Mathematical Inequalities & Applications
Volume 19, Number 2 (2016), 709–719
doi:10.7153/mia-19-51

TURÁN TYPE INEQUALITIES FOR GENERAL BESSEL FUNCTIONS

ÁRPÁD BARICZ, SAMINATHAN PONNUSAMY AND SANJEEV SINGH

(*Communicated by T. Erdélyi*)

Abstract. In this paper some Turán type inequalities for the general Bessel function, monotonicity and bounds for its logarithmic derivative are derived. Moreover we find the series representation and the relative extrema of the Turánian of general Bessel functions. The key tools in the proofs are the recurrence relations together with some asymptotic relations for Bessel functions.

1. Introduction and the main results

The Turán type inequalities for orthogonal polynomials and special functions have been studied extensively in the last 70 years. Usually, these orthogonal polynomials and special functions are solutions of some second order differential equations. The log-concave/log-convex nature of orthogonal polynomials and special functions have attracted many researchers, and the topic seems to be interesting still nowadays. Some of the results on modified Bessel functions of the first and second kind have been used recently in different problems of applied mathematics and this motivated new researches in this topic. See for example the paper [1] and the references therein for more details. In this paper we focus on general Bessel functions, sometimes called cylinder functions. The main motivation to write this paper emerges from the fact about the Bessel functions of the first kind J_ν, Bessel functions of the second kind Y_ν and the zeros $c_{\nu,n}$ of general Bessel function satisfying some Turán type inequalities (see [3, 5, 7, 9, 13]). It is natural to ask whether the general Bessel functions C_ν, defined by $C_\nu(x) = (\cos\alpha)J_\nu(x) - (\sin\alpha)Y_\nu(x),\ 0 \leqslant \alpha < \pi$, has some similar properties. As we can see below, from the point of view of Turán type inequalities, the general Bessel function C_ν behaves like J_ν and Y_ν. The results presented in this paper complement the picture about the Turán type inequalities for Bessel functions of the first and second kind. The case $\alpha = 0$ corresponds to the case of Bessel function J_ν, while $\alpha = \pi/2$ corresponds to the case of Bessel function Y_ν. See [3, 5, 7, 13] for more details. Note that in the proof of our main results we used ideas from the papers [2, 3, 5, 7, 11, 13], the recurrence relations, asymptotics and the differential equation play an important role in the proof of the main results.

Mathematics subject classification (2010): 39B62, 33C10, 42A05.
Keywords and phrases: General Bessel functions, Bessel functions of the first and second kind, Turán type inequalities, recurrence relations, asymptotic relations.
The research of Á. Baricz was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The second author is on leave from the IIT Madras. The research of S. Singh was supported by the fellowship of the University Grants Commission, India. The authors wish to acknowledge the referee's comments and suggestions which enhanced this paper.

THEOREM 1. *The following assertions are valid:*

a. *If* $\nu > 0$, $0 < \alpha < \pi$ *and* $x \geqslant c_{\nu,1}$ *where* $c_{\nu,1}$ *is the first positive zero of the general Bessel function* C_ν*, then the following Turán type inequality holds*

$$\Delta_\nu(x) = C_\nu^2(x) - C_{\nu-1}(x)C_{\nu+1}(x) > \frac{1}{\nu+1}C_\nu^2(x). \tag{1}$$

Moreover, for $\alpha = 0$ *the above Turán type inequality holds true for all* $x > 0$ *and* $\nu > 0$.

b. *If* $\nu > 1$, $0 < \alpha < \pi$ *and* $x_\nu \in (0, c_{\nu,1})$ *is the unique root of the equation*

$$C_\nu^2(x) - C_{\nu-1}(x)C_{\nu+1}(x) = \frac{1}{\nu+1}C_\nu^2(x),$$

then the Turán type inequality (1) *holds true for all* $x > x_\nu$. *Moreover, the inequality* (1) *is reversed for* $0 < x < x_\nu$.

