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ISOPERIMETRIC–TYPE INEQUALITIES FOR GENERALIZED CENTROID BODIES

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Abstract. We extend the Orlicz centroid body for probability measures to multivariate case and establish the affine isoperimetric-type inequality for the generalized centroid body. Moreover, the concepts of centroid body and mean zonoid are unified through the generalized definition, which is much more significant.

1. Introduction

Centroid bodies were defined and investigated by Petty [25]. They are special sets which play an important role in the Brunn-Minkowski theory of convex bodies and have been proven to be a remarkably powerful tool in establishing a number of fundamental affine isoperimetric inequalities (see, e.g., [4, 12, 13, 19, 26]).

With the rapid development of the L_p-Brunn-Minkowski theory and its dual in the last two decades, The L_p-analogues of centroid bodies quickly became a central focus and affine isoperimetric inequalities for L_p-centroid body were established. See, for example, [2, 14, 15, 18, 27, 29]. Moreover, The L_p-centroid bodies have turned out to be a useful tool in the study of asymptotic geometric analysis (see, e.g., [6, 10, 21, 24]) and the theory of stable distributions (e.g. [20]).

Recently, Orlicz Brunn-Minkowski theory which extended the L_p Brunn-Minkowski theory emerged and the fundamental notions of L_p centroid body, L_p projection body and L_p addition were extended to an Orlicz setting. See, e.g., [5, 8, 11, 16, 17, 28]. In this paper, we generalize the Orlicz centroid body for probability measures and establish the centroid inequality.

Let $\mathscr{K}^n$ denote the class of convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^n$. We consider convex functions $\phi:\mathbb{R}\to[0,\infty)$ such that $\phi(0)=0$. This means that ϕ must be decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. We require that ϕ is either strictly decreasing on $(-\infty,0]$ or strictly increasing on $[0,\infty)$. The class of such ϕ is denoted by $\mathscr{C}$. Let $\mathscr{C}_0$ be the class of Young functions, i.e., convex, strictly increasing functions $\phi:[0,\infty)\to[0,\infty)$ such that $\phi(0)=0$.

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Let $\phi \in \mathscr{C}$ and $K \in \mathscr{K}^n$ with $\operatorname{vol}_n(K) = 1$, the Orlicz centroid body $\Gamma_\phi(K)$ of K which is introduced by Lutwak, Yang and Zhang [16] is the convex body whose support function is given by

$$h_{\Gamma_\phi(K)}(x) = \inf\left\{\lambda > 0 : \int_K \phi\left(\frac{\langle x,y\rangle}{\lambda}\right) dy \leqslant 1\right\}, \tag{1}$$

where $\langle x,y\rangle$ denotes the standard inner product of x and y in $\mathbb{R}^n$ and the integration is with respect to Lebesgue measure on $\mathbb{R}^n$.

The following Orlicz centroid inequality is also given by Lutwak, Yang and Zhang [16],

$$\operatorname{vol}_n(\Gamma_\phi(K)) \geqslant \operatorname{vol}_n(\Gamma_\phi(D_n)), \tag{2}$$

where D_n is the Euclidean ball of volume one. Equality holds if and only if K is an ellipsoid centered at the origin.

Let $\mathscr{P}_{[n]}$ be the class of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. By the probabilistic take, Paouris and Pivovarov [23] extended the symmetric case of (1) to the class of $\mathscr{P}_{[n]}$.

Let $\phi \in \mathscr{C}_0, \mu \in \mathscr{P}_{[n]}$. Define the Orlicz centroid body $\Gamma_\phi(\mu)$ of μ corresponding to ϕ by its support function

$$h(\Gamma_\phi(\mu),y) = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\left(\frac{|\langle x,y\rangle|}{\lambda}\right) d\mu(x) \leqslant 1\right\}. \tag{3}$$

If f denotes the density of μ and if $\|f\|_\infty \leqslant 1$, then

$$\operatorname{vol}_n(\Gamma_\phi(\mu)) \geqslant \operatorname{vol}_n(\Gamma_\phi(\lambda_{D_n})), \tag{4}$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n.

Recently, the asymmetric case of (3) was discussed in [9] and the centroid inequality analogous to (4) was also established.

