Mathematical
Inequalities
& Applications
Volume 19, Number 2 (2016), 769–779 doi:10.7153/mia-19-57

A NOTE ON CONVEXITY, CONCAVITY, AND GROWTH CONDITIONS IN DISCRETE FRACTIONAL CALCULUS WITH DELTA DIFFERENCE

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(*Communicated by M. Bohner*)

Abstract. We demonstrate that some recent results regarding the connection between the convexity of the map $t \mapsto f(t)$ and the sign of $\Delta_a^\nu f(t)$, with $2 < \nu < 3$, can be improved. In particular, by utilizing a recent inequality due to Jia, Erbe, and Peterson, we are able to improve some of the existing results in the literature. As part of this study we illustrate the improvements that our results afford by providing several specific examples of their application.

1. Introduction

The discrete fractional calculus has recently received a great deal of attention. Since the initial works of Atici and Eloe [2, 3, 4], a steadily increasing number of researchers have begun investigating various questions in the area. Some of the most intriguing open questions concern in what way the fractional difference affects and is related to either the monotonicity or convexity of the maps on which it operates. Especially this interest arises from the nonlocal nature of the fractional difference and sum. In particular, recall that these are defined as follows; note that we first define the so-called falling factorial map, denoted $t \mapsto t^{\underline{\nu}}$, as it plays a key role in the definition. For further information about these and related fundamental definitions, the reader may consult the textbook by Goodrich and Peterson [18].

DEFINITION 1. We define the *falling factorial function*, denoted $t \mapsto t^{\underline{\nu}}$, by

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} := 0$.

Mathematics subject classification (2010): Primary: 26A51, 39A12, 39B62; Secondary: 26A33, 39A70, 39A99.

Keywords and phrases: Discrete fractional calculus, convexity, concavity, Taylor monomial, delta difference.

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DEFINITION 2. The ν*-th fractional sum*, $\nu > 0$, of a function $f : \mathbb{N}_a \to \mathbb{R}$, where $a \in \mathbb{R}$ is given, is

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. We also define the ν*-th fractional difference* of f, for $\nu > 0$, by

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{\nu-N} f(t),$$

where $t \in \mathbb{N}_{a-\nu+N}$ and $N \in \mathbb{N}_1$ is the unique number satisfying $N-1 < \nu \leqslant N$.

Regarding the notation in Definitions 1–2 and throughout the remainder of this note we use the following conventions.

REMARK 1. Given numbers n_1, $n_2 \in \mathbb{R}$ with $n_1 \leqslant n_2$ and $n_2 - n_1 \in \mathbb{N}_0$, we define by $\mathbb{N}_{n_1}^{n_2}$ the set

$$\mathbb{N}_{n_1}^{n_2} := \{n_1, n_1+1, \ldots, n_2\}.$$

Furthermore, given $r_1 \in \mathbb{R}$ we put

$$\mathbb{N}_{r_1} := \{r_1, r_1+1, r_1+2, \ldots\}.$$

The most important observation that we can make about Definition 2 is the nonlocal nature of the operator $\Delta_a^{-\nu}$, and, thus, of Δ_a^{ν}. More specifically, whereas the map $t \mapsto \Delta f(t)$ involves only $f(t+1)$ and $f(t)$ and, thus, is a *local* construction, the map $t \mapsto \Delta_a^{\nu} f(t)$ involves the entire collection of values $\{f(a), f(a+1), \ldots, f(t+\nu)\}$, for $t \in \mathbb{N}_{a+N-\nu}$, and thus is decidedly *nonlocal* in character. For this reason we sometimes say that these fractional operators possess a "memory property" since they "remember" all of the previous values that f has attained.

Among the most interesting consequences of the nonlocal structure of the fractional difference is its relationship to the monotonicity and convexity of f. It is a triviality that if $\Delta f(t) \geqslant 0$ for $t \in \mathbb{N}_a$, then f is increasing on $\mathbb{N}_a$. It is similarly trivial that if $\Delta^2 f(t) > 0$ for $t \in \mathbb{N}_a$, then f is convex on $\mathbb{N}_a$. But the relationship between the sign of $\Delta_a^{\nu} f(t)$ and the monotonicity or convexity of f is much more subtle and complicated. Some recent works by Atici and Uyanik [6], Dahal and Goodrich [9, 10], Jia, Erbe, and Peterson [20, 21, 22], Baoguo, Erbe, Goodrich, and Peterson [7], and Goodrich [16] have slowly begun to address these questions.

