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AN INEQUALITY FOR t–GEOMETRIC MEANS

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This paper dedicated to the 60th Birthday of my supervisor, Professor Tikhonov Oleg Evgenievich

(*Communicated by J.-C. Bourin*)

Abstract. Let A_i, B_i $(i = 1, \ldots, m)$ be positive definite matrices, $r \geqslant 1$, $t \in [0,1]$ and $s > 0$. Then for any unitarily invariant norm $|||\cdot|||$

$$\begin{aligned} |||\sum_{i=1}^{m}(A_i \sharp_t B_i)^r||| &\leqslant |||((\sum_{i=1}^{m} B_i)^{rts/2}(\sum_{i=1}^{m} A_i)^{(1-t)rs}(\sum_{i=1}^{m} B_i)^{rts/2})^{1/s}||| \\ &\leqslant |||((\sum_{i=1}^{m} A_i)^{(1-t)rs/2}(\sum_{i=1}^{m} B_i)^{rts/2})^{1/s}|||. \end{aligned}$$

A recent result of Audenaert [2] immediately follows from the above inequalities.

1. Introduction

Let M_n be the set of $n \times n$ matrices over $\mathbb{C}$ and M_n^+ the set of positive definite matrices in M_n. Recall that a norm $|||\cdot|||$ on M_n is *unitarily invariant* if $|||UAV||| = |||A|||$ for any unitary matrices $U, V \in M_n$ and any $A \in M_n$. For $t \in [0,1]$, the t-*geometric mean* of $A, B \in M_n^+$ is defined as:

$$A \sharp_t B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

For $A \in M_n$ with positive eigenvalues, let $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$ denote the vector of eigenvalues of A such that $\lambda_1(A) \geqslant \ldots \geqslant \lambda_n(A)$. For $A, B \in M_n^+$ the denotation $\lambda(A) \prec_{\log} \lambda(B)$ means

$$\sum_{i=1}^{k}\lambda_i(A) \leqslant \sum_{i=1}^{k}\lambda_i(B), \quad k = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n}\lambda_i(A) = \sum_{i=1}^{n}\lambda_i(B).$$

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Bourin and Uchiyama [4] proved that for positive semidefinite matrices A_i $(i = 1,\cdots,m)$, for every non-negative convex function f on $[0,\infty)$ with $f(0) = 0$ and for any unitarily invariant norm $|||\cdot|||$ on M_n

$$|||f(A_1)+f(A_2)+\cdots f(A_m)||| \leqslant |||f(A_1+A_2+\cdots+A_m)|||. \tag{1.1}$$

This is a noncommutative version of the well-known inequality for nonnegative concave function f on $[0,\infty)$

$$f(a+b) \leqslant f(a)+f(b), \quad a,b \geqslant 0.$$

In [5] Bourin asked a related question: Given $A,B \geqslant 0$ and $p,q > 0$, is it true that

$$|||A^{p+q}+B^{p+q}||| \leqslant |||(A^p+B^p)(A^q+B^q)|||\,?$$

Bourin also wondered whether the following stronger inequality

$$|||A^{p+q}+B^{p+q}||| \leqslant |||(A^p+B^p)^{1/2}(A^q+B^q)(A^p+B^p)^{1/2}|||$$

holds true. Hayajneh and Kittaneh [6] gave an affirmatively answer for the trace norm $||\cdot||_1$ and the Hilbert-Schmidt norm $||\cdot||_2$. Recently, Audenaert [2] proved that for any positive semidefinite matrices A_i,B_i $(i = 1,\cdots,m)$ such that $A_iB_i = B_iA_i$ and for any unitarily invariant norm $|||\cdot|||$ on M_n,

$$|||\sum_{i=1}^{m} A_iB_i||| \leqslant |||(\sum_{i=1}^{m} A_i^{1/2}B_i^{1/2})^2||| \leqslant |||(\sum_{i=1}^{m} A_i)(\sum_{i=1}^{m} B_i)|||. \tag{1.2}$$

In particular, this result confirms a conjecture of Hayajneh and Kittaneh in [6] and answers the mentioned above question of Bourin. Very recently Lin [7] gave another proof of inequality (1.2). In the next section, based on a result of Bourin and Uchiyama in [4] we prove an inequality for t-geometric means that immediately implies (1.2).

2. Main result

Denote by $s(A) = (s_1(A),\ldots,s_n(A))$ the vector of singular values of $A \in M_n$ in descending order. The following proposition is a generalized version of [8, Proposition 2.2].