A zonoid in $\mathbb{R}^n$ is an origin-symmetric convex body that can be approximated (in the Hausdorff metric) by finite Minkowski sums of line segments. Clearly, the centroid body is a zonoid. Zhang [29] defined a mean zonoid $\widetilde{Z}K$ of a convex body K with $\operatorname{vol}_n(K) = 1$ by

$$h_{\widetilde{Z}K}(u) = \int_K \int_K |\langle u, x-y\rangle| dxdy \quad \text{for all } u \in S^{n-1}, \tag{5}$$

and showed that the volume of $\widetilde{Z}K$ satisfies

$$\operatorname{vol}_n(\widetilde{Z}K) \geqslant \operatorname{vol}_n(\widetilde{Z}D_n), \tag{6}$$

equality holds if and only if K is an ellipsoid.

The L_p version and the Orlicz version of mean zonoid and inequalities were obtained recently, see [7, 27].

Inspired by the notion of mean zonoid, we generalize the definition of Orlicz centroid body for probability measure, and make it possible to unify the notions of centroid body and mean zonoid.

DEFINITION 1. Let $m \geqslant 1$ be an integer, $\phi \in \mathscr{C}_0, \mu_i \in \mathscr{P}_{[n]}$, $i = 1,2,\cdots,m$. Let f_i denote the density of $\mu_i, i = 1,2,\cdots,m$. We define the *generalized centroid body* $\Gamma_\phi(\mu_1,\cdots,\mu_m)$ of $\mu_1,\cdots,\mu_m$ corresponding to ϕ as the convex body whose support function at $y \in \mathbb{R}^n$ is given by

$$\begin{aligned}&h(\Gamma_\phi(\mu_1,\cdots,\mu_m),y)\\&=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+\cdots+x_m,y\rangle|}{\lambda}\right)d\mu_1(x_1)\cdots d\mu_m(x_m)\leqslant 1\right\}.\end{aligned}\tag{7}$$

If $m = 1$, (7) is just the Orlicz centroid body defined by (3). If $m = 2$, $\phi = |t|$, $f_1 = \mathbf{1}_K$, $f_2 = \mathbf{1}_{-K}$ where $K \in \mathscr{K}^n$ with $\mathrm{vol}_n(K) = 1$, then $\Gamma_\phi(\mu_1,\mu_2) = \widetilde{Z}K$, the mean zonoid (5) defined by Zhang [29]. That is, the mean zonoid is one of the special cases of the generalized centroid body defined above.

We will also establish the following affine isoperimetric inequality for the generalized centroid bodies.

THEOREM 1. *Let $m \geqslant 1$ be an integer, $\phi \in \mathscr{C}_0$, $\mu_i \in \mathscr{P}_{[n]}$, $i = 1,2,\cdots,m$. Let f_i denote the density of μ_i with $\|f_i\|_\infty \leqslant 1$, $i = 1,2,\cdots,m$. then*

$$vol_n(\Gamma_\phi(\mu_1,\cdots,\mu_m)) \geqslant vol_n(\Gamma_\phi(\lambda_{D_n},\cdots,\lambda_{D_n})),\tag{8}$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n.

Choosing proper values of m, ϕ and f_i in the above theorem, we can get the affine isopermetric inequalities obtained in [7, 22, 23, 27, 29] (see corollaries in section 3). Especially, if $m = 2$, we can define the general mean zonoid which generalizes the mean zonoid (5).

COROLLARY 1. *Let $\phi \in \mathscr{C}_0$, $\mu_1,\mu_2 \in \mathscr{P}_{[n]}$, Define the* general mean zonoid *$Z_\phi(\mu_1,\mu_2)$ of μ_1 and μ_2 corresponding to ϕ by its support function*

$$\begin{aligned}h(Z_\phi(\mu_1,\mu_2),y) :&= h(\Gamma_\phi(\mu_1,\mu_2),y)\\&=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+x_2,y\rangle|}{\lambda}\right)d\mu_1(x_1)d\mu_2(x_2)\leqslant 1\right\}.\end{aligned}$$

If f_i denotes the density of μ_i with $\|f_i\|_\infty \leqslant 1$, $i = 1,2$. Then

$$vol_n(Z_\phi(\mu_1,\mu_2)) \geqslant vol_n(Z_\phi(\lambda_{D_n},\lambda_{D_n})),$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n.

