mathscr{Z}^\alpha} = \|f\|_{\mathscr{B}^\alpha}, \quad f \in \mathscr{Z}^\alpha.$$

Moreover, we have

$$D_\alpha S_\alpha = id_{\mathscr{B}^\alpha} : \mathscr{B}^\alpha \to \mathscr{B}^\alpha, \tag{1}$$

while

$$S_\alpha D_\alpha \neq id_{\mathscr{Z}^\alpha} : \mathscr{Z}^\alpha \to \mathscr{Z}^\alpha.$$

Indeed,

$$(S_\alpha D_\alpha f)(z) = f(z) - f(0), \quad f \in \mathscr{Z}^\alpha, z \in \mathbb{D}, \tag{2}$$

where $f(0)$ is the constant function $z \mapsto f(0)$ in $\mathscr{Z}^\alpha$. About the operator $S_\alpha : \mathscr{B}^\alpha \to \mathscr{Z}^\alpha$, it is worth mentioning that $S_\alpha(\mathscr{B}^\alpha) \subseteq \widetilde{\mathscr{Z}^\alpha}$. Also, if we restrict domain of the operator $D_\alpha : \mathscr{Z}^\alpha \to \mathscr{B}^\alpha$ to $\widetilde{\mathscr{Z}^\alpha}$, by (2), we have

$$S_\alpha \left(D_\alpha|_{\widetilde{\mathscr{Z}^\alpha}}\right) = id_{\widetilde{\mathscr{Z}^\alpha}} : \widetilde{\mathscr{Z}^\alpha} \to \widetilde{\mathscr{Z}^\alpha}. \tag{3}$$

Regarding (3), in the next theorem we show that for an arbitrary bounded operator $U : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$, in order to get estimates for $\|U\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta}$ one can focus on $\|U|_{\widetilde{\mathscr{Z}^\alpha}}\|_{e,\widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta}$.

THEOREM 1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $U : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$ be a bounded operator. Then, $\|U\|_{e,\widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta} = \|U\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta}$.*

*Proof.* We also denote the operator $U|_{\widetilde{\mathscr{Z}^\alpha}} : \widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta$ by $U$. It is easy to see that $\|U\|_{e,\widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta} \leqslant \|U\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta}$. Indeed, for every compact operator $T : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$, the operator $T|_{\widetilde{\mathscr{Z}^\alpha}} : \widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta$ is also compact and therefore

$$\begin{aligned}
\|U\|_{e,\widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta} &\leqslant \|U - T|_{\widetilde{\mathscr{Z}^\alpha}}\|_{\widetilde{\mathscr{Z}^\alpha} \to \mathscr{Z}^\beta} \\
&= \sup_{f \in \widetilde{\mathscr{Z}^\alpha}, \|f\|_{\mathscr{Z}^\alpha} \leqslant 1} \|Uf - Tf\|_{\mathscr{Z}^\beta} \\
&\leqslant \sup_{f \in \mathscr{Z}^\alpha, \|f\|_{\mathscr{Z}^\alpha} \leqslant 1} \|Uf - Tf\|_{\mathscr{Z}^\beta} \\
&= \|U - T\|_{\mathscr{Z}^\alpha \to \mathscr{Z}^\beta},
\end{aligned}$$

which implies that

$$\|U\|_{e,\widetilde{\mathscr{Z}^\alpha}\to\mathscr{Z}^\beta} \leqslant \|U\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}. \tag{4}$$