PROPOSITION 2.1. Let $A,B \in M_n^+$, $r \geqslant 1$, $t \in [0,1]$ and $s > 0$. Then for all unitarily invariant norms $|||\cdot|||$ on M_n,

$$|||(A\sharp_t B)^r||| \leqslant |||\left(B^{rts/2}A^{(1-t)rs}B^{rts/2}\right)^{1/s}||| \leqslant |||\left(A^{(1-t)rs}B^{rts}\right)^{1/s}|||. \tag{2.1}$$

Proof. Since $A\sharp_t B \in M_n^+$ for all $t \in [0,1]$,

$$s(A\sharp_t B) = \lambda(A\sharp_t B) \prec_{\log} \lambda\left(B^{ts/2}A^{(1-t)s}B^{ts/2}\right)^{1/s}$$

for all $t \in [0,1]$ and $s > 0$. By Weyl's theorem [3] on the singular values and the eigenvalues of a matrix, $\lambda\left(B^{ts/2}A^{(1-t)s}B^{ts/2}\right)^{1/s} \prec_{\log} s\left(B^{ts/2}A^{(1-t)s}B^{ts/2}\right)^{1/s}$. Then apply Ky Fan Dominance Theorem to have $|||A\sharp_t B||| \leqslant |||\left(B^{ts/2}A^{(1-t)s}B^{ts/2}\right)^{1/s}|||$. Then we have

$$|||(A\sharp_t B)^r||| \leqslant |||\left(B^{ts/2}A^{(1-t)s}B^{ts/2}\right)^{r/s}||| \leqslant |||\left(B^{rts/2}A^{(1-t)rs}B^{rts/2}\right)^{1/s}|||. \tag{2.2}$$

The first inequality in (2.2) follows from the convexity and monotonicity of the function $f(t) = t^r$, the second inequality in (2.2) follows from the Araki-Lieb-Thirring inequality.

The second inequality of (2.1) follows from

$$s\left(B^{rts/2}A^{(1-t)rs}B^{rts/2}\right)^{1/s} = \lambda\left(B^{rts/2}A^{(1-t)rs}B^{rts/2}\right)^{1/s} = \lambda\left(A^{(1-t)rs}B^{rts}\right)^{1/s} \prec_{\log} s\left(A^{(1-t)rs}B^{rts}\right)^{1/s}$$

and Ky Fan Dominance Theorem. □

Our main theorem states as follows.

THEOREM 2.2. *Let* $A_i, B_i \in M_n^+$ $(i = 1, \ldots, m)$, $r \geqslant 1$, $t \in [0,1]$ *and* $s > 0$. *Then for any unitarily invariant norm* $|||\cdot|||$ *on* M_n

$$\begin{aligned} |||\sum_{i=1}^m (A_i\sharp_t B_i)^r||| &\leqslant |||((\sum_{i=1}^m B_i)^{rts/2}(\sum_{i=1}^m A_i)^{(1-t)rs}(\sum_{i=1}^m B_i)^{rts/2})^{1/s}||| \\ &\leqslant |||((\sum_{i=1}^m A_i)^{(1-t)rs/2}(\sum_{i=1}^m B_i)^{rts/2})^{1/s}|||. \end{aligned} \tag{2.3}$$

Proof. Since the function x^r is convex and monotone increasing, by the mentioned above inequality (1.1) and the concavity of the t-geometric means, we have

$$|||\sum_{i=1}^m (A_i\sharp_t B_i)^r||| \leqslant |||(\sum_{i=1}^m A_i\sharp_t B_i)^r||| \leqslant |||\left((\sum_{i=1}^m A_i)\sharp_t(\sum_{i=1}^m B_i)\right)^r|||. \tag{2.4}$$

On account of Proposition 2.1

$$\begin{aligned} |||\left((\sum_{i=1}^m A_i)\sharp_t(\sum_{i=1}^m B_i)\right)^r||| &\leqslant |||((\sum_{i=1}^m B_i)^{rts/2}(\sum_{i=1}^m A_i)^{(1-t)rs}(\sum_{i=1}^m B_i)^{rts/2})^{1/s}||| \\ &\leqslant |||((\sum_{i=1}^m A_i)^{(1-t)rs/2}(\sum_{i=1}^m B_i)^{rts/2})^{1/s}|||. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get (2.3). □

REMARK 2.3. For the commuting matrices A_k and B_k, we have $(A_k \sharp_t B_k)^r = A_k^{(1-t)r} B_k^{rt}$. Applying Theorem 2.2 for $r = 2$, $s = 1$, $t = 1/2$, we obtain (1.2).

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