The idea and the techniques of Paouris and Pivovarov [23], especially the law of large numbers and the methods of approximation and limiting arguments, play a critical role throughout the paper.

2. Preliminaries

We work in $\mathbb{R}^n$, which is equipped with an Euclidean structure $\langle\cdot,\cdot\rangle$. Let S^{n-1} denote the unit sphere, B_2^n be the unit Euclidean ball and D_n be the Euclidean ball of volume one. Let $\mathrm{vol}_n(\cdot)$ be n-dimensional Lebesgue measure. Write $\mathscr{K}^n$ for the class of compact convex sets in $\mathbb{R}^n$ with non-empty interior and $\mathscr{K}_o^n$ for the class of members of $\mathscr{K}^n$ containing the origin in their interiors.

For each $K \in \mathscr{K}^n$, its support function $h(K,\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(K,y) = \max\{\langle x,y\rangle : \ x \in K\}.$$

From the definition of the support function it follows immediately that for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, the support function of the image $TK = \{Tx : x \in K\}$ of $K \in \mathscr{K}^n$ is given by

$$h(TK,y) = h(K,T^t y) \tag{9}$$

for any $y \in S^{n-1}$, where T^t is adjoint of operator T.

The Hausdorff metric δ between sets $K,L \in \mathscr{K}^n$ is defined by

$$\delta(K,L) := \min\{\lambda > 0 : \ K \subset L + \lambda B_2^n, \ L \subset K + \lambda B_2^n\},$$

or equivalently, by

$$\delta(K,L) = \max_{u \in S^{n-1}} |h(K,u) - h(L,u)|.$$

The polar body K° of $K \in \mathscr{K}^n$ is the convex body define by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1 \quad \text{for all} \quad y \in K\}.$$

We write $\|\cdot\|_K$ for the Minkowski functional

$$\|x\|_K = \min\{t \geqslant 0 : x \in tK\} \tag{10}$$

induced to $\mathbb{R}^n$ by K. One can easily check that the body K° satisfies

$$h(K^\circ,y) = \|y\|_K, \tag{11}$$

for all $y \in \mathbb{R}^n$.

Recall that $\mathscr{P}_{[n]}$ is the class of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. If $N \geqslant n$, and $X_1,\cdots,X_N$ are independent random points with X_i distributed according to $\mu_i \in \mathscr{P}_{[n]}$, Applying the $n \times N$ radom matrix $[X_1,\cdots,X_N]$ to a convex body $K \subset \mathbb{R}^N$ produces a random convex set in $\mathbb{R}^n$, i.e.,

$$[X_1,\cdots,X_N]K = \left\{\sum_{i=1}^N c_i X_i : (c_i) \subset K\right\}. \tag{12}$$

We adopt the common convention that all random vectors are defined on a common underlying probability space $(\Omega,\Sigma,\mathbb{P})$ and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

We say that a sequence of random convex bodies $\{K_N\}_{N=1}^{\infty}$ converges (in the Hausdorff metric) to a convex body K almost surely (a.s.) as $N \to \infty$ means that

$$\mathbb{P}\left(\lim_{N\to\infty} \delta(K_N, K) = 0\right) = 1.$$

The law of large numbers is one of the most important results in probability theory that describes the results of performing the same experiment of a large number of times. For independent, identically distributed real random variables $\{X_j\}$, $j = 1,2,\cdots$, if $E|X_1| < \infty$, then the strong law of large numbers holds, that is, the empirical average converges to the expected value almost surely, i.e.,

$$\lim_{N\to\infty} \frac{X_1+\cdots+X_N}{N} \to \mathbb{E}X_1 \qquad \text{(a.s.)}. \tag{13}$$

We refer the reader to [4] and [26] for additional background material on convex geometry and to [3, Chapter 8] for basic on laws of large numbers. See [1] for a law of large numbers for random compact sets.