Of particular relevance to this note is the recent paper by Jia, Erbe, and Peterson [21]. In that work, generalizing and improving upon a result of Goodrich [16], the authors deduced results regarding the sign of $\Delta_a^{\nu} f(t)$, in the case where $2 < \nu < 3$, and the convexity and concavity of the map $t \mapsto f(t)$. In particular, they obtained the following result.

THEOREM 1. *Assume that the function* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\Delta_a^{\nu} f(t) \geqslant 0$, *for each* $t \in \mathbb{N}_{3+a-\nu}$, *where* $2 < \nu < 3$. *If, in addition, it holds that* $f(a) \leqslant 0$, $\Delta f(a) \geqslant 0$, *and* $\Delta^2 f(a) \geqslant 0$, *then* $\Delta^2 f(t) \geqslant 0$, *for* $t \in \mathbb{N}_{a+1}$.

Their proof of Theorem 1 relied on the following very important lemma, which was also discovered by Jia, Erbe, and Peterson [21]. Note that in [21], the statement of Lemma 1 contains a very slight misprint – i.e., it should read $\Delta_a^\nu f(a+3-\nu+k)$, as we have here, rather than $\Delta_a^\nu f(a+k)$ as they have. An examination of the proof Lemma 1 as stated in [21] reveals this to be the case. Other than this typographical error, the results of [21] are perfectly correct and valid.

LEMMA 1. *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\Delta_a^\nu f(a+3-\nu+k) \geqslant 0$, *for each* $k \in \mathbb{N}_0$, *where* $2 < \nu < 3$. *Define the map* $(t,a) \mapsto h_\nu(t,a)$ *by*

$$h_\nu(t,a) := \frac{(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)}. \tag{1}$$

Then, for each $k \in \mathbb{N}_0$,

$$\Delta^2 f(a+k+1) \geqslant -h_{-\nu}(a+3-\nu+k,a)f(a) - h_{-\nu+1}(a+3-\nu+k,a)\Delta f(a) - \sum_{i=0}^{k} h_{-\nu+1}(a+3-\nu+k,a+i+1)\Delta^2 f(a+i).$$

REMARK 2. The map $(t,s) \mapsto h_\nu(t,s)$ defined in (1) in Lemma 1 is known as the "Taylor monomial of degree ν based at a" – see [18].

In particular, then, we notice that in Theorem 1 one has a somewhat unusual collection of hypotheses – namely, that each of the following must hold.

$$f(a) \leqslant 0 \qquad \Delta f(a) \geqslant 0 \qquad \Delta^2 f(a) \geqslant 0 \qquad \Delta_a^\nu f(t) \geqslant 0, \text{ for each } t \in \mathbb{N}_{3+a-\nu}$$

This certainly stands in bold relief when compared to the integer-order setting. Thus, it seems natural to wonder whether some of these conditions can be eliminated. Or, if that is not possible in general, whether in special cases can we relax certain of these conditions.

The main result of this note, Theorem 2, demonstrates that we may give a partial affirmative answer to the preceding questions. In particular, by using Lemma 1 we can, in some cases, eliminate some of the conditions used in Theorem 1. While we still obtain results that are not as simple as in the integer-order setting, we, nonetheless, are able to recover results that are simpler to apply and do not require some of conditions in Theorem 1. For example, we are able, in certain circumstances, to

- eliminate the "initial nonpositivity" condition – i.e., that $f(a) \leqslant 0$; and
- eliminate the "initial convexity" condition – i.e., that $\Delta^2 f(a) \geqslant 0$.

And we will illustrate these facts explicitly with a variety of examples.