c. *The function* $x \mapsto xC'_\nu(x)/C_\nu(x)$ *is strictly decreasing on* $(c_{\nu,1}, \infty) \setminus \mathscr{Z}$ *for all* $\nu > 0$, $0 < \alpha < \pi$*, where* $\mathscr{Z} = \{c_{\nu,n}\}_{n \geqslant 1}$*, and* $c_{\nu,n}$ *denote the* n*th positive zeros of the general Bessel functions* C_ν. *Moreover, if* $\alpha = 0$ *then* $x \mapsto xC'_\nu(x)/C_\nu(x)$ *is strictly decreasing on* $(0, \infty) \setminus \mathscr{Z}$ *for all* $\nu > 0$ *and if* $0 < \alpha < \pi$ *then* $x \mapsto xC'_\nu(x)/C_\nu(x)$ *is strictly decreasing on* $(x_\nu, \infty) \setminus \mathscr{Z}$ *for all* $\nu > 1$. *Furthermore, the following inequality holds true for* $\nu > 0$, $0 < \alpha < \pi$ *and* $x \in (c_{\nu,1}, \infty) \setminus \mathscr{Z}$

$$\left[\frac{xC'_\nu(x)}{C_\nu(x)}\right]^2 > \nu^2 - \frac{\nu}{\nu+1}x^2. \tag{2}$$

If $\alpha = 0$*, then the inequality* (2) *is valid for all* $\nu > 0$, $x \in (0, \infty) \setminus \mathscr{Z}$. *However, if* $0 < \alpha < \pi$*, then the inequality* (2) *is valid for all* $\nu > 1$, $x \in (x_\nu, \infty) \setminus \mathscr{Z}$*, and for* $x \in (0, x_\nu)$ *it is reversed. The following inequality is also valid for* $\nu > 1$, $0 < \alpha < \pi$ *and* $x \in (x_\nu, c_{\nu,1})$

$$\frac{xC'_\nu(x)}{C_\nu(x)} < -\sqrt{\nu^2 - \frac{\nu}{\nu+1}x_\nu^2}. \tag{3}$$

The proof of Theorem 1 will be presented in Section 2.

Now we let $\mu = \frac{\nu}{\nu+1}$ and denote by $j_{\nu,n}$, the nth positive zero of the Bessel function J_ν. We would like to mention that by using the particular case of (2) when $\alpha = 0$, it can be shown that for $\nu > 0$, $x \in (0, \sqrt{\nu(\nu+1)}]$ such that $x \neq j_{\nu-1,n}$, $n \in \mathbb{N}$, we have

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \leqslant \frac{\nu - \sqrt{\nu^2 - \mu x^2}}{\mu x} \leqslant \frac{\nu + \sqrt{\nu^2 - \mu x^2}}{\mu x}, \tag{4}$$

and this inequality corrects the inequality [3, eq. 2.20]

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \geqslant \frac{\nu + \sqrt{\nu^2 - \mu x^2}}{\mu x},$$

where $\nu > 0$, $x \in (0, \sqrt{\nu(\nu+1)}]$ such that $x \neq j_{\nu-1,n}$, $n \in \mathbb{N}$.

Indeed, to prove inequality (4), we first consider the inequality [3, eq. 2.17]

$$[xJ_\nu'(x)]^2 + (\mu x^2 - \nu^2)J_\nu^2(x) \geqslant 0, \qquad \nu > -1 \text{ and } x \in \mathbb{R},$$

which implies

$$\left[\frac{xJ_\nu'(x)}{J_\nu(x)}\right]^2 \geqslant \nu^2 - \mu x^2, \qquad \nu > -1,\ x \in \mathbb{R},\ x \neq j_{\nu,n},\ n \in \mathbb{N}.$$

In view of the limit

$$\lim_{x\to 0} \frac{xJ_\nu'(x)}{J_\nu(x)} = \nu > 0$$

and the fact that smallest positive zero $j_{\nu,1}'$ of J_ν' satisfies the inequality $j_{\nu,1}' > \sqrt{\nu(\nu+2)}$ for $\nu > 0$ [14, p. 487], it follows that

$$\frac{xJ_\nu'(x)}{J_\nu(x)} > 0, \qquad \nu > 0 \text{ and } x \in (0, \sqrt{\nu(\nu+2)}],\ x \neq j_{\nu,n}.$$

Hence for $\nu > 0$, $x \in (0, \sqrt{\nu(\nu+1)}]$ such that $x \neq j_{\nu,n}$, $n \in \mathbb{N}$ we have

$$\frac{xJ_\nu'(x)}{J_\nu(x)} \geqslant \sqrt{\nu^2 - \mu x^2}$$

which on using the recurrence relation

$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x) \tag{5}$$

gives the inequality

$$\sqrt{\nu^2 - \mu x^2} \leqslant \frac{xJ_{\nu-1}(x)}{J_\nu(x)} - \nu.$$

Rewriting the last inequality as

$$0 < \frac{\nu + \sqrt{\nu^2 - \mu x^2}}{x} \leqslant \frac{J_{\nu-1}(x)}{J_\nu(x)},$$

we obtain

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \leqslant \frac{x}{\nu + \sqrt{\nu^2 - \mu x^2}} = \frac{\nu - \sqrt{\nu^2 - \mu x^2}}{\mu x},$$

where $x \in (0, \sqrt{\nu(\nu+1)}]$ such that $x \neq j_{\nu-1,n}$, $n \in \mathbb{N}$. This completes the proof of inequality (4).