Recall that for an integer $m \geqslant 1$, $\phi \in \mathscr{C}_0$, $\mu_i \in \mathscr{P}_{[n]}$, $i = 1,2,\cdots,m$, the generalized centroid body $\Gamma_\phi(\mu_1,\cdots,\mu_m)$ of $\mu_1,\cdots,\mu_m$ corresponding to ϕ is defined by its support function at $y \in \mathbb{R}^n$

$$\begin{aligned} &h(\Gamma_\phi(\mu_1,\cdots,\mu_m),y)\\ =&\inf\left\{\lambda>0:\ \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+\cdots+x_m,y\rangle|}{\lambda}\right)d\mu_1(x_1)\cdots d\mu_m(x_m)\leqslant 1\right\}. \end{aligned} \tag{14}$$

Since ϕ is convex and strictly increasing, and the fact that $|\cdot|$ is subadditivity, it is not hard to prove that $h(\Gamma_\phi(\mu_1,\cdots,\mu_m),\cdot)$ is a sublinear function and hence is a support function of a convex body (see e.g., [16, Lemma 2.2]). And from our definition, we can see $\Gamma_\phi(\mu_1,\cdots,\mu_m)$ is symmetric. Set

$$B_{\phi/N} = \left\{t = (t_1,\cdots,t_N) \in \mathbb{R}^N : \frac{1}{N}\sum_{i=1}^{N}\phi(|t_i|) \leqslant 1\right\}. \tag{15}$$

One can check that $B_{\phi/N}$ is convex, symmetric, bounded and the origin is an interior point, hence from the Minkowski functional (10) we have

$$\|t\|_{B_{\phi/N}} = \inf\{\lambda > 0 : t \in \lambda B_{\phi/N}\}.$$

Together with (11), we know that $\|\cdot\|_{B_{\phi/N}}$ is the support function for $B^o_{\phi/N} = \{x \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1\ \forall y \in B_{\phi/N}\}$.

Let $x_{i,1}, x_{i,2}, \cdots$ $(i = 1,2,\cdots,m)$ be sequences of vectors in $\mathbb{R}^n$, and let $T_N = T_N(x_{1,1}+\cdots+x_{m,1},\cdots,x_{1,N}+\cdots+x_{m,N})$ be the $n\times N$ matrix with columns $x_{1,1}+\cdots+x_{m,1},\cdots,x_{1,N}+\cdots+x_{m,N}$. Then we can use (9), (15) and (12) to express the support function of $T_N B^\circ_{\phi/N}$ as

$$\begin{aligned} h(T_N B^\circ_{\phi/N}, y) &= h(B^\circ_{\phi/N}, T_N^t y) = \|T_N^t y\|_{B_{\phi/N}} \\ &= \|(\langle x_{1,1} + \cdots + x_{m,1}, y\rangle, \cdots, \langle x_{1,N} + \cdots + x_{m,N}, y\rangle)\|_{B_{\phi/N}} \\ &= \inf\left\{\lambda > 0 : \ \frac{1}{N}\sum_{i=1}^{N} \phi\left(\frac{|\langle x_{1,i} + \cdots + x_{m,i}, y\rangle|}{\lambda}\right) \leqslant 1\right\}, \end{aligned} \tag{16}$$

for all $y \in S^{n-1}$.

3. Proof of main results

Let $m \geqslant 1$ be an integer, $1 \leqslant t \leqslant m$, assume that $\{\mu_{t,i}\}_{i=1}^{\infty}$ are m sequences of probability measures in $\mathscr{P}_{[n]}$ and let $f_{t,i}$ denote the density of $\mu_{t,i}$ for $i = 1, 2, \cdots$. Suppose we have the following sequences of independent random vectors:

(1) $X_{t,1}, X_{t,2}, \cdots$ with $X_{t,i}$ distributed according to $f_{t,i}$;

(2) $Y_1, Y_2, \cdots$ with Y_i distributed according to $\mathbf{1}_{D_n}$.

The following lemma obtained in [23] is important to the proof of the Theorem 1 in section 1.