To conclude this section, let us end by mentioning briefly some of the current literature in discrete fractional calculus. In addition to the works already mentioned above that relate to monotonicity and convexity, other works such as one by Holm have addressed operational properties of the fractional difference and sum (e.g., composing various fractional operators) [19]; chaos in discrete fractional dynamical systems has been considered by Wu and Baleanu [23]; Ferreira [11] has produced a version of Gronwall's inequality in the setting of discrete fractional calculus; Ferreira and Goodrich have investigated discrete fractional boundary and initial value problems [12, 13, 14, 15, 17]; exponential functions in discrete fractional calculus have been treated by Acar and Atici [1]; an application to tumor growth modeling was provided by Atici and Şengül [5]; and extensions of fractional calculus to other time scales have been investigated by Bastos, Mozyrska, and Torres [8]. Thus, there have been and continue to exist a number of different areas of investigation in discrete fractional calculus. Finally, for readers interested in more details regarding discrete fractional calculus, the recent text by Goodrich and Peterson [18] may be consulted as a general reference.

2. Convexity and concavity results for fractional delta differences

We begin this section by stating and proving the primary result of the note. The proof makes use of the fundamental inequality in Lemma 1. We then demonstrate that several corollaries follow from Theorem 2. Throughout this section we provide specific examples to illustrate the use and applicability of Theorem 2 and its associated corollaries, and, moreover the way in which these results generalize Theorem 1, as discussed in Section 1. Throughout this section we invoke the convention that, unless otherwise stated, f is a map satisfying $f : \mathbb{N}_a \to \mathbb{R}$.

THEOREM 2. *Fix* $\nu \in (2,3)$ *and suppose that* $\Delta_a^{\nu} f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{3+a-\nu}$. *If for each* $k \in \mathbb{N}_{-1}$ *it holds that*

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+2+k}{(\nu-1)(3+k)} f(a+1) - \frac{\nu}{(3+k)(4+k)} f(a) \leqslant 0, \tag{2}$$

then $\Delta^2 f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{a+1}$.

Proof. We first establish that $\Delta^2 f(a+1) \geqslant 0$ holds. To see that this claim is true, with the help of Lemma 1 and the definition of the Taylor monomial in (1) we write

$$\begin{aligned}
\Delta^2 f(a+1) &\geqslant -h_{-\nu}(a+3-\nu,a)f(a) - h_{-\nu+1}(a+3-\nu,a)\Delta f(a) \\
&\quad - h_{-\nu+1}(a+3-\nu,a+1)\Delta^2 f(a) \\
&= \underbrace{-\frac{\Gamma(3-\nu)}{\Gamma(-\nu+1)}}_{<0} \left[\frac{1}{-\nu+1} f(a+2) + \left[\frac{3-\nu}{2(-\nu+1)} - \frac{2}{-\nu+1} \right] f(a+1) \right. \\
&\quad \left. + \left[\frac{3-\nu}{6} - \frac{3-\nu}{2(-\nu+1)} + \frac{1}{-\nu+1} \right] f(a) \right] \\
&= -\frac{\Gamma(3-\nu)}{\Gamma(-\nu+1)} \underbrace{\left[\frac{1}{-\nu+1} f(a+2) + \frac{\nu+1}{2(\nu-1)} f(a+1) - \frac{\nu}{6} f(a) \right]}_{\leqslant 0} \\
&\geqslant 0,
\end{aligned}$$

where the final inequality follows from an application of inequality (2) in case $k=-1$.

Having established the base case, we now use induction to complete the proof. Therefore, assume that $\Delta^2 f(a+k)\geqslant 0$ for each $k\in\mathbb{N}_1^{k_0+1}$ for some $k_0\geqslant 0$. We prove that $\Delta^2 f(a+k_0+2)\geqslant 0$. To this end, we write

$$\begin{aligned}\Delta^2 f(a+k_0+2) \geqslant & -h_{-\nu}(a+3-\nu+(1+k_0),a)f(a)\\ & -h_{-\nu+1}(a+3-\nu+(1+k_0),a)\Delta f(a)\\ & -\sum_{j=0}^{k_0+1} h_{-\nu+1}(a+3-\nu+(1+k_0),a+j+1)\Delta^2 f(a+j)\\ \geqslant & -h_{-\nu}(a+3-\nu+(1+k_0),a)f(a)\\ & -h_{-\nu+1}(a+3-\nu+(1+k_0),a)\Delta f(a)\\ & -h_{-\nu+1}(a+3-\nu+(1+k_0),a+1)\Delta^2 f(a),\end{aligned} \tag{3}$$

