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SOME INEQUALITIES INVOLVING OPERATOR MONOTONE FUNCTIONS AND OPERATOR MEANS

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Abstract. In this paper we show that if $f:[0,\infty)\to[0,\infty)$ is an operator monotone function and A,B are positive operators such that $0<pA\leqslant B\leqslant qA$, then for all $\alpha\in[0,1]$

$$f(A)\sharp_\alpha f(B)\leqslant \max\{S(p),S(q)\}f(A\sharp_\alpha B),$$

where $S(t)$ is the so called Specht's ratio, and $\sharp_\alpha$ is α-geometric mean.

Moreover, we present some majorization and norm inequalities for operator monotone functions. Operator monotone decreasing functions are also discussed.

1. Introduction

Let $\mathscr{B}(H)$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator $A\in\mathscr{B}(H)$ is called *positive* if $\langle Ax,x\rangle\geqslant 0$ for every $x\in H$ and then we write $A\geqslant 0$. For self-adjoint operators $A,B\in\mathscr{B}(H)$, we say $A\leqslant B$ if $B-A\geqslant 0$. Also we say A is *positive definite* and we write $A>0$, if $\langle Ax,x\rangle>0$ for every $x\in H$. Let f be a continuous real function on $(0,\infty)$. Then f is said to be *operator monotone* (more precisely, operator monotone increasing) if $A\geqslant B$ implies $f(A)\geqslant f(B)$ for positive definite operators A,B, and *operator monotone decreasing* if $-f$ is operator monotone or $A\geqslant B$ implies $f(A)\leqslant f(B)$.

Also, f is said to be *operator convex* if $f(\alpha A+(1-\alpha)B)\leqslant \alpha f(A)+(1-\alpha)f(B)$ for all positive definite operators A,B and $\alpha\in[0,1]$, and *operator concave* if $-f$ is operator convex.

For each $\alpha\in[0,1]$ the α*-arithmetic* and the α*-harmonic* means are defined as $A\bigtriangledown_\alpha B:=(1-\alpha)A+\alpha B$ and $A!_\alpha B:=((1-\alpha)A^{-1}+\alpha B^{-1})^{-1}$ for positive definite operators A,B. Also, the α*-geometric* mean is

$$A\sharp_\alpha B=A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.$$

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For $\alpha = \frac{1}{2}$ one obtains the arithmetic mean $A \bigtriangledown B = \dfrac{A+B}{2}$, harmonic mean $A!B = \left(\dfrac{A^{-1}+B^{-1}}{2}\right)^{-1}$ and geometric mean $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$.

Basic properties of the arithmetic, harmonic and geometric means can be found in [1].

It is well-known the Young inequality

$$A\sharp_\alpha B \leqslant (1-\alpha)A + \alpha B \tag{1}$$

for positive definite operators A and B.

The constant Specht's ratio [7, 9] is defined as $S(t) = \dfrac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}}$ for a positive real number t. Note that $\lim_{t\to 1} S(t) = 1$ and $S(t) = S(1/t) > 1$ for $t \neq 1$, $t > 0$.

One of reverse inequalities for (1) is given by M. Tominaga in [9], using the Specht's ratio, as follows:

If $0 < mI \leqslant A, B \leqslant MI$ with $h = M/m$ and $\alpha \in [0,1]$, then

$$(1-\alpha)A + \alpha B \leqslant S(h)(A\sharp_\alpha B). \tag{2}$$

This paper is organized as follows. In section 2 we study an analogue of geometric concavity property

$$f(a)\sharp_\alpha f(b) \leqslant f(a\sharp_\alpha b),$$

for operator functions, by using several revers Young's inequalities. More precisely, we show that if $f : [0,\infty) \to [0,\infty)$ is an operator monotone function and $0 < pA \leqslant B \leqslant qA$, then for all $\alpha \in [0,1]$

$$f(A)\sharp_\alpha f(B) \leqslant \max\{S(p), S(q)\} f(A\sharp_\alpha B),$$

where $S(t)$ is the so called Specht's ratio.

As an immediate result we have: if $f : [0,\infty) \longrightarrow [0,\infty)$ is operator monotone function and $0 < mI \leqslant A, B \leqslant MI$ with $h = \frac{M}{m}$, then for all $\alpha \in [0,1]$

$$f(A)\sharp_\alpha f(B) \leqslant S(h) f(A\sharp_\alpha B).$$

At the end of this section, we also present a norm inequality involving Specht's ratio $S(h)$, for operator convex functions.