In order to prove the converse of (4), consider an arbitrary compact operator $\widetilde{T}:\widetilde{\mathscr{Z}^\alpha}\to\mathscr{Z}^\beta$. Let $T:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$ be the (compact) extension of $\widetilde{T}$ to $\mathscr{Z}^\alpha$ defined by

$$T(f) = T\left(f - f(0) - f'(0)z\right) + f(0) + f'(0)z,$$

for each $f\in\mathscr{Z}^\alpha$. Also, regarding operators $U$ and $T$, consider related operators

$$U_0:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta,$$

$$T_0:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta,$$

defined by

$$U_0(f) = U\left(f(0)+f'(0)z\right) = f(0)U1 + f'(0)Uz,$$
$$T_0(f) = T\left(f(0)+f'(0)z\right) = f(0)T1 + f'(0)Tz,$$

for each $f\in\mathscr{Z}^\alpha$. Note that $T_0:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$ and $U_0:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$ are compact operators. Therefore,

$$\begin{aligned}
\|U\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta} \leqslant & \|U-(T-T_0+U_0)\|_{\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}\\
= & \sup_{f\in\mathscr{Z}^\alpha,\|f\|_{\mathscr{Z}^\alpha}\leqslant 1} \|(U-U_0)f-(T-T_0)f\|_{\mathscr{Z}^\beta}\\
= & \sup_{f\in\mathscr{Z}^\alpha,\|f\|_{\mathscr{Z}^\alpha}\leqslant 1} \|U(f-f(0)-f'(0)z)-T(f-f(0)-f'(0)z)\|_{\mathscr{Z}^\beta}\\
\leqslant & \sup_{g\in\widetilde{\mathscr{Z}^\alpha},\|g\|_{\mathscr{Z}^\alpha}\leqslant 1} \|U(g)-T(g)\|_{\mathscr{Z}^\beta}\\
= & \sup_{g\in\widetilde{\mathscr{Z}^\alpha},\|g\|_{\mathscr{Z}^\alpha}\leqslant 1} \|U(g)-\widetilde{T}(g)\|_{\mathscr{Z}^\beta}\\
= & \|U-\widetilde{T}\|_{\widetilde{\mathscr{Z}^\alpha}\to\mathscr{Z}^\beta}.
\end{aligned} \tag{5}$$

Since the compact operator $\widetilde{T}:\widetilde{\mathscr{Z}^\alpha}\to\mathscr{Z}^\beta$ was arbitrary, (5) implies that

$$\|U\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta} \leqslant \|U\|_{e,\widetilde{\mathscr{Z}^\alpha}\to\mathscr{Z}^\beta},$$

and completes the proof. □

REMARK 1. The result of Theorem 1 is proved in [2, Lemma 3.1] for (bounded) weighted composition operators between Zygmund type spaces. But note that Theorem 1, besides revising the proof of [2, Lemma 3.1] on page 481, generalizes its result to an arbitrary (bounded) operator between Zygmund type spaces.

It is worth mentioning that the result of Theorem 1 is also valid if we replace $\widetilde{\mathscr{Z}^\alpha}$ with the closed subspace of all functions $f\in\mathscr{Z}^\alpha$ for which $f(0)=0$.

In the next theorem, we state the result of Theorem 1 for Bloch type spaces.

THEOREM 2. *Let* $0<\alpha<\infty$, $0<\beta<\infty$ *and* $U:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ *be a bounded operator. Then,* $\|U\|_{e,\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta}=\|U\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$.

*Proof.* The proof is similar to the proof of Theorem 1 with minor modifications. We next sketch an outline of the proof.

Let $U$ also denote the restriction operator $U|_{\widetilde{\mathscr{B}}^\alpha}:\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta$. Then, as in the proof of Theorem 1, one can see that

$$\|U\|_{e,\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta}\leqslant\|U\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}. \tag{6}$$

In order to prove the converse of (6), let $\widetilde{T}:\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta$ be an arbitrary compact operator. Define $T:\mathscr{B}^\alpha\to\mathscr{B}^\beta$, the (compact) extension of $\widetilde{T}$ to $\mathscr{B}^\alpha$, by

$$T(f)=T(f-f(0))+f(0),$$

for each $f\in\mathscr{B}^\alpha$. Also, consider related compact operators

$$U_0:\mathscr{B}^\alpha\to\mathscr{B}^\beta,$$

$$T_0:\mathscr{B}^\alpha\to\mathscr{B}^\beta,$$

defined by

$$U_0(f)=U(f(0))=f(0)U1,$$

$$T_0(f)=T(f(0))=f(0)T1,$$

for each $f\in\mathscr{B}^\alpha$. Then, as in the proof of Theorem 1, one can see that