We also note that the monotonicity of $x \mapsto xC_\nu'(x)/C_\nu(x)$ has been proved already by Spigler [12] (as it is mentioned in the paper of Elbert and Siafarikas [4]), but only for the intervals $(c_{\nu,n}, c_{\nu,n+1})$, $n \in \mathbb{N}$. Our proof, which is based on Turán type inequalities, is completely different and we proved the above monotonicity property for $x \in (x_\nu, c_{\nu,1})$ and also for $x \in (c_{\nu,n}, c_{\nu,n+1})$, $n \in \mathbb{N}$.

We recall the inequality from part **c** of Theorem 1

$$\left[\frac{xC_\nu'(x)}{C_\nu(x)}\right]^2 < \nu^2 - \frac{\nu}{\nu+1}x^2,$$

where $\nu > 1$, $0 < \alpha < \pi$ and $x \in (0, x_\nu)$. We may rewrite the above inequality as

$$-\sqrt{\nu^2 - \frac{\nu}{\nu+1}x^2} < \frac{xC_\nu'(x)}{C_\nu(x)} < \sqrt{\nu^2 - \frac{\nu}{\nu+1}x^2}, \tag{6}$$

where $\nu > 1$, $0 < \alpha < \pi$ and $x \in (0, x_\nu)$. We would like to mention that for $\nu > 1$, $0 < \alpha < \pi$ and $x \in (0, x_\nu)$ the right-hand side of (6) is better than an earlier inequality of Laforgia [6, p. 76]

$$\frac{xC_\nu'(x)}{C_\nu(x)} < \nu - \frac{x^2}{2(\nu+1)}, \tag{7}$$

which is valid for $\nu > 0$, $0 \leqslant \alpha < \pi$ and $x \in (0, c_{\nu,1})$. This can be verified by comparing the corresponding right-hand sides of the last two inequalities. We also note that for $\nu > 1$ and $0 < \alpha < \pi$ we have $x_\nu < \sqrt{\nu(\nu+1)}$, since the expression in the square root in (3) is positive.

Again, it is worth to mention that the inequality (3) is better than the inequality (7) for all $\nu > 1$, $0 < \alpha < \pi$ and $x \in (x_\nu, \min\{\sqrt{\nu(\nu+1)}, c_{\nu,1}\})$ as the right-hand side of (3) is negative and the right-hand side of (7) is positive on $(x_\nu, \min\{\sqrt{\nu(\nu+1)}, c_{\nu,1}\})$.

The next theorem is about the series representation of the Turánian of general Bessel functions. Clearly, this result implies inequality (1).

THEOREM 2. *For $0 \leqslant \alpha < \pi$, $\nu > 0$ and $x > c_{\nu,1}$, the following identity holds*

$$C_\nu^2(x) - C_{\nu-1}(x)C_{\nu+1}(x) = \frac{1}{\nu+1}C_\nu^2(x) + 2\nu\sum_{i\geqslant 1}\frac{C_{\nu+i}^2(x)}{(\nu+i)^2-1}. \tag{8}$$

The next result, whose proof will also be given in Section 2, is about the relative extrema of the Turánian of general Bessel functions and is a generalization of the main result from [7]. Figure 1 illustrates this result for $\alpha = \pi/6$ and $\nu = 3/2$.

THEOREM 3. *For $0 \leqslant \alpha < \pi$ and $\nu > 0$, the relative maxima (denoted by $M_{\nu,k}$) of the function $x \mapsto \Delta_\nu(x)$ occurs at the zeros of the function $C_{\nu-1}(x)$ and the relative minima (denoted by $m_{\nu,k}$) occurs at the zeros of the function $C_{\nu+1}(x)$. Since the values of $M_{\nu,k}$ and $m_{\nu,k}$ can be expressed as $M_{\nu,k} = \Delta_\nu(c_{\nu-1,k}) = C_\nu^2(c_{\nu-1,k}) > 0$ and $m_{\nu,k} = \Delta_\nu(c_{\nu+1,k}) = C_\nu^2(c_{\nu+1,k}) > 0$, for $\nu > 0$ and $x \geqslant c_{\nu-1,1}$, the following Turán type inequality is valid:*

$$C_\nu^2(x) - C_{\nu-1}(x)C_{\nu+1}(x) > 0.$$

We also mention that for $x \in (0, \nu]$ such that $x \neq j_{\nu-1,n}$, $n \in \mathbb{N}$, the following inequality is valid:

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} < \frac{\nu - \sqrt{\nu^2 - x^2}}{x} < \frac{\nu + \sqrt{\nu^2 - x^2}}{x}. \tag{9}$$

Note that the above inequality corrects the recent known inequality [3, eq 2.16]

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \geqslant \frac{\nu+\sqrt{\nu^2-x^2}}{x},$$

where $\nu > 0$, $x \in (0,\nu]$ such that $x \neq j_{\nu-1,n}$, $n \in \mathbb{N}$.

To prove the first inequality (9), it is suffices to observe the following inequality

$$\sqrt{\nu^2-x^2} < \sqrt{\nu^2-\mu x^2}, \qquad x \in (0,\nu),$$

and consequently we have

$$\frac{\nu-\sqrt{\nu^2-\mu x^2}}{\mu x} = \frac{x}{\nu+\sqrt{\nu^2-\mu x^2}} < \frac{x}{\nu+\sqrt{\nu^2-x^2}}, \qquad x \in (0,\nu),$$

which in view of left-hand side of (4) implies left-hand side of (9). This proves inequality (9).

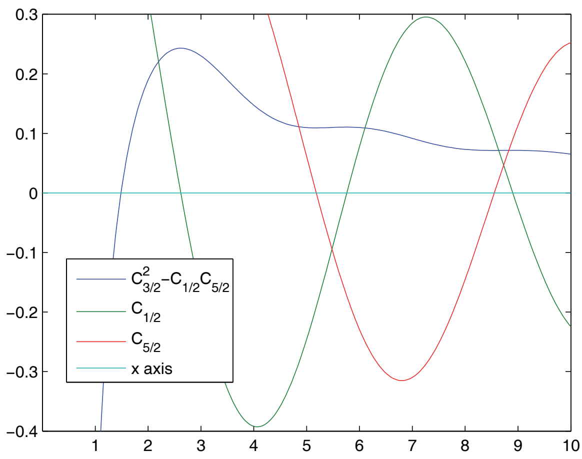

Figure 1: *The graph of the functions* $\Delta_{3/2}$, $C_{1/2}$ *and* $C_{5/2}$ *for* $\alpha = \pi/6$ *on* $[0,10]$.

Finally, it is worth to mention that the Turánian

$$\Delta_\alpha(x) = C^2_{\nu,\alpha}(x) - C_{\nu,\alpha-1}(x)C_{\nu,\alpha+1}(x),$$

where as above $C_{\nu,\alpha}(x) = (\cos\alpha)J_\nu(x) - (\sin\alpha)Y_\nu(x)$, $0 \leqslant \alpha < \pi$, is in fact independent of α. Namely, by using some elementary trigonometric identities it can be shown that $\Delta_\alpha(x) = (\sin^2 1)\left(J^2_\nu(x) + Y^2_\nu(x)\right)$, which is clearly strictly positive for all real ν and x.

2. Proofs of the main results

Proof of Theorem 1. **a.** We first recall the recurrence relation and the derivative formula for general Bessel functions [10, p. 222], which will be used in the sequel

$$C_{\nu-1}(x)+C_{\nu+1}(x)=\frac{2\nu}{x}C_{\nu}(x), \tag{10}$$

and

$$\frac{d}{dx}(x^{-\nu}C_{\nu}(x))=-x^{-\nu}C_{\nu+1}(x). \tag{11}$$

Define the normalized general Bessel function by $\Phi_{\nu}(x)=2^{\nu}x^{-\nu}\Gamma(\nu+1)C_{\nu}(x)$, where $\nu>-1$ and $x>0$. Since $C_{\nu}(x)$ is the solution of the Bessel differential equation

$$x^2y''(x)+xy'(x)+(x^2-\nu^2)y(x)=0,$$

we see that $\Phi_{\nu}(x)$ satisfies the differential equation

$$x^2\Phi_{\nu}''(x)+(2\nu+1)x\Phi_{\nu}'(x)+x^2\Phi_{\nu}(x)=0. \tag{12}$$

Now, if we consider the Turánian $\Theta_{\nu}(x)=\Phi_{\nu}^2(x)-\Phi_{\nu-1}(x)\Phi_{\nu+1}(x)$, then the Turán type inequality (1) is equivalent to $\Theta_{\nu}(x)>0$. Using the definition of the normalized general Bessel function $\Phi_{\nu}(x)$, we can write

$$x^{2\nu+2}\Theta_{\nu}(x)=2^{2\nu}\Gamma(\nu)\Gamma(\nu+2)\left[x^2\left(C_{\nu}^2(x)-C_{\nu-1}(x)C_{\nu+1}(x)\right)-\frac{x^2}{\nu+1}C_{\nu}^2(x)\right]. \tag{13}$$