Section 3 is devoted to prove some majorization inequalities for operator monotone functions. Let A be a finite rank operator, we always denote the eigenvalues of $|A| = (A^*A)^{1/2}$ by $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A)$ listed in decreasing order with multiplicities and put $s(A) = (s_1(A), \cdots, s_n(A))$. These are called the singular values of A. The *weakly log-majorization* $s(A) \prec_{w\log} s(B)$ means that

$$\prod_{j=1}^{k} s_j(A) \leqslant \prod_{j=1}^{k} s_j(B), \qquad k = 1, 2, \ldots, n.$$

We refer to [3, 8], for a detailed study. In this section, we show that if f is a nonnegative operator monotone function on $(0,\infty)$, then for every $1 \leqslant k \leqslant n$

$$\prod_{j=1}^{k} s_j(f(A!B)) \leqslant \prod_{j=1}^{k} s_j^{1/2}(f(A)) \cdot s_j^{1/2}(f(B)).$$

Then we deduce some norm and determinantal inequalities from this majorization relation. We also give the counterpart of this results for operator monotone decreasing functions.

2. Operator monotone functions and ratio type reverse inequalities

We first show the following converse ratio-type inequality for the Young's inequality. The sketch of proof is similar to that of [9, Theorem 2.1].

LEMMA 1. *Let* $0 < pA \leqslant B \leqslant qA,\ p,q > 0$ *and* $\alpha \in [0,1]$. *Then*

$$(1-\alpha)A + \alpha B \leqslant \max\{S(p), S(q)\}(A\sharp_\alpha B), \tag{3}$$

where $S(t)$ *is the so called Specht's ratio.*

Proof. From [9, Lemma 2.3] if b is a positive number and $\alpha \in [0,1]$, then

$$(1-\alpha)b + \alpha \leqslant S(b)b^{1-\alpha}.$$

Thus for the invertible positive operator $0 < pI \leqslant C \leqslant qI$, we have

$$(1-\alpha)C + \alpha I \leqslant \max_{p \leqslant t \leqslant q} S(t) C^{1-\alpha}.$$

Putting $C = B^{-1/2}AB^{-1/2}$ we get

$$(1-\alpha)B^{-1/2}AB^{-1/2} + \alpha I \leqslant \max_{p \leqslant t \leqslant q} S(t)(B^{-1/2}AB^{-1/2})^{1-\alpha}.$$

Multiplying $B^{1/2}$ to the both sides in the above inequality, and using this fact that $\max_{p \leqslant t \leqslant q} S(t) = max\{S(p), S(q)\}$, inequality (3) is obtained. Since $A\sharp_\alpha B = B\sharp_{1-\alpha}A$. □

REMARK 1. Note that Tominaga's inequality (2), can be derived from Lemma 1. Because if $0 < mI \leqslant A, B \leqslant MI$, then $\frac{m}{M}A \leqslant B \leqslant \frac{M}{m}A$. Since $S(h) = S(\frac{1}{h})$, we obtain inequality (2) by letting $p = \frac{m}{M}$ and $q = \frac{M}{m}$ in Lemma 1.

THEOREM 1. *Let* $f : [0,\infty) \longrightarrow [0,\infty)$ *be operator monotone function and* $0 < pA \leqslant B \leqslant qA$. *Then for all* $\alpha \in [0,1]$

$$f(A)\sharp_\alpha f(B) \leqslant max\{S(p), S(q)\} f(A\sharp_\alpha B),$$

where $S(t)$ *is the so called Specht's ratio.*

Proof. First note that since f is analytic on $(0,\infty)$, we may assume that $f(x) > 0$ for all $x > 0$; otherwise f is identically zero. Also, since f is operator monotone function on $[0,\infty)$, so it is operator concave function [3, Theorem V.2.5]. For the convenience we put $M = max\{S(p),S(q)\}$. So $M \geqslant 1$. Now compute

$$\begin{aligned} f(A)\sharp_\alpha f(B) &\leqslant (1-\alpha)f(A)+\alpha f(B) \leqslant f((1-\alpha)A+\alpha B)) \\ &\leqslant f(M(A\sharp_\alpha B)) \leqslant Mf(A\sharp_\alpha B), \end{aligned}$$

where the first inequality follows from inequality (1), the second follows from operator concavity of f and the third follows from Lemma 1 and monotony of f. □