$$\|U\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}\leqslant\|U-\widetilde{T}\|_{\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta},$$

and since $\widetilde{T}:\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta$ was an arbitrary compact operator, we get

$$\|U\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}\leqslant\|U\|_{e,\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta},$$

which completes the proof. □

In the next theorem we give the upper estimate for the essential norm of the operator $C_\varphi^g:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$, in the general case of $0<\alpha<\infty$ and $0<\beta<\infty$. Applying Theorems 1 and 2, in order to get the desired upper estimate, we focus on the essential norms of $C_\varphi^g:\widetilde{\mathscr{Z}}^\alpha\to\mathscr{Z}^\beta$ and $gC_\varphi:\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta$.

THEOREM 3. *Let* $0<\alpha<\infty$, $0<\beta<\infty$ *and* $C_\varphi^g:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$ *be a bounded operator. Then,* $\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}\leqslant\|gC_\varphi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$.

*Proof.* By Theorems 1 and 2, it is enough to show that

$$\|C_\varphi^g\|_{e,\widetilde{\mathscr{Z}}^\alpha\to\mathscr{Z}^\beta}\leqslant\|gC_\varphi\|_{e,\widetilde{\mathscr{B}}^\alpha\to\mathscr{B}^\beta}.$$

For each $0<\alpha<\infty$ and $0<\beta<\infty$, if $C_\varphi^g : \widetilde{\mathscr{Z}}^\alpha \to \mathscr{Z}^\beta$ is a well-defined (bounded) operator, then

$$D_\beta C_\varphi^g S_\alpha = gC_\varphi : \widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta,$$

is also a well-defined (bounded) operator. Note that, by (2), we have

$$S_\alpha D_\alpha f = f, \quad f \in \widetilde{\mathscr{Z}}^\alpha. \tag{7}$$

Also, since $\left(C_\varphi^g f\right)(0)=0$, we have

$$S_\beta D_\beta C_\varphi^g f = C_\varphi^g f, \quad f \in \mathscr{Z}^\alpha,$$

which along (7) implies that

$$S_\beta D_\beta C_\varphi^g S_\alpha D_\alpha f = C_\varphi^g f, \quad f \in \widetilde{\mathscr{Z}}^\alpha. \tag{8}$$

Now, let $T : \widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta$ be an arbitrary compact operator. Then, boundedness of the operators $D_\alpha : \widetilde{\mathscr{Z}}^\alpha \to \widetilde{\mathscr{B}}^\alpha$ and $S_\beta : \mathscr{B}^\beta \to \mathscr{Z}^\beta$ implies that

$$K = S_\beta T D_\alpha : \widetilde{\mathscr{Z}}^\alpha \to \mathscr{Z}^\beta,$$

is also a compact operator. Therefore, by applying (8), for each $f \in \widetilde{\mathscr{Z}}^\alpha$ we get

$$\begin{aligned}
\left\|C_\varphi^g f - Kf\right\|_{\mathscr{Z}^\beta} &= \left\|S_\beta D_\beta C_\varphi^g S_\alpha D_\alpha f - S_\beta T D_\alpha f\right\|_{\mathscr{Z}^\beta} \\
&= \left\|S_\beta \left(D_\beta C_\varphi^g S_\alpha - T\right) D_\alpha f\right\|_{\mathscr{Z}^\beta} \\
&= \left\|\left(gC_\varphi - T\right) D_\alpha f\right\|_{\mathscr{B}^\beta} \\
&\leqslant \left\|gC_\varphi - T\right\|_{\widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta} \left\|D_\alpha f\right\|_{\mathscr{B}^\alpha} \\
&\leqslant \left\|gC_\varphi - T\right\|_{\widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta} \|f\|_{\mathscr{Z}^\alpha}.
\end{aligned}$$