where we use the induction hypothesis together with the fact that

$$h_{-\nu+1}(a+3-\nu+(1+k_0),a+j+1)\leqslant 0$$

for each $0\leqslant j\leqslant k_0+1$. Continuing, then, from estimate (3) we deduce that

$$\begin{aligned}&\Delta^2 f(a+k_0+2)\\ &\quad\geqslant -\frac{\Gamma(4-\nu+k_0)}{\Gamma(-\nu+1)}\left[\frac{4-\nu+k_0}{(4+k_0)!}f(a)\right.\\ &\qquad+\frac{4-\nu+k_0}{(-\nu+1)(3+k_0)!}[f(a+1)-f(a)]\\ &\qquad\left.+\frac{1}{(-\nu+1)(2+k_0)!}[f(a+2)-2f(a+1)+f(a)]\right]\\ &\quad\geqslant -\frac{\Gamma(4-\nu+k_0)}{\Gamma(-\nu+1)(2+k_0)!}\left[\frac{4-\nu+k_0}{(3+k_0)(4+k_0)}f(a)\right.\\ &\qquad+\frac{4-\nu+k_0}{(-\nu+1)(3+k_0)}[f(a+1)-f(a)]\\ &\qquad\left.+\frac{1}{-\nu+1}[f(a+2)-2f(a+1)+f(a)]\right]\\ &\quad= -\frac{\Gamma(4-\nu+k_0)}{\Gamma(-\nu+1)(2+k_0)!}\left[\frac{1}{-\nu+1}f(a+2)+\left(\frac{\nu+2+k_0}{(\nu-1)(3+k_0)}\right)f(a+1)\right.\\ &\qquad\left.+\underbrace{\frac{(4-\nu+k_0)(-3-\nu-k_0)+(4+k_0)(3+k_0)}{(-\nu+1)(3+k_0)(4+k_0)}}_{=\frac{\nu^2-\nu}{(-\nu+1)(3+k_0)(4+k_0)}}f(a)\right].\end{aligned} \tag{4}$$

Thus, from estimate (4) we obtain the inequality

$$\begin{aligned}\Delta^2 f(a+k_0+2) \geqslant & -\frac{\Gamma(4-\nu+k_0)}{\Gamma(-\nu+1)(2+k_0)!}\left[\frac{1}{-\nu+1}f(a+2)\right.\\ &+\frac{\nu+2+k_0}{(\nu-1)(3+k_0)}f(a+1)\\ &\left.-\frac{\nu}{(3+k_0)(4+k_0)}f(a)\right]\\ \geqslant & \, 0,\end{aligned} \tag{5}$$

where to obtain the final inequality in (5) we utilize hypothesis (2). Thus, by the arbitrariness of $k_0 \in \mathbb{N}_{-1}$, we conclude that $\Delta^2 f(t) \geqslant 0$ for each $t \in \mathbb{N}_{a+1}$, and this completes the proof. □

We give an example to demonstrate that, as discussed in Section 1, an advantage of Theorem 2 over the previously known results is that here we do not require that $\Delta^2 f(a) \geqslant 0$ holds; in addition, we do not require that $f(a) \leqslant 0$ holds. Thus, in these ways, Theorem 2 improves the known results in the literature.