Taking into account (10) we have

$$\frac{x^2\Phi_{\nu+1}(x)}{4\nu(\nu+1)}=\Phi_{\nu}(x)-\Phi_{\nu-1}(x), \tag{14}$$

and consequently, in view of (11),

$$\Phi_{\nu}'(x)=-\frac{x\Phi_{\nu+1}(x)}{2(\nu+1)}=\frac{2\nu}{x}\left(\Phi_{\nu-1}(x)-\Phi_{\nu}(x)\right). \tag{15}$$

By using the left-hand side of (15) for $\nu-1$ instead of ν and the right-hand side of (15) for $\nu+1$ instead of ν, we have

$$\begin{aligned}
x\Theta'_\nu(x) &= 2x\Phi_\nu(x)\Phi'_\nu(x) - x\Phi_{\nu+1}(x)\Phi'_{\nu-1}(x) - x\Phi_{\nu-1}(x)\Phi'_{\nu+1}(x) \\
&= 2x\Phi_\nu(x)\left[\frac{2\nu}{x}\left(\Phi_{\nu-1}(x)-\Phi_\nu(x)\right)\right] - x\Phi_{\nu+1}(x)\left[-\frac{x\Phi_\nu(x)}{2\nu}\right] \\
&\quad -x\Phi_{\nu-1}(x)\left[\frac{2(\nu+1)}{x}\left(\Phi_\nu(x)-\Phi_{\nu+1}(x)\right)\right] \\
&= 2(\nu-1)\Phi_\nu(x)\Phi_{\nu-1}(x) - 4\nu\Phi_\nu^2(x) + \frac{x^2}{2\nu}\Phi_\nu(x)\Phi_{\nu+1}(x) \\
&\quad +2(\nu+1)\Phi_{\nu-1}(x)\Phi_{\nu+1}(x) \\
&= 2(\nu-1)\Phi_\nu(x)\Phi_{\nu-1}(x) - 4\nu\Phi_\nu^2(x) + \Phi_\nu(x)\left[2(\nu+1)(\Phi_\nu(x)-\Phi_{\nu-1}(x))\right] \\
&\quad +2(\nu+1)\Phi_{\nu-1}(x)\Phi_{\nu+1}(x) \\
&= -4\Phi_\nu(x)\Phi_{\nu-1}(x) - 2(\nu-1)\Phi_\nu^2(x) + 2(\nu+1)\Phi_{\nu-1}(x)\Phi_{\nu+1}(x) \\
&= -4\Phi_\nu(x)\left[\frac{x}{2\nu}\Phi'_\nu(x)+\Phi_\nu(x)\right] - 2(\nu-1)\Phi_\nu^2(x) + 2(\nu+1)\Phi_{\nu-1}(x)\Phi_{\nu+1}(x) \\
&= -\frac{2x}{\nu}\Phi_\nu(x)\Phi'_\nu(x) - 2(\nu+1)\left[\Phi_\nu^2(x) - \Phi_{\nu-1}(x)\Phi_{\nu+1}(x)\right].
\end{aligned}$$

Thus, we obtain

$$x\Theta'_\nu(x) = -\frac{2x}{\nu}\Phi_\nu(x)\Phi'_\nu(x) - 2(\nu+1)\Theta_\nu(x),$$

which on multiplying by $x^{2\nu+1}$ both sides can be written as

$$x^{2\nu+2}\Theta'_\nu(x) + (2\nu+2)x^{2\nu+1}\Theta_\nu(x) = -\frac{2}{\nu}x^{2\nu+2}\Phi_\nu(x)\Phi'_\nu(x).$$

Thus, we have

$$\frac{d}{dx}(x^{2\nu+2}\Theta_\nu(x)) = -\frac{2}{\nu}x^{2\nu+2}\Phi_\nu(x)\Phi'_\nu(x).$$