This implies that

$$\|C_\varphi^g\|_{e,\widetilde{\mathscr{Z}}^\alpha \to \mathscr{Z}^\beta} \leqslant \|C_\varphi^g - K\|_{\widetilde{\mathscr{Z}}^\alpha \to \mathscr{Z}^\beta} \leqslant \left\|gC_\varphi - T\right\|_{\widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta}. \tag{9}$$

Since the compact operator $T : \widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta$ in (9) was arbitrary, we conclude that

$$\|C_\varphi^g\|_{e,\widetilde{\mathscr{Z}}^\alpha \to \mathscr{Z}^\beta} \leqslant \|gC_\varphi\|_{e,\widetilde{\mathscr{B}}^\alpha \to \mathscr{B}^\beta}. \quad \square$$

In the following theorem we give the lower estimate for the essential norm of the operator $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$, in the general case of $0<\alpha<\infty$ and $0<\beta<\infty$. Note that, as mentioned in (1), for each $0<\alpha<\infty$ we have

$$D_\alpha S_\alpha = id_{\mathscr{B}^\alpha} : \mathscr{B}^\alpha \to \mathscr{B}^\alpha.$$

As a consequence of this fact, unlike Theorem 3, proof of the next theorem is independent of focusing on $\widetilde{\mathscr{Z}}^\alpha$ or using Theorems 1 and 2.

THEOREM 4. *Let* $0<\alpha<\infty$, $0<\beta<\infty$ *and* $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$ *be a bounded operator. Then,* $\|gC_\varphi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} \leqslant \|C_\varphi^g\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}$.

*Proof.* For each $0<\alpha<\infty$ and $0<\beta<\infty$, boundedness of the operators $D_\beta : \mathscr{Z}^\beta \to \mathscr{B}^\beta$ and $S_\alpha : \mathscr{B}^\alpha \to \mathscr{Z}^\alpha$ implies that

$$D_\beta C_\varphi^g S_\alpha = gC_\varphi : \mathscr{B}^\alpha \to \mathscr{B}^\beta,$$

is also a well-defined (bounded) operator. Moreover, we have

$$S_\beta gC_\varphi D_\alpha = C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta.$$

For any compact operator $T : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$, the operator

$$K = D_\beta T S_\alpha : \mathscr{B}^\alpha \to \mathscr{B}^\beta,$$

is also a compact operator and for each $f\in\mathscr{B}^\alpha$ we have

$$\begin{aligned}\left\|gC_\varphi f - Kf\right\|_{\mathscr{B}^\beta} &= \left\|D_\beta S_\beta gC_\varphi D_\alpha S_\alpha f - D_\beta T S_\alpha f\right\|_{\mathscr{B}^\beta}\\ &= \left\|D_\beta\left(S_\beta gC_\varphi D_\alpha - T\right) S_\alpha f\right\|_{\mathscr{B}^\beta}\\ &\leqslant \left\|\left(C_\varphi^g - T\right) S_\alpha f\right\|_{\mathscr{Z}^\beta}\\ &\leqslant \left\|C_\varphi^g - T\right\|_{\mathscr{Z}^\alpha\to\mathscr{Z}^\beta} \|S_\alpha f\|_{\mathscr{Z}^\alpha}\\ &= \left\|C_\varphi^g - T\right\|_{\mathscr{Z}^\alpha\to\mathscr{Z}^\beta} \|f\|_{\mathscr{B}^\alpha}.\end{aligned}$$

This implies that

$$\|gC_\varphi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} \leqslant \|gC_\varphi - K\|_{\mathscr{B}^\alpha\to\mathscr{B}^\beta} \leqslant \left\|C_\varphi^g - T\right\|_{\mathscr{Z}^\alpha\to\mathscr{Z}^\beta},$$

and since the compact operator $T : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$ was arbitrary, we get

$$\|gC_\varphi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} \leqslant \|C_\varphi^g\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}. \quad \square$$

As a consequence of Theorems 3 and 4, by applying [13, Theorems 3 and 4] and the proof of [14, Theorem 8], we get the following estimates for the essential norm of the operator $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$ in different cases of $0<\alpha<\infty$ and $0<\beta<\infty$.