EXAMPLE 1. Put $f(a)=0$, $f(a+1)=1$, and $f(a+2)=1.9$ and fix $\nu=\frac{5}{2}\in(2,3)$. Then we calculate

$$\frac{1}{-\nu+1}f(a+2)+\frac{\nu+1}{2(\nu-1)}f(a+1)-\frac{\nu}{6}f(a)=-\frac{2}{3}\cdot 1.9+\frac{7}{6}\cdot 1-\frac{5}{12}\cdot 0=-\frac{1}{10}<0,$$

which shows that inequality (2) is satisfied in case $k=-1$; in fact, it can be shown that (2) is satisfied for each $k\in\mathbb{N}_{-1}$. In addition, it is a straightforward exercise to argue that a function with initial values as above can satisfy $\Delta_a^{2.5} f(t)\geqslant 0$, for each $t\in\mathbb{N}_{a+0.5}$; that is to say, we can define $f(k)$ for $k\in\mathbb{N}_a$ in such a way that the initial values hold and that $\Delta_a^{2.5} f(t)\geqslant 0$, $t\in\mathbb{N}_{a+0.5}$, also holds. For example, any map that satisfies $f(a+3)\geqslant 2.875$ and $f(a+4)\geqslant 3.9375$ will also satisfy $\Delta_a^{2.5} f(t)\geqslant 0$ for $t\in\mathbb{N}_{a+0.5}^{a+1.5}$. In any case, we nonetheless calculate

$$\Delta^2 f(a)=-\frac{1}{10}<0,$$

which establishes that f is actually "initially concave".

Similarly, if we put $g(a)=1$, $g(a+1)=2.8$, and $g(a+2)=4.5$ as well as again taking $\nu=\frac{5}{2}$, then we see that $\Delta^2 g(a)=-\frac{1}{10}<0$. Yet at the same time we calculate

$$\frac{1}{-\nu+1}g(a+2)+\frac{\nu+1}{2(\nu-1)}g(a+1)-\frac{\nu}{6}g(a)=-\frac{2}{3}\cdot 4.5+\frac{7}{6}\cdot 2.8-\frac{5}{12}\cdot 1=-\frac{3}{20}<0,$$

which shows that inequality (2) is satisfied in case $k=-1$, and, as can again be easily shown, it holds for $k\in\mathbb{N}_{-1}$. Once again, we can also easily argue that if $g(a)=1$, $g(a+1)=2.8$, and $g(a+2)=4.5$, then $\Delta_a^{2.5} g(t)\geqslant 0$ can hold for each $t\in\mathbb{N}_{a+0.5}$.

All in all, then, we see that condition (2) may be satisfied even if the map $t \mapsto f(t)$ is not convex "at" $t = a$. In particular, this means that Theorem 2 does *not* require that the map $t \mapsto f(t)$ be "initially convex".

We next state a few corollaries that specialize condition (2) if one knows *a priori* the sign of either $f(a)$, $f(a+1)$, or $f(a+2)$. In particular, these corollaries demonstrate that if we know some information about the "initial" pointwise values of f, then we can replace inequality (2) with more specific conditions that may prove to be easier to use and apply in practice. As part of stating and proving these corollaries, we also provide some examples to explicate their use and application.

COROLLARY 1. *Fix $\nu \in (2,3)$ and suppose that $\Delta_a^{\nu} f(t) \geqslant 0$ for each $t \in \mathbb{N}_{3+a-\nu}$. In addition, assume that:*

1. $f(a) \leqslant 0$*; and*
2. $f(a+1) \geqslant 0$.

If it holds that

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+1}{2(\nu-1)} f(a+1) - \frac{\nu}{6} f(a) \leqslant 0, \tag{6}$$

then $\Delta^2 f(t) \geqslant 0$ for each $t \in \mathbb{N}_{a+1}$.

Proof. For $\nu \in (2,3)$ fixed, consider the map $G_\nu : [-1,+\infty) \to \mathbb{R}$ defined by

$$G_\nu(k) := \frac{\nu+2+k}{(\nu-1)(3+k)}.$$

We easily calculate

$$\frac{dG_\nu}{dk}(k) = \frac{1-\nu}{(\nu-1)(3+k)^2},$$

which is nonpositive on its domain. Consequently, it holds that

$$\sup_{k \in \mathbb{N}_{-1}} \frac{\nu+2+k}{(\nu-1)(3+k)} = \frac{\nu+1}{2(\nu-1)}, \tag{7}$$