LEMMA 1. *Suppose that $(C_N)_{N=n}^{\infty}$ is a sequence of convex bodies with $C_N \subset \mathbb{R}^N$. Let*

$$T_N^{(1)} = [X_{1,1} + \cdots + X_{m,1}, \cdots, X_{1,N} + \cdots + X_{m,N}], \quad T_N^{(2)} = [Y_1, \cdots, Y_N].$$

Suppose $\mathscr{C}^{(1)}, \mathscr{C}^{(2)}$ are (random) convex bodies in $\mathbb{R}^n$ defined as

$$\mathscr{C}^{(1)} := \lim_{N\to\infty} T_N^{(1)} C_N \quad (a.s.) \qquad \mathscr{C}^{(2)} := \lim_{N\to\infty} T_N^{(2)} C_N \quad (a.s.), \tag{17}$$

where the convergence is in the Hausdoeff metric. Suppose further that there exist $R_1, R_2 > 0$ such that for any $N \geqslant n$,

$$T_N^{(1)} C_N \subseteq R_1 B_2^n, \quad (a.s.) \qquad T_N^{(2)} C_N \subseteq R_2 B_2^n, \quad (a.s.), \tag{18}$$

and if $\|f_{t,i}\|_\infty \leqslant 1$ for each $1 \leqslant t \leqslant m$, $i = 1, 2, \cdots$, then

$$\mathbb{E}vol_n\left(\mathscr{C}^{(1)}\right) \geqslant \mathbb{E}vol_n\left(\mathscr{C}^{(2)}\right). \tag{19}$$

Next following lemma plays a key role in the proof of the main theorem.

LEMMA 2. *Let $m \geqslant 1$ be an integer, $\mu_1, \cdots, \mu_m \in \mathscr{P}_{[n]}$. Let $x_{t,1}, x_{t,2}, \cdots, 1 \leqslant t \leqslant m$ be m sequences of vectors in $\mathbb{R}^n$ and suppose that*

$$span\{x_{t,1}, \cdots, x_{t,n}\} = \mathbb{R}^n, \qquad for\ each\ 1 \leqslant t \leqslant m. \tag{20}$$

Let $\phi \in \mathscr{C}_0$. Assume that for each $y \in S^{n-1}$ and each $\lambda > 0$, we have

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{|\langle x_{1,i}+\cdots+x_{m,i},y\rangle|}{\lambda}\right)\right.$$
$$\left.-\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+\cdots+x_m,y\rangle|}{\lambda}\right)d\mu_1(x_1)\cdots d\mu_m(x_m)\right|=0. \tag{21}$$

Let $T_N = T_N(x_{1,1}+\cdots+x_{m,1},\cdots,x_{1,N}+\cdots+x_{m,N})$ be the $n\times N$ matrix with columns $x_{1,1}+\cdots+x_{m,1},\cdots,x_{1,N}+\cdots+x_{m,N}$. Then

$$\Gamma_\phi(\mu_1,\cdots,\mu_m)=\lim_{N\to\infty}T_N B^o_{\phi/N}.$$

Proof. Since pointwise convergence of support functions implies uniform convergence in the Hausdorff metric (see e.g., [26, Theorem 1.8.15]), it is sufficient to show that for each $y \in S^{n-1}$, we have

$$\lim_{N\to\infty}h(T_N B^o_{\phi/N},y)=h(\Gamma_\phi(\mu_1,\cdots,\mu_m),y). \tag{22}$$

Fix $y \in S^{n-1}$. By (20), we can get

$$\text{span}\{x_{1,1}+\cdots+x_{m,1},\cdots,x_{1,n}+\cdots+x_{m,n}\}=\mathbb{R}^n,$$

so there exists $i\in\{1,\cdots,n\}$ such that $\langle x_{1,i}+\cdots+x_{m,i},y\rangle\neq 0$, hence

$$\frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{|\langle x_{1,i}+\cdots+x_{m,i},y\rangle|}{\lambda}\right)>0.$$

For simplicity of notation, for each $N\geqslant n$, let $g_N:(0,\infty)\to(0,\infty)$ be defined by

$$g_N(\lambda):=\frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{|\langle x_{1,i}+\cdots+x_{m,i},y\rangle|}{\lambda}\right)>0.$$

Consider also $g:(0,\infty)\to(0,\infty)$ defined by

$$g(\lambda):=\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+\cdots+x_m,y\rangle|}{\lambda}\right)d\mu_1(x_1)\cdots d\mu_m(x_m).$$

Since ϕ is convex and strictly increasing, g and g_N are continuous and strictly decreasing.