From the above expression, at the roots of $\Phi_\nu(x) = 0$ we have

$$\frac{d^2}{dx^2}(x^{2\nu+2}\Theta_\nu(x)) = -\frac{2}{\nu}x^{2\nu+2}\left(\Phi'_\nu(x)\right)^2 < 0$$

and at the roots of $\Phi'_\nu(x) = 0$, by using (12) we obtain

$$\frac{d^2}{dx^2}(x^{2\nu+2}\Theta_\nu(x)) = \frac{2}{\nu}x^{2\nu+2}\left(\Phi_\nu(x)\right)^2 > 0.$$

These two inequalities show that the relative extrema of $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ occurs at the roots of $\Phi_\nu(x) = 0$ and $\Phi'_\nu(x) = 0$, respectively. At the roots of $\Phi_\nu(x) = 0$, by (14) we have

$$\Theta_\nu(x) = -\Phi_{\nu-1}(x)\Phi_{\nu+1}(x) = \frac{x^2}{4\nu(\nu+1)}\Phi_{\nu+1}^2(x) > 0,$$

and in view of (15) at the roots of $\Phi'_\nu(x) = 0$ we have $\Theta_\nu(x) = \Phi^2_\nu(x) > 0$. On the other hand, $\Phi_\nu(x) = 0$ if and only if $C_\nu(x) = 0$ and in view of (15) we have that

$$\Phi'_\nu(x) = 0 \Longleftrightarrow \Phi_{\nu+1}(x) = 0 \Longleftrightarrow C_{\nu+1}(x) = 0.$$

Therefore, the first relative extrema of $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ occurs at $x = c_{\nu,1}$, as $c_{\nu,1} < c_{\nu+1,1}$ (since $\nu \mapsto c_{\nu,k}$ is increasing function of ν [14, p. 508]). Since $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ has all its relative extrema positive, it follows that $x^{2\nu+2}\Theta_\nu(x) > 0$ for all $x \geqslant c_{\nu,1}$ and $\nu > 0$, which implies that $\Theta_\nu(x) > 0$ and consequently the Turán type inequality (1) follows for all $\nu > 0$ and $x \geqslant c_{\nu,1}$. Since the first relative extrema of $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ occurs at $x = c_{\nu,1}$, which is the point of relative maxima, we conclude that $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ is strictly increasing on $(0, c_{\nu,1})$. Now, if we take $\alpha = 0$, then $C_\nu(x) = J_\nu(x)$ and using the fact that $J_\nu(0) = 0$ for $\nu > 0$ we have $\lim_{x\to 0^+} x^{2\nu+2}\Theta_\nu(x) = 0$. Hence $x^{2\nu+2}\Theta_\nu(x) > 0$ on $(0, c_{\nu,1})$ and consequently, $\Theta_\nu(x) > 0$. Therefore in this case, the Turán type inequality (1) holds true for all $\nu > 0$ and $x > 0$.

b. In view of (13) and the fact that $J_\nu(0) = 0$ for $\nu > 0$, the asymptotic formula which is valid for $\nu > 0$ fixed and $x \to 0$ [10, p. 223]

$$Y_\nu(x) \sim -\frac{1}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu},$$

and the limit (see [3, p. 316])

$$\lim_{x\to 0^+} x^2\left(Y^2_\nu(x) - Y_{\nu-1}(x)Y_{\nu+1}(x)\right) = -\infty,$$

where $\nu > 1$ is fixed, we obtain that

$$\lim_{x\to 0^+} x^{2\nu+2}\Theta_\nu(x) = -\infty.$$

Therefore, since $x^{2\nu+2}\Theta_\nu(x)$ is positive at $x = c_{\nu,1}$, $x^{2\nu+2}\Theta_\nu(x)$ tends to $-\infty$ as $x \to 0^+$ and $x \mapsto x^{2\nu+2}\Theta_\nu(x)$ is strictly increasing on $(0, c_{\nu,1})$, it follows that there exists an unique $x_\nu \in (0, c_{\nu,1})$ such that

$$\begin{cases} x^{2\nu+2}\Theta_\nu(x) < 0 & \text{for } x \in (0, x_\nu), \\ x^{2\nu+2}\Theta_\nu(x) = 0 & \text{for } x = x_\nu, \\ x^{2\nu+2}\Theta_\nu(x) > 0 & \text{for } x \in (x_\nu, c_{\nu,1}). \end{cases}$$

Hence $\Theta_\nu(x) > 0$ for $x \in (x_\nu, c_{\nu,1})$. Consequently, by part **a** of this theorem, the Turán type inequality (1) is indeed true for all $\nu > 1$ and $x > x_\nu$. We also note that $\Theta_\nu(x) < 0$ for $x \in (0, x_\nu)$ and hence in this case inequality (1) is reversed.