for each $\nu \in (2,3)$. In addition, it evidently holds that

$$\sup_{k \in \mathbb{N}_{-1}} \frac{\nu}{(3+k)(4+k)} = \frac{\nu}{6},$$

for each $\nu \in (2,3)$. Putting the preceding calculations together with the fact that $f(a+1) \geqslant 0$ and $f(a) \leqslant 0$ we estimate

$$\begin{aligned}
&\frac{1}{-\nu+1}f(a+2)+\frac{\nu+2+k}{(\nu-1)(3+k)}f(a+1)-\frac{\nu}{(3+k)(4+k)}f(a)\\
&\leqslant \frac{1}{-\nu+1}f(a+2)+\sup_{k\in\mathbb{N}_{-1}}\frac{\nu+2+k}{(\nu-1)(3+k)}\underbrace{f(a+1)}_{\geqslant 0}\\
&\quad+\sup_{k\in\mathbb{N}_{-1}}\frac{\nu}{(3+k)(4+k)}\underbrace{(-f(a))}_{\geqslant 0}\\
&\leqslant \frac{1}{-\nu+1}f(a+2)+\frac{\nu+1}{2(\nu-1)}f(a+1)-\frac{\nu}{6}f(a)\\
&\leqslant 0,
\end{aligned}$$

where the final inequality follows from assumption (6). Thus, the conclusion of Theorem 2 may be invoked to deduce that $\Delta^2 f(t) \geqslant 0$ on $\mathbb{N}_{a+1}$, as desired. □

EXAMPLE 2. In this example we demonstrate that Corollary 1 allows for the inequality $\Delta^2 f(a) < 0$ to hold in some cases. In particular, suppose that we put $f(a) = -0.1$ and $f(a+1) = 0.5$. Let us, as in Example 1, take $\nu = \frac{5}{2}$. Obviously, with this choice for $f(a)$ and $f(a+1)$, inequality (6) becomes

$$-\frac{2}{3}f(a+2)+\frac{7}{6}\cdot\frac{1}{2}-\frac{5}{12}\cdot-\frac{1}{10}\leqslant 0.$$

Rearranging and simplifying the above inequality results in the lower bound

$$f(a+2)\geqslant\frac{15}{16}.$$

In other words, in order for inequality (6) to be satisfied, it must be the case that $f(a+2) \geqslant \frac{15}{16}$ holds. But this clearly allows $\Delta^2 f(a) < 0$ in some cases. For example, if we take $f(a+2) = 1$, then we calculate that $\Delta^2 f(a) = -\frac{1}{10} < 0$. Moreover, one can show that the assumption $f(a) = -0.1$, $f(a+1) = 0.5$, and $f(a+2) = 1$ is compatible with the assumption $\Delta_a^{2.5} f(t) \geqslant 0$ for each $t \in \mathbb{N}_{a+0.5}$. Thus, all in all, we conclude that Corollary 1 admits maps f that

- are *not* “initially convex” insofar as $\Delta^2 f(a) < 0$; and
- are *not* “initially nonnegative” insofar as $f(a) < 0$.

And so we again see that Theorem 2 generates results that are more general and widely applicable than the existing results in the literature.

COROLLARY 2. *Fix* $\nu \in (2,3)$ *and suppose that* $\Delta_a^\nu f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{3+a-\nu}$. *In addition, assume that:*

1. $-\dfrac{6}{\nu(\nu-1)}\Delta f(a+1) \leqslant f(a) \leqslant 0$;

2. $f(a+1) \leqslant 0$; *and*

3. $\Delta f(a+1) \geqslant 0$.

Then it holds that $\Delta^2 f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{a+1}$.