Hence, from (16) and the definition of $\Gamma_\phi(\mu_1,\cdots,\mu_m)$, set

$$\lambda(N):=h(T_N B^o_{\phi/N},y)=\inf\{\lambda>0:g_N(\lambda)\leqslant 1\} \tag{23}$$

and

$$\lambda_0:=h(\Gamma_\phi(\mu_1,\cdots,\mu_m),y)=\inf\{\lambda>0:g(\lambda)\leqslant 1\}. \tag{24}$$

We assume that (22) is false and we need to get a contradiction. Then there exists $\varepsilon_0 > 0$ and a subsequence $(N_j)_{j=1}^{\infty} \subset \mathbb{N}$ such that either

(i) $\lambda(N_j) \geqslant \lambda_0 + \varepsilon_0$ for each $j = 1, 2, \cdots$, or

(ii) $\lambda(N_j) \leqslant \lambda_0 - \varepsilon_0$ for each $j = 1, 2, \cdots$.

First, consider the case (i). Let

$$\lambda_* := \inf_j \lambda(N_j), \tag{25}$$

then by the assumption, we have

$$\lambda_* \geqslant \lambda_0 + \varepsilon_0. \tag{26}$$

Let $\eta > 0$. For each $j = 1, 2, \cdots$, from (23), (25) and the fact that g_{N_j} is decreasing, it follows that

$$1 < g_{N_j}(\lambda(N_j) - \eta) \leqslant g_{N_j}(\lambda_* - \eta).$$

Thus by (21), we have

$$1 \leqslant \lim_{j \to \infty} g_{N_j}(\lambda_* - \eta) = g(\lambda_* - \eta).$$

As $\eta > 0$ is arbitrary, and g is continuous, we can get $1 \leqslant g(\lambda_*)$. If $1 < g(\lambda_*)$, then from the definition of λ_0 there is $\lambda_* < \lambda_0$, contradicting (26). While if $1 = g(\lambda_*)$, then from (24) and the fact that g is a strictly decreasing continuous function, we have $\lambda_* = \lambda_0$, contradicting (26).

Now consiger the case (ii). Let

$$\lambda^* := \sup_j \lambda(N_j), \tag{27}$$

from the assumption, there is

$$\lambda^* \leqslant \lambda_0 - \varepsilon_0. \tag{28}$$

Let $\eta > 0$. For each $j = 1, 2, \cdots$, from (23), (27) and the fact that g_{N_j} is decreasing, we have

$$g_{N_j}(\lambda^* + \eta) \leqslant g_{N_j}(\lambda(N_j) + \eta) \leqslant 1.$$

Therefore by (21), we have

$$g(\lambda^* + \eta) = \lim_{j \to \infty} g_{N_j}(\lambda^* + \eta) \leqslant 1.$$

Together with (24), we can get $\lambda_0 \leqslant \lambda^* + \eta$. As $\eta > 0$ is arbitrary, there is $\lambda_0 \leqslant \lambda^*$, contradicting (28). □

Now using the above lemmas and the strong law of large numbers, we prove the theorem introduced in the section 1.

THEOREM 1. *Let $m \geqslant 1$ be an integer, $\phi \in \mathscr{C}_0$, $\mu_i \in \mathscr{P}_{[n]}$, $i = 1,2,\cdots,m$. Let f_i denote the density of μ_i with $\|f_i\|_\infty \leqslant 1$, $i = 1,2,\cdots,m$. then*

$$vol_n(\Gamma_\phi(\mu_1,\cdots,\mu_m)) \geqslant vol_n(\Gamma_\phi(\lambda_{D_n},\cdots,\lambda_{D_n})), \tag{29}$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n.

Proof. First we show that the assumptions in the theorem satisfy (18) in Lemma 1.

For each $1 \leqslant t \leqslant m$, take $\mu_{t,i} = \mu_t$ for each $i = 1,2,\cdots$. Hence $X_{t,1}, X_{t,2},\cdots$ are independent and identically distributed according to f_t, the density of μ_t. By standard approximation arguments, we can assume that $\mu_1,\cdots\mu_m$ are compactly supported, that is to say there exists $R_1,\cdots,R_m > 0$ such that

$$\text{supp}(\mu_1) \subset R_1 B_2^n, \ \cdots, \ \text{supp}(\mu_m) \subset R_m B_2^n,$$

Let $R = \max\{R_1,\cdots,R_m\}$, then $\text{supp}(\mu_i) \subset RB_2^n$, $i = 1,2,\cdots,m$. It follows that

$$\langle X_{t,i}, y\rangle \leqslant R \tag{30}$$

for all $1 \leqslant t \leqslant m$, $i \in \mathbb{N}$ and $y \in S^{n-1}$.

