

## ON JORDAN'S, REDHEFFER'S AND WILKER'S INEQUALITY

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*Abstract.* In this paper, the authors offer new Jordan, Redheffer and Wilker type inequalities, along with refinements and converses. Connections with Euler's gamma function are pointed out, too.

### 1. Introduction

In the recent years, the refinements of the inequalities of trigonometric functions such as Adamović-Mitrinović, Cusa-Huygens, Jordan inequality, Redheffer inequality, Becker-Stark inequality, Wilker inequality, Huygens inequality, and Kober inequality have been studied extremely by numerous authors, e.g., see [4, 5, 10, 18, 19, 23, 24, 25, 26, 32, 33, 34, 35, 36] and the references therein. Motivated by these rapid studies, in this paper we make a contribution to the subject by refining the Cusa-Huygens, Jordan and Redheffer inequality, and establish a Wilker type inequality. In all cases, we give the upper and lower bound of  $\sin(x)/x$  in terms of simple functions. Meanwhile, we give some Redheffer type inequalities for trigonometric functions, which refine the existing results in the literature.

For the representation of trigonometric functions in terms of gamma function, we define for  $x, y > 0$  the classical *gamma function*  $\Gamma$ , the *digamma function*  $\psi$  and the *beta function*  $B(\cdot, \cdot)$  by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. We denote the *trigamma function*  $\psi'$  by  $K$ , and defined as,

$$K(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}, \quad K'(x) = -2 \sum_{n=0}^{\infty} \frac{1}{(x+n)^3}. \quad (1.1)$$

We refer to reader to see [39, 40, 41, 43] for more properties and relations of  $K$ . The functions  $\Gamma$  and  $\psi$  satisfy the recurrence relation

$$\Gamma(1+z) = z\Gamma(z), \quad \psi(1+z) = z\psi(z). \quad (1.2)$$

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The following relation

$$\psi(1+z) - \psi(z) = \frac{1}{z} \tag{1.3}$$

follows from (1.2). Differentiating both sides of (1.3), one has

$$K(1+z) - K(z) = -\frac{1}{z^2}. \tag{1.4}$$

The following Euler’s reflection formula [1, 6.1.17]

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}, \quad 0 < t < 1, \tag{1.5}$$

can be written as

$$\frac{x}{\sin(x)} = \Gamma\left(1 + \frac{x}{\pi}\right)\Gamma\left(1 - \frac{x}{\pi}\right) = B\left(1 + \frac{x}{\pi}, 1 - \frac{x}{\pi}\right), \quad 0 < x < \pi. \tag{1.6}$$

The logarithmic differentiation to both sides of (1.5) gives the following reflection formula,

$$\psi(1-t) - \psi(t) = \frac{\pi}{\tan(\pi t)}. \tag{1.7}$$

Replacing  $t$  by  $t + 1/2$  in (1.5), we get

$$\frac{x}{\cos(x)} = \frac{x}{\pi}\Gamma\left(\frac{1}{2} + \frac{x}{\pi}\right)\Gamma\left(\frac{1}{2} - \frac{x}{\pi}\right), \quad 0 < x < \frac{\pi}{2}. \tag{1.8}$$

Next we recall the Adamović-Mitrinović [21, p.238] and Cusa-Huygens [36] inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < |x| < \frac{\pi}{2}, \tag{1.9}$$

For the refinement of (1.9), see [18, 23, 25, 35, 36, 51].

Our first main result refines the inequalities in (1.9) as follows:

**THEOREM 1.** *For  $x \in (0, \pi)$ , the following inequalities hold,*

$$\frac{2 - (x/\pi)^3 + \cos(x)}{3} < \frac{\sin(x)}{x} < \frac{2 - (x/\pi)^4 + \cos(x)}{3}. \tag{1.10}$$

In [46], Wilker asked a question to find the largest constant  $c$  such that the inequality

$$a(x) > cb(x), \quad 0 < |x| < \frac{\pi}{2}, \tag{1.11}$$

holds true, where

$$a(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - 2, \quad \text{and} \quad b(x) = x^3 \tan(x).$$

The inequality (1.11) is known as the Wilker inequality in the literature. Anglesio [44] proved that the ratio  $a(x)/b(x)$  is decreasing in  $x \in (0, \pi/2)$ , and he answered the question by showing the following two sided inequality,

$$2 + \frac{8}{45}x^3 \tan(x) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{16}{\pi^4}x^3 \tan(x), \quad (1.12)$$

with the best possible constants  $8/45$  and  $16/\pi^4$ . For the new proofs and refinement of (1.12), see [13, 28, 50, 49, 52, 55]. Thereafter the following Wilker type inequalities,

$$\begin{aligned} 3 + \frac{3}{20}x^3 \tan(x) &< 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \frac{16}{\pi^4}x^3 \tan(x), \quad 0 < x < \frac{\pi}{2}, \\ 2 + \frac{2}{45}x^3 \sin(x) &< \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < 2 + \left(\frac{2}{\pi} - \frac{16}{\pi^3}\right)x^3 \sin(x), \quad 0 < x < \frac{\pi}{2}, \\ \left(\frac{\sinh(x)}{x}\right)^2 + \frac{\tanh(x)}{x} &> 2 + \frac{8}{45}x^3 \tanh(x), \quad x > 0, \\ \left(\frac{x}{\sinh(x)}\right)^2 + \frac{x}{\tanh(x)} &< 2 + \frac{2}{45}x^3 \sinh(x), \quad x > 0, \end{aligned}$$

were established by Chen-Sándor [11], Wang [50], Zhu [56], and Sun-Zhu [57], respectively.

We establish an other Wilker type inequality by giving the following result.

**THEOREM 2.** For  $x \in (0, \pi)$ , we have

$$3 + \left(\frac{x}{\pi}\right)^4 \frac{x}{\sin(x)} < 2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} < 3 + \left(\frac{x}{\pi}\right)^3 \frac{x}{\sin(x)}.$$

The following inequality

$$2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} > 2, \quad 0 < |x| < \frac{\pi}{2}, \quad (1.13)$$

has been recently established by Wu and Srivastava [48], which is sometime known as the second Wilker inequality [25]. Inequality (1.13) can be written as,

$$\frac{\sin(x)}{x} < \frac{1}{2} \left( \frac{x}{\sin(x)} + \cos(x) \right), \quad 0 < |x| < \frac{\pi}{2}.$$

It was shown in [25] that this inequality is weaker than the Cusa-Huygens inequality (1.9), as follows:

$$\frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3} < \frac{1}{2} \left( \frac{x}{\sin(x)} + \cos(x) \right), \quad 0 < |x| < \frac{\pi}{2},$$

here second inequality is equivalent to

$$3 \frac{x}{\sin(x)} + \cos(x) > 4, \quad 0 < |x| < \frac{\pi}{2}. \quad (1.14)$$

Recently, Mortici [22] refined this inequality (1.14) as follows:

$$3 \frac{x}{\sin(x)} + \cos(x) > 4 + \frac{1}{10}x^4 + \frac{1}{210}x^6, \quad 0 < |x| < \frac{\pi}{2}. \quad (1.15