Proof. Using the fact that

$$\inf_{k\in\mathbb{N}_{-1}} \frac{\nu+2+k}{(\nu-1)(3+k)} = \lim_{k\to\infty} \frac{\nu+2+k}{(\nu-1)(3+k)} = \frac{1}{\nu-1},$$

we merely compute

$$\begin{aligned}
&\frac{1}{-\nu+1}f(a+2) + \frac{\nu+2+k}{(\nu-1)(3+k)}f(a+1) - \frac{\nu}{(3+k)(4+k)}f(a)\\
&\leqslant \frac{1}{-\nu+1}f(a+2) + \frac{1}{\nu-1}f(a+1) - \frac{\nu}{6}f(a)\\
&= \frac{1}{-\nu+1}\Delta f(a+1) - \frac{\nu}{6}f(a)\\
&= -\frac{1}{\nu-1}\Delta f(a+1) - \frac{\nu}{6}f(a)\\
&\leqslant 0,
\end{aligned}$$

which holds for each $k \in \mathbb{N}_{-1}$, and then invoke Theorem 2. □

REMARK 3. Observe that since $\Delta f(a+1) \geqslant 0$ and $-\frac{6}{\nu(\nu-1)} < 0$, it follows that

$$\left[-\frac{6}{\nu(\nu-1)}\Delta f(a+1), 0\right] \neq \varnothing$$

so that condition (1) in Corollary 2 is not vacuous.

COROLLARY 3. *Fix* $\nu \in (2,3)$ *and suppose that* $\Delta_a^\nu f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{3+a-\nu}$. *In addition, assume that:*

1. $f(a) \geqslant 0$; *and*

2. $f(a+1) \geqslant 0$.

If it holds that

$$\frac{1}{-\nu+1}f(a+2) + \frac{\nu+1}{2(\nu-1)}f(a+1) \leqslant 0, \tag{8}$$

then $\Delta^2 f(t) \geqslant 0$ *for each* $t \in \mathbb{N}_{a+1}$.

Proof. For each $k \in \mathbb{N}_{-1}$ we write

$$\begin{aligned}&\frac{1}{-\nu+1}f(a+2)+\frac{\nu+2+k}{(\nu-1)(3+k)}f(a+1)-\frac{\nu}{(3+k)(4+k)}f(a)\\&\quad\leqslant\frac{1}{-\nu+1}f(a+2)+\frac{\nu+1}{2(\nu-1)}f(a+1),\end{aligned}$$

where we have utilized the calculation in (7). Then condition (8) implies inequality (2) for each $k \in \mathbb{N}_{-1}$, and so, an application of Theorem 2 yields the desired conclusion and completes the proof. □

REMARK 4. The condition (8) appearing in Corollary 3 is equivalent to the inequality

$$f(a+2) \geqslant \frac{\nu+1}{2}f(a+1). \tag{9}$$

Since $\nu > 2$ we see that (9) implies that the hypotheses of Corollary 3 force $\Delta f(a+1) > 0$ to hold – namely, that

$$f(a+2) > \frac{3}{2}f(a+1).$$

However, as remarked earlier it, nonetheless, need *not* hold that $\Delta^2 f(a) \geqslant 0$.

EXAMPLE 3. Suppose that we set $f(a) = 1$, $f(a+1) = 3$, and $f(a+2) = 4.9$. Moreover, put $\nu := \frac{21}{10}$. Then one finds that $f(a)$, $f(a+1) \geqslant 0$ and, moreover, that

$$\frac{1}{-\nu+1}f(a+2)+\frac{\nu+1}{2(\nu-1)}f(a+1) = -\frac{10}{11}\cdot\frac{49}{10}+\frac{31}{22}\cdot 3 < 0,$$

so that condition (8) holds. If it then holds, in addition, that $\Delta_a^{2.1} f(t) \geqslant 0$ for each $t \in \mathbb{N}_{a+0.9}$, then Corollary 3 may be invoked to deduce that $\Delta^2 f(t) \geqslant 0$, for each $t \in \mathbb{N}_{a+1}$. Observe, in particular, that

$$\Delta^2 f(a) = -\frac{1}{10} < 0.$$

Thus, we conclude that Corollary 3 does *not* require that f possess any "initial convexity". In addition, we see that Corollary 3 does *not* require that f possess any "initial nonpositivity".

REMARK 5. It is clearly possible to recast the results of this note in terms of concavity instead of convexity – i.e., establishing conditions under which $\Delta^2 f(t) < 0$ or, more generally, $\Delta^2 f(t) \leqslant 0$. Since these are obvious generalizations (see the techniques in, for example, [16] or [18]), we omit them.

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(Received November 1, 2015)

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