For convenience, let

$$\overline{\lambda} := \frac{mR}{\phi^{-1}(1)}.$$

Observe that ϕ is strictly increasing, then, for any N and $y \in S^{n-1}$, by (30), we have

$$\frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{|\langle X_{1,i}+\cdots+X_{m,i}, y\rangle|}{\overline{\lambda}}\right) \leqslant \frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{mR}{\overline{\lambda}}\right) = \frac{1}{N}\sum_{i=1}^{N}\phi(\phi^{-1}(1)) = 1.$$

As the notation of $T_N^{(1)}$ defined in Lemma 1, together with (16), it follows that

$$h(T_N^{(1)}B^\circ_{\phi/N}, y) = \left\|(T_N^{(1)})^t y\right\|_{B_{\phi/N}} \leqslant \overline{\lambda}.$$

Therefore, we have

$$T_N^{(1)}B^\circ_{\phi/N} \subset \overline{\lambda}B_2^n,$$

for any N.

The same reasoning applies to $T_N^{(2)}$ and $\mathbf{1}_{D_n}$, it is not hard to prove that there exists $\lambda' > 0$ such that for any N,

$$T_N^{(2)}B^\circ_{\phi/N} \subset \lambda' B_2^n.$$

This shows that (18) in Lemma 1 is satisfied.

Next, we prove that the assumptions in the theorem satisfy (17) in Lemma 1 as well. Fix $y \in S^{n-1}$, and $\lambda > 0$. Let $X_i = \phi\left(\frac{|\langle X_{1,i}+\cdots+X_{m,i}, y\rangle|}{\lambda}\right)$ for $i \in \mathbb{N}$. Since ϕ is strictly increasing, $\mu_t(\mathbb{R}^n) = 1$, for $t = 1, 2, \cdots, m$, we have

$$\begin{aligned}\mathbb{E}|X_1| &= \mathbb{E}\phi\left(\frac{|\langle X_{1,1} + \cdots + X_{m,1}, y\rangle|}{\lambda}\right) \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \phi\left(\frac{|\langle x_1 + \cdots + x_m, y\rangle|}{\lambda}\right) d\mu_1(x_1) \cdots d\mu_m(x_m) \\ &\leqslant \phi\left(\frac{R}{\lambda}\right) < \infty.\end{aligned}$$

Thus the strong law of large numbers (13) holds, that is

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N} \phi\left(\frac{|\langle x_{1,i} + \cdots + x_{m,i}, y\rangle|}{\lambda}\right) = \frac{X_1 + \cdots + X_N}{N} \to \mathbb{E}|X_1| \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \phi\left(\frac{|\langle x_1 + \cdots + x_m, y\rangle|}{\lambda}\right) d\mu_1(x_1) \cdots d\mu_m(x_m), \qquad \text{(a.s.)}\end{aligned}$$

Therefore, $(X_{t,i})$'s satisfy (21) in Lemma 2 almost surely. By Lemma 2, in the Hausdorff metric, we obtain

$$\Gamma_\phi(\mu_1, \cdots, \mu_m) = \lim_{N\to\infty} T_N^{(1)} B^o_{\phi/N}$$

almost surely.

The same reasoning applies to T_N^2 and $\mathbf{1}_{D_n}$, it is not hard to prove that

$$\Gamma_\phi(\lambda_{D_n}, \cdots, \lambda_{D_n}) = \lim_{N\to\infty} T_N^{(2)} B^o_{\phi/N} \qquad \text{(a.s.)}.$$

So (17) in Lemma 1 holds as well.