As an immediate result we have the following corollary:

COROLLARY 1. *Let $f:[0,\infty)\longrightarrow[0,\infty)$ be operator monotone function and let $0 < mI \leqslant A,B \leqslant MI$. Then for all $\alpha \in [0,1]$*

$$f(A)\sharp_\alpha f(B) \leqslant S(h)f(A\sharp_\alpha B),$$

where $S(h)$ is the so called Specht's ratio and $h = \frac{M}{m}$.

COROLLARY 2. *Let g be a non-negative operator monotone decreasing function on $(0,\infty)$ and let $0 < mI \leqslant A,B \leqslant MI$. Then for all $\alpha \in [0,1]$*

$$g(A\sharp_\alpha B) \leqslant S(h)(g(A)\sharp_\alpha g(B)),$$

where $S(h)$ is the so called Specht's ratio and $h = \frac{M}{m}$.

Proof. Since g is operator monotone decreasing on $(0,\infty)$, so $1/g$ is operator monotone on $(0,\infty)$. Now by applying Corollary 1 for $f = 1/g$, we have

$$(g(A)\sharp_\alpha g(B))^{-1} = g(A)^{-1}\sharp_\alpha g(B)^{-1} \leqslant S(h)g(A\sharp_\alpha B)^{-1}.$$

By reversing both sides, the alleged inequality is obtained. □

Furuichi and Minculete in [7], gave another reverse inequalities for Young's inequality without using Specht's ratio as follows:

LEMMA 2. *Let $0 < mI \leqslant A \leqslant B \leqslant MI \leqslant I$ with $h = \frac{M}{m}$. Then for all $\alpha \in [0,1]$*

$$(1-\alpha)A+\alpha B \leqslant exp\Big(\alpha(1-\alpha)\Big(1-\frac{1}{h}\Big)^2\Big)A\sharp_\alpha B. \tag{4}$$

Using this lemma we state the next theorem.

THEOREM 2. *Let $f:[0,\infty)\longrightarrow[0,\infty)$ be operator monotone function and let $0 < mI \leqslant A \leqslant B \leqslant MI \leqslant I$ with $h = \frac{M}{m}$. Then for all $\alpha \in [0,1]$*

$$f(A)\sharp_\alpha f(B) \leqslant exp\Big(\alpha(1-\alpha)\Big(1-\frac{1}{h}\Big)^2\Big)f(A\sharp_\alpha B).$$

Proof. The assertion is obtained similar to the proof of Theorem 1, with applying inequality (4) instead of inequality (3). Note that the factor $exp\big(\alpha(1-\alpha)(1-\frac{1}{h})^2\big) \geqslant 1$. $\square$

In the remaining part of this section, we let H be a finite dimensional Hilbert space and $\alpha = \frac{1}{2}$. We show a norm inequality for operator convex function g involving Specht's ratio $S(h)$. For this purpose, the following subadditivity results are needed. Recall that the norm $\|\cdot\|_u$ is said a unitarily invariant norm, if satisfies $\|A\|_u = \|UAV\|_u$ for all A and all unitaries U,V.

LEMMA 3. [6, Theorem 1.2] *Let $A,B \geqslant 0$ and let $g:[0,\infty) \longrightarrow [0,\infty)$ be a convex function with $g(0)=0$. Then for every unitarily invariant norm $\|\cdot\|_u$*

$$\|g(A)+g(B)\|_u \leqslant \|g(A+B)\|_u.$$

LEMMA 4. [4, Theorem 2.1] *Let $f:[0,\infty) \longrightarrow [0,\infty)$ be a concave function. Let $A \geqslant 0$ and let Z be expansive. Then for every unitarily invariant norm $\|\cdot\|_u$*

$$\|f(Z^*AZ)\|_u \leqslant \|Z^*f(A)Z\|_u.$$

THEOREM 3. *Let $g:[0,\infty) \longrightarrow [0,\infty)$ be an operator convex function with $g(0)=0$ and let $0 < mI \leqslant A,B \leqslant MI$. Then for every unitarily invariant norm $\|\cdot\|_u$*

$$\frac{\|g(A)\sharp g(B)\|_u}{\|A\sharp B\|_u} \leqslant 2S(h)^2 \left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u,$$

where $S(h)$ is the so called Specht's ratio and $h = \frac{M}{m}$.