Before stating the next theorem we note that, for real scalars $A$ and $B$, the notation $A \asymp B$ means that $cB \leqslant A \leqslant CB$ for some positive constants $c$ and $C$. Also, in order to simplify the notation in the statement of our results, we use the following simplifications (see, for example, [13]):

$$A(g,\varphi,\alpha,\beta) = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|g(z)|,$$

$$B(g,\varphi,\beta) = \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta |g(z)| \log\frac{1}{1-|\varphi(z)|^2}.$$

THEOREM 5. *Let* $0 < \alpha < \infty$, $0 < \beta < \infty$ *and* $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{Z}^\beta$ *be a bounded operator. Then,* $\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta} = \|gC_\varphi\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta}$ *and consequently*

*(i) if* $0 < \alpha < 1$, *then*

$$\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta} \asymp A(g\varphi', \varphi, \alpha, \beta),$$

*(ii)*

$$\|C_\varphi^g\|_{e,\mathscr{Z} \to \mathscr{Z}^\beta} \asymp \max\Big\{A(g\varphi', \varphi, 1, \beta), B(g', \varphi, \beta)\Big\},$$

*(iii) if* $1 < \alpha < \infty$, *then*

$$\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha \to \mathscr{Z}^\beta} \asymp \max\Big\{A(g\varphi', \varphi, \alpha, \beta), A(g', \varphi, \alpha - 1, \beta)\Big\}.$$

REMARK 2. In order to get lower estimates in different parts of Theorem 5, instead of using Theorem 4, one can also use the classical approach of using test functions in different cases of $0 < \alpha < \infty$ and $0 < \beta < \infty$ (see, for example, [2, 17, 21] for such classic approach). It is worth mentioning that, in order to apply test function approach and deal with limsup-terms of Theorem 5, for an arbitrary sequence $(z_n)$ in $\mathbb{D}$ with $|\varphi(z_n)| \to 1$, one can consider sequences of functions $\big(g_{\varphi(z_n)}\big)$ and $\big(h_{\varphi(z_n)}\big)$ described below, as suitable test functions in $\mathscr{Z}_0^\alpha$:

$(i)$ If $0 < \alpha < 1$, then

$$g_a(z) = f_a(z) - k_a(z),$$

where,

$$f_a(z) = \frac{1}{\bar{a}^2}\left(\frac{(1-|a|^2)^2}{(1-\bar{a}z)^\alpha} - \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha-1}}\right),$$

and

$$k_a(z) = \frac{1}{\bar{a}}\int_0^z \frac{1-|a|^2}{(1-\bar{a}w)^\alpha}dw.$$

$(ii)$ If $\alpha = 1$, then

$$g_a(z) = \frac{f(\bar{a}z)}{\bar{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1} - \left(\int_0^z \ln^3\frac{1}{1-\bar{a}w}dw\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-2},$$

and

$$h_a(z) = \frac{f(\bar{a}z)}{\bar{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1},$$

where,

$$f(z) = (z-1)\left(\left(1+\ln\frac{1}{1-z}\right)^2 + 1\right).$$

$(iii)$ If $1<\alpha<\infty$, then the function $g_a$ is as described in part $(i)$ and

$$h_a(z)=\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\alpha}}.$$

We next show that the given approach for obtaining essential norm estimates of $C_\varphi^g:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$ can also be applied to get the essential norm of $C_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$, for any $0<\alpha<\infty$ and $0<\beta<\infty$.