Finally, Lemma 1 implies that

$$\mathbb{E}\mathrm{vol}_n(\Gamma_\phi(\mu_1, \cdots, \mu_m)) \geqslant \mathbb{E}\mathrm{vol}_n(\Gamma_\phi(\lambda_{D_n}, \cdots, \lambda_{D_n})).$$

From the definition of the generalized centroid body $\Gamma_\phi(\mu_1, \cdots, \mu_m)$, for given $\mu_1, \cdots, \mu_m$, $\Gamma_\phi(\mu_1, \cdots, \mu_m)$ is not a random set, and it is obvious that $\Gamma_\phi(\lambda_{D_n}, \cdots, \lambda_{D_n})$ is a non-random set, thus we complete the proof. □

If $m = 1$, $f_1 = \mathbf{1}_K$, where $K \in \mathscr{K}^n$ with $\mathrm{vol}_n(K) = 1$, then (16) and (29) are the symmetric Orlicz centroid body and Orlicz centroid inequality introduced by Lutwak, Yang and Zhang [16].

COROLLARY 1. *Let $\phi \in \mathscr{C}_0$ and $K \in \mathscr{K}^n$ with $vol_n(K) = 1$, the Orlicz centroid body $\Gamma_\phi(K)$ of K is defined by*

$$h_{\Gamma_\phi(K)}(y) = h(\Gamma_\phi(\mathbf{1}_K), y) = \inf\left\{\lambda > 0 : \int_K \phi\left(\frac{|\langle x, y\rangle|}{\lambda}\right) dx \leqslant 1\right\},$$

where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. And the following Orlicz centroid inequality holds,

$$vol_n(\Gamma_\phi(K)) \geqslant vol_n(\Gamma_\phi(D_n)),$$

where D_n is the Euclidean ball of volume one.

Let $m=2$, $\phi\in\mathscr{C}_0$, $\mu_1,\mu_2\in\mathscr{P}_{[n]}$, we call $Z_\phi(\mu_1,\mu_2)$ the general mean zonoid for probability measures, whose support function is given by

$$\begin{aligned} h(Z_\phi(\mu_1,\mu_2),y) &:= h(\Gamma_\phi(\mu_1,\mu_2),y) \\ &= \inf\left\{\lambda>0: \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi\left(\frac{|\langle x_1+x_2,y\rangle|}{\lambda}\right)d\mu_1(x_1)d\mu_2(x_2)\leqslant 1\right\}. \end{aligned} \tag{31}$$

As a corollary of Theorem 1, the following affine isoperimetric inequality for general mean zonoid is established.

COROLLARY 2. *Let $\phi\in\mathscr{C}_0$, $\mu_1,\mu_2\in\mathscr{P}_{[n]}$. Let f_i denote the density of μ_i with $\|f_i\|_\infty\leqslant 1$, $i=1,2$. Then*

$$vol_n(Z_\phi(\mu_1,\mu_2))\geqslant vol_n(Z_\phi(\lambda_{D_n},\lambda_{D_n})), \tag{32}$$

where λ_{D_n} is the restriction of Lebesgue measure to D_n.

Take $\phi=|t|$, $f_1=\mathbf{1}_K$, $f_2=\mathbf{1}_{-K}$ in Corollary 2 where $K\in\mathscr{K}^n$ with $\mathrm{vol}_n(K)=1$, then $Z_\phi(\mu_1,\mu_2)=\widetilde{Z}K$, the mean zonoid (5) defined by Zhang [29].

Let $f_1=\mathbf{1}_K$, $f_2=\mathbf{1}_{-K}$ where $K\in\mathscr{K}^n$ with $\mathrm{vol}_n(K)=1$, then Orlicz mean zonoid (see [7]) and corresponding inequality can be obtained from (31) and (32). In addition, take $\phi=|t|^p$, thus (31) is the L_p mean zonoid (see [27]).

COROLLARY 3. *If $\phi\in\mathscr{C}_0$ and $K\in\mathscr{K}^n$ with $vol_n(K)=1$, then the support function of Orlicz mean zonoid $Z_\phi K$ is given by*

$$\begin{aligned} h_{Z_\phi K}(y) &:= h(Z_\phi(\mathbf{1}_K,\mathbf{1}_{-K}),y) \\ &= \inf\left\{\lambda>0: \int_K\int_K\phi\left(\frac{|\langle x_1-x_2,y\rangle|}{\lambda}\right)dx_1dx_2\leqslant 1\right\}, \end{aligned}$$

and the following affine isopermetric inequality holds,

$$vol_n(Z_\phi K)\geqslant vol_n(Z_\phi D_n).$$

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