c. By the recurrence relations [10, p. 222]

$$C'_\nu(x) = C_{\nu-1}(x) - \frac{\nu}{x}C_\nu(x) \quad \text{and} \quad C'_\nu(x) = -C_{\nu+1}(x) + \frac{\nu}{x}C_\nu(x), \tag{16}$$

the Turán expression $\Delta_\nu(x)$ can be written as

$$\Delta_\nu(x) = C^2_\nu(x) - C_{\nu-1}(x)C_{\nu+1}(x) = \left(1 - \frac{\nu^2}{x^2}\right)C^2_\nu(x) + [C'_\nu(x)]^2. \tag{17}$$

Now, by (17) and the Bessel differential equation we get

$$\left(\frac{xC_\nu'(x)}{C_\nu(x)}\right)' = -\frac{x\Delta_\nu(x)}{C_\nu^2(x)}.$$

Thus, in view of parts **a** and **b** of Theorem 1, the monotonicity of $x \mapsto xC_\nu'(x)/C_\nu(x)$ follows. We note that the inequality (2) follows from (1) by using the above recurrence relations in (16).

Since $x \mapsto xC_\nu'(x)/C_\nu(x)$ is strictly decreasing on $(x_\nu, c_{\nu,1})$ for all $\nu > 1$, we see that

$$\frac{xC_\nu'(x)}{C_\nu(x)} < \lim_{x\to x_\nu} \frac{xC_\nu'(x)}{C_\nu(x)}, \qquad \text{for all } x \in (x_\nu, c_{\nu,1}). \tag{18}$$

Using relation (17) we obtain

$$\left[\frac{xC_\nu'(x)}{C_\nu(x)}\right]^2 = \frac{x^2\Delta_\nu(x)}{C_\nu^2(x)} + (\nu^2 - x^2),$$

which together with the fact that (see [6, p. 78])

$$C_\nu(x) > 0 \text{ and } C_\nu'(x) < 0 \text{ for } 0 < x < c_{\nu,1}$$

implies

$$\frac{xC_\nu'(x)}{C_\nu(x)} = -\sqrt{\frac{x^2\Delta_\nu(x)}{C_\nu^2(x)} + (\nu^2 - x^2)}, \qquad \text{for all } x \in (x_\nu, c_{\nu,1}).$$

Taking the limit $x \to x_\nu$ in the above equation and using (18) and (13), we get inequality (3). □

Before we give the proof of Theorem 2, recall a result of Ross [11, Theorem 3].

LEMMA 1. *Let I be an interval and let $\{y_n\}_{n\geqslant 0}$ be a sequence of functions of real variable x, which is uniformly bounded in n for each $x \in I$. If these functions satisfy*

$$y_n(x) = B_n y_{n+1}(x) + C_n y_{n-1}(x),$$

where B_n and C_n are functions of x, $x \in I$, with the property that $C_n(x) \neq 0$, $B_n(x) \to 0$ and $\prod_{i=1}^n |B_i(x)/C_i(x)|$ converges as $n \to \infty$ for all $x \in I$, then

$$y_n^2(x) - y_{n-1}(x)y_{n+1}(x) = -\frac{\delta C_n}{C_n} y_n^2(x) - \sum_{i\geqslant 1} \frac{B_{n+1}B_{n+2}\cdots B_{n+i+1}}{C_nC_{n+1}\cdots C_{n+i}} \delta(B_{n+i-1}C_{n+i}) y_{n+i}^2(x), \tag{19}$$

where δ is the forward difference operator defined by $\delta y_n = y_{n+1} - y_n$.

We note here that (as mentioned in [2]) in formula (i) of [11, p. 28] the expression $B_n y_n$ should be written as $B_{n+1} y_n$, and in the main formula of [11, Theorem 3] the expression B_{n+i-1} should be written as B_{n+i+1}, just like in (19).

Proof of Theorem 2. We recall one of the main results of Landau [8, p. 151]: the magnitude of the general Bessel function of order ν is decreasing in ν at all of its positive stationary points. We note that here by stationary points we mean actually zeros of $C_\nu'(x)$. This means that if $\nu > 0$ and $x > c_{\nu,1}$, then we have that $|C_\nu(x)| < |C_0(x)| < |\tau|$, where $\tau = C_0(x_1)$ and x_1 is the abscissa of the first minimum point of $C_0(x)$. Hence in view of the recurrence relation (10), all the conditions of Lemma 1 are satisfied and consequently we obtain identity (8). □