Indeed, in the case of Bloch type spaces, the operators $D_\alpha$ and $S_\alpha$ are defined as follows:

$$D_\alpha:\mathscr{B}^\alpha\to H_\alpha^\infty,\quad (D_\alpha f)(z)=f'(z),$$

$$S_\alpha:H_\alpha^\infty\to\mathscr{B}^\alpha,\quad (S_\alpha f)(z)=\int_0^z f(\xi)d\xi.$$

Operators $D_\alpha:\mathscr{B}^\alpha\to H_\alpha^\infty$ and $S_\alpha:H_\alpha^\infty\to\mathscr{B}^\alpha$ are bounded operators and

$$\|D_\alpha f\|_{H_\alpha^\infty}\leqslant\|f\|_{\mathscr{B}^\alpha},\quad f\in\mathscr{B}^\alpha,$$

$$\|S_\alpha f\|_{\mathscr{B}^\alpha}=\|f\|_{H_\alpha^\infty},\quad f\in H_\alpha^\infty.$$

Using Theorem 2 and applying a similar approach as in the case of $C_\varphi^g:\mathscr{Z}^\alpha\to\mathscr{Z}^\beta$, one can show that the essential norm of $C_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is equal to the essential norm of $gC_\varphi:H_\alpha^\infty\to H_\beta^\infty$. This, along with [15, Theorem 2.1], leads to the following result.

THEOREM 6. *Let* $0<\alpha<\infty$, $0<\beta<\infty$ *and* $C_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ *be a bounded operator. Then,* $\|C_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}=\|gC_\varphi\|_{e,H_\alpha^\infty\to H_\beta^\infty}$ *and consequently*

$$\|C_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}=\limsup_{|\varphi(z)|\to 1}|g(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

REMARK 3. Since differentiation operator $D_\alpha$ and integration operator $S_\alpha$ lead to a natural isometry $\widetilde{\mathscr{Z}^\alpha}\simeq\widetilde{\mathscr{B}^\alpha}$ satisfying $D_\beta C_\varphi^g S_\alpha=gC_\varphi$ and $S_\beta gC_\varphi D_\alpha=C_\varphi^g$, one can immediately conclude that

$$\|C_\varphi^g\|_{e,\widetilde{\mathscr{Z}^\alpha}\to\widetilde{\mathscr{Z}^\beta}}=\|gC_\varphi\|_{e,\widetilde{\mathscr{B}^\alpha}\to\widetilde{\mathscr{B}^\beta}}.$$

The same argument is valid for the equality

$$\|C_\varphi^g\|_{e,\widetilde{\mathscr{B}^\alpha}\to\widetilde{\mathscr{B}^\beta}}=\|gC_\varphi\|_{e,H_\alpha^\infty\to H_\beta^\infty}.$$

It is worth mentioning that, this argument is *not* valid for obtaining the essential norms of such operators on the *original spaces*, that is $\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}$ and $\|C_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$. See for example [15, Theorem 2.3] where the author, in order to apply this argument, considered "*Bloch type spaces modulo constant functions*" instead of "*Bloch type spaces*".

But the approach given in this paper, leading to Theorems 5 and 6, gives the equality of essential norms on the original spaces, that is

$$\|C_\varphi^g\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta} = \|gC_\varphi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta},$$

and

$$\|C_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} = \|gC_\varphi\|_{e,H_\alpha^\infty\to H_\beta^\infty}.$$

REMARK 4. The approach leading to Theorems 5 and 6, can also be applied to show that essential norms of $C_\varphi^g : \mathscr{Z}^\alpha \to \mathscr{B}^\beta$ and $C_\varphi^g : \mathscr{B}^\alpha \to \mathscr{Z}^\beta$ are equal to essential norms of $gC_\varphi : \mathscr{B}^\alpha \to H_\beta^\infty$ and $gC_\varphi : H_\alpha^\infty \to \mathscr{B}^\beta$, respectively.

Recently, there has been growing interest in stating essential norm estimates of the type $A(g,\varphi,\alpha,\beta)$ (or $B(g,\varphi,\beta)$) in terms of $g$ and $\varphi^n$. See, for example, [3, 14, 17] for such new estimates. Essential norm estimates of the operators $C_\varphi^g$ given in this paper, can also be stated in terms of $g$ and $\varphi^n$ using Theorems 5 and 6 and applying related estimates for the essential norms of weighted composition operators $gC_\varphi$ in terms of $g$ and $\varphi^n$.

Finally, it is worth mentioning that, as a consequence of essential norm estimates of the operators $C_\varphi^g$ given in this paper, one can get necessary and sufficient conditions for the compactness of such operators [6].