Proof. From revers Young's inequality (2), for $\alpha = \frac{1}{2}$, we have

$$A+B \leqslant 2S(h)(A\sharp B). \tag{5}$$

Since $g(t)$ is operator convex with $g(0)=0$, then $f(t)=g(t)/t$ is operator monotone function on $(0,\infty)$. [3, Theorem V.2.9]. Therefore,

$$\frac{g(A+B)}{A+B} \leqslant \frac{g(2S(h)(A\sharp B))}{2S(h)(A\sharp B)}. \tag{6}$$

Now compute

$$\begin{aligned}
\frac{\|g(A)+g(B)\|_u}{\|A+B\|_u} &\leqslant \frac{\|g(A+B)\|_u}{\|A+B\|_u} \leqslant \left\|\frac{g(A+B)}{A+B}\right\|_u \\
&\leqslant \left\|\frac{g(2S(h)(A\sharp B))}{2S(h)(A\sharp B)}\right\|_u \\
&\leqslant 2S(h)\left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u,
\end{aligned} \tag{7}$$

where the first inequality follows from applying Lemma 3 for $g(t)$, and the last inequality follows from applying Lemma 4 for $f(t)$. The second inequality is due to submultiplicativity of unitarily invariant norms. Also applying with $AM-GM$ inequality for $g(A)$ and $g(B)$, we get

$$2\|g(A)\sharp g(B)\|_u \leqslant \|g(A)+g(B)\|_u. \tag{8}$$

Combining left side of (7) with (8), we deduce

$$\frac{\|g(A)\sharp g(B)\|_u}{\|A+B\|_u} \leqslant S(h)\left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u. \tag{9}$$

On the other hand, from inequality (5)

$$\frac{1}{2S(h)\|A\sharp B\|_u} \leqslant \frac{1}{\|A+B\|_u}, \tag{10}$$

So, inequalities (9) and (10), give the assertion as follows

$$\frac{\|g(A)\sharp g(B)\|_u}{\|A\sharp B\|_u} \leqslant 2S(h)^2\left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u. \qquad \square$$

3. Operator monotone functions and majorization inequalities

In this section, we obtain some majorization and norm inequalities for operator monotone and operator monotone decreasing functions on $(0,\infty)$. Throughout this section, Hilbert space H is assumed to be finite dimensional.

Let A and B, be positive definite operators. It is known that $A!B \leqslant A\sharp B$. So for every $1 \leqslant k \leqslant n$ we have

$$\prod_{j=1}^{k} s_j(A!B) \leqslant \prod_{j=1}^{k} s_j(A\sharp B) \leqslant \prod_{j=1}^{k} s_j(A)\sharp \prod_{j=1}^{k} s_j(B), \tag{11}$$

where the second inequality follows from [5, Proposition 3.5].

The following proposition is the extension of inequality (11) to operator monotone functions on $(0,\infty)$.

PROPOSITION 1. *Let f be a non-negative operator monotone function on $(0,\infty)$. Then for every $A,B>0$ and $1 \leqslant k \leqslant n$*

$$\prod_{j=1}^{k} s_j(f(A!B)) \leqslant \prod_{j=1}^{k} s_j^{1/2}(f(A)) \cdot s_j^{1/2}(f(B)). \tag{12}$$

Proof. By [2, Theorem 3.7], if $f \geqslant 0$ is an operator monotone function on $(0,\infty)$, then

$$\begin{bmatrix} f(A) & f(A!B) \\ f(A!B) & f(B) \end{bmatrix} \geqslant 0.$$

So by [3, Theorem IX.5.9] there exist a contraction K such that $f(A!B)=f(A)^{1/2}Kf(B)^{1/2}$. Hence

$$\prod_{j=1}^{k} s_j(f(A!B)) = \prod_{j=1}^{k} s_j(f(A)^{1/2}Kf(B)^{1/2}) \leqslant \prod_{j=1}^{k} s_j(f(A))^{1/2} s_j(f(A))^{1/2},$$

where the inequality is an immediate result of Horn's log-majorization and this fact that $s_j(K) \leqslant 1$ for all j. □