Proof of Theorem 3. Using the recurrence relations in (16), we have

$$\begin{aligned}\Delta_\nu'(x) &= 2C_\nu(x)C_\nu'(x) - C_{\nu-1}'(x)C_{\nu+1}(x) - C_{\nu-1}(x)C_{\nu+1}'(x)\\ &= C_\nu(x)\left[C_{\nu-1}(x) - C_{\nu+1}(x)\right] - C_{\nu+1}(x)\left[-C_\nu(x) + \frac{\nu-1}{x}C_{\nu-1}(x)\right]\\ &\quad -C_{\nu-1}(x)\left[C_\nu(x) - \frac{\nu+1}{x}C_{\nu+1}(x)\right].\end{aligned}$$

Therefore

$$\Delta_\nu'(x) = \frac{2}{x}C_{\nu-1}(x)C_{\nu+1}(x). \tag{20}$$

Hence the relative extrema of $x \mapsto \Delta_\nu(x)$ occurs at the zeros of $C_{\nu-1}(x)$ and $C_{\nu+1}(x)$. From (20), by using the second recurrence relation in (16) for $\nu-1$ instead of ν, and (10), we get

$$\Delta_\nu''(x)\big|_{x=c_{\nu-1,k}} = -\frac{4\nu}{c_{\nu-1,k}^2}C_\nu^2(c_{\nu-1,k}) < 0.$$

Similarly, by using the first recurrence relation in (16) for $\nu+1$ instead of ν, and (10), we get

$$\Delta_\nu''(x)\big|_{x=c_{\nu+1,k}} = \frac{4\nu}{c_{\nu+1,k}^2}C_\nu^2(c_{\nu+1,k}) > 0.$$

The desired conclusion follows. □

REFERENCES

[1] Á. BARICZ, *Bounds for Turánians of modified Bessel functions*, Expo. Math. **23**, 2 (2015), 223–251.

[2] Á. BARICZ, *Turán type inequalities for regular Coulomb wave functions*, J. Math. Anal. Appl. **430**, 1 (2015), 166–180.

[3] Á. BARICZ AND T. K. POGÁNY *Turán determinants of Bessel functions*, Forum Math. **26**, (2014), 295–322.

[4] Á. ELBERT AND P. D. SIAFARIKAS, *On the zeros of $\alpha C_\nu(x) + xC_\nu'(x)$, where $C_\nu(x)$ is a cylinder function*, J. Math. Anal. Appl. **164**, (1992), 21–33.

[5] C. M. JOSHI AND S. K. BISSU, *Some inequalities of Bessel and modified Bessel functions*, J. Aust. Math. Soc. Ser. A **50**, (1991), 333–342.

[6] A. LAFORGIA, *Inequalities for Bessel functions*, J. Comput. Appl. Math. **15**, (1986), 75–81.

[7] S. K. LAKSHMANA RAO, *On the relative extrema of the Turán expression for Bessel functions*, Proc. Indian Acad. Sci. Sect. A **53**, (1961), 239–243.

[8] L. LANDAU, *Monotonicity and bounds on Bessel functions*, Electron. J. Differ. Equ. Conf. **4**, (2000), 147–154.

[9] L. LORCH, *Turánians and Wronskians for the zeros of Bessel functions*, SIAM J. Math. Anal. **11**, (1980), 223–227.
[10] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, Cambridge, 2010.
[11] D. K. ROSS, *Inequalities and identities for* $y_n^2 - y_{n-1}y_{n+1}$, Aequationes Math. **20**, (1980), 23–32.
[12] R. SPIGLER, *Sulle radici dell'equazione,* $AC_\nu(x) + BxC'_\nu(x)$, Atti Sem. Mat. Fis. Univ. Modena **27**, (1978), 153–166.
[13] O. SZÁSZ, *Identities and inequalities concerning orthogonal polynomials and Bessel functions*, J. Analyse Math. **1**, (1951), 116–134.
[14] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1922.

(Received September 19, 2015)

Árpád Baricz
Institute of Applied Mathematics
Óbuda University
1034 Budapest, Hungary
and
Department of Economics
Babeş-Bolyai University
Cluj-Napoca 400591, Romania
e-mail: bariczocsi@yahoo.com

Saminathan Ponnusamy
Indian Statistical Institute, Chennai Centre
Society for Electronic Transactions and Security
MGR Knowledge City, CIT Campus
Taramani, Chennai 600113, India
e-mail: samy@iitm.ac.in,
samy@isichennai.res.in

Sanjeev Singh
Department of Mathematics
Indian Institute of Technology Madras
Chennai 600036, India
e-mail: sanjeevsinghiitm@gmail.com