*Acknowledgement.* The author is grateful to the anonymous referee whose valuable comments and suggestions improved the manuscript.

## REFERENCES

[1] P. L. DUREN, *Theory of $H^p$ Spaces*, Pure and Applied Mathematics **38**, Academic Press, San Diego, 1970.

[2] K. ESMAEILI AND M. LINDSTRÖM, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integr. Equ. Oper. Theory **75** (2013), 473–490.

[3] O. HYVÄRINEN, M. KEMPPAINEN, M. LINDSTRÖM, A. RAUTIO AND E. SAUKKO, *The essential norm of weighted composition operators on weighted Banach spaces of analytic functions*, Integr. Equ. Oper. Theory **72** (2012), 151–157.

[4] O. HYVÄRINEN AND M. LINDSTRÖM, *Estimates of essential norms of weighted composition operators between Bloch type spaces*, J. Math. Anal. Appl. **393** (2012), 38–44.

[5] S. LI AND S. STEVIĆ, *Volterra-type operators on Zygmund spaces*, J. Inequal. Appl. (2007), Art. ID 32124, 10 pp.

[6] S. LI AND S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338** (2008), 1282–1295.

[7] S. LI AND S. STEVIĆ, *Products of Volterra type operator and composition operator from $H^\infty$ and Bloch spaces to Zygmund spaces*, J. Math. Anal. Appl. **345** (2008), 40–52.

[8] S. LI AND S. STEVIĆ, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Appl. Math. Comput. **206** (2008), 825–831.

[9] S. LI AND S. STEVIĆ, *Integral-type operators from Bloch-type spaces to Zygmund-type spaces*, Appl. Math. Comput. **215** (2009), 464–473.

[10] S. LI AND S. STEVIĆ, *On an integral-type operator from $\omega$-Bloch spaces to $\mu$-Zygmund spaces*, Appl. Math. Comput. **215** (2010), 4385–4391.

[11] S. LI AND S. STEVIĆ, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput. **217** (2010), 3144–3154.

[12] M. LINDSTRÖM AND A. H. SANATPOUR, *Derivative-free characterizations of compact generalized composition operators between Zygmund type spaces*, Bull. Aust. Math. Soc. **81** (2010), 398-408.

[13] B. MACCLUER AND R. ZHAO, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), 1437–1458.

[14] J. S. MANHAS AND R. ZHAO, *New estimates of essential norms of weighted composition operators between Bloch type spaces*, J. Math. Anal. Appl. **389** (2012), 32–47.

[15] A. MONTES-RODRÍGUEZ, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. Lond. Math. Soc. **61** (2000), 872–884.

[16] S. OHNO, K. STROETHOFF AND R. ZHAO, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), 191–215.

[17] A. H. SANATPOUR AND M. HASSANLOU, *Essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces*, Turkish J. Math. **38** (2014), 872–882.

[18] S. STEVIĆ, *On an integral operator from the Zygmund space to the Bloch-type space on the unit ball*, Glasg. Math. J. **51** (2009), 275–287.

[19] S. STEVIĆ, *Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball*, Sib. Math. J. **50** (2009), 1098–1105.

[20] S. STEVIĆ, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstr. Appl. Anal. (2010), Art. ID 198608, 7 pp.

[21] S. UEKI, *On the Li-Stević integral type operators from weighted Bergman spaces into $\beta$-Zygmund spaces*, Integr. Equ. Oper. Theory **74** (2012), 137–150.

[22] W. YANG, Y. LUO AND X. ZHU, *Differences of generalized composition operators between Bloch type spaces*, Math. Inequal. Appl. **17** (2014), 977–987.

[23] X. ZHU, *A new characterization of the generalized weighted composition operator from $H^\infty$ into the Zygmund space*, Math. Inequal. Appl. **18** (2015), 1135–1142.

(Received August 19, 2015)

*Amir H. Sanatpour*
*Department of Mathematics*
*Kharazmi University*
*Tehran, Iran*
*e-mail:* a_sanatpour@khu.ac.ir
a.sanatpour@gmail.com