COROLLARY 3. *Let f be a non-negative operator monotone function on $(0,\infty)$ and let $A,B>0$. Then*

$$\det f(A!B) \leqslant (\det f(A))^{1/2} \cdot (\det f(B))^{1/2}.$$

LEMMA 5. *Let f be a non-negative operator monotone function on $(0,\infty)$ and let $A,B>0$. Then for every unitarily invariant norm $\|\cdot\|_u$*

$$\|f(A!B)\|_u \leqslant \|f(A)\|_u^{1/2} \cdot \|f(B)\|_u^{1/2} \leqslant \|f(A)\|_u \bigtriangledown \|f(B)\|_u.$$

Proof. Since weak log-majorization implies weak majorization [8], from inequality (12) we have

$$\sum_{j=1}^{k} s_j(f(A!B)) \leqslant \sum_{j=1}^{k} s_j^{1/2}(f(A)) \cdot s_j^{1/2}(f(B)).$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a sequence with decreasing nonnegative entries. Define $\|X\|_\alpha = \sum_{j=1}^{k} \alpha_j s_j(X)$ for $X \in \mathscr{B}(H)$. Compute

$$\begin{aligned}\|f(A!B)\|_\alpha &= \sum_{j=1}^{k} \alpha_j s_j(f(A!B)) \leqslant \sum_{j=1}^{k} \alpha_j s_j^{1/2}(f(A)) \cdot s_j^{1/2}(f(B)) \\ &\leqslant \Big(\sum_{j=1}^{k} \alpha_j s_j(f(A))\Big)^{1/2} \cdot \Big(\sum_{j=1}^{k} \alpha_j s_j(f(B))\Big)^{1/2} = \|f(A)\|_\alpha^{1/2} \cdot \|f(B)\|_\alpha^{1/2},\end{aligned}$$

where the Cauchy-Schwarz inequality is used in the second inequality. As α is arbitrarily chosen, the first alleged inequality follows from [8, Corollary 3.5.9], and the second, follows from $AM-GM$ inequality. □

The next proposition is the counterpart of Proposition 1 for operator monotone decreasing functions.

PROPOSITION 2. *Let g be a non-negative operator monotone decreasing function on $(0,\infty)$. Then for every $A,B>0$ and $1 \leqslant k \leqslant n$*

$$\prod_{j=1}^{k} s_j(g(A \bigtriangledown B)) \leqslant \prod_{j=1}^{k} s_j(g(A))^{1/2} \cdot \prod_{j=1}^{k} s_j(g(B))^{1/2}. \tag{13}$$

Proof. The proof is similar to that of Proposition 1, by using this fact that for every operator monotone decreasing function $g \geqslant 0$, the operator matrix $\begin{bmatrix} g(A) & g(A \bigtriangledown B) \\ g(A \bigtriangledown B) & g(B) \end{bmatrix}$ is positive [2, Theorem 3.1]. □

COROLLARY 4. *Let g be a non-negative operator monotone decreasing function on $(0,\infty)$ and let $A,B > 0$. Then*

$$\det g(A \bigtriangledown B) \leqslant \det(g(A))^{1/2} \cdot \det(g(B))^{1/2}.$$

LEMMA 6. *Let g be a non-negative operator monotone decreasing function on $(0,\infty)$ and let $A,B > 0$. Then for every unitarily invariant norm $\|\cdot\|_u$*

$$\|g(A \bigtriangledown B)\|_u \leqslant \|g(A)\|_u^{1/2} \cdot \|g(B)\|_u^{1/2} \leqslant \|g(A)\|_u \bigtriangledown \|g(B)\|_u.$$

Proof. The proof is similar to that of Lemma 5, by applying inequality (13). Also, we can get the result in a direct way as follows: It is well known that $\|A\sigma B\|_u \leqslant \|A\|_u \sigma \|B\|_u$ for every unitarily invariant norm $\|\cdot\|_u$ and every operator mean σ. Since $g \geqslant 0$ is operator monotone decreasing on $(0,\infty)$, then it is operator log-convex [2, Theorem 2.1]. Hence

$$\|g(A \bigtriangledown B)\|_u \leqslant \|g(A \sharp B)\|_u \leqslant \|g(A)\|_u^{1/2} \cdot \|g(B)\|_u^{1/2}. \quad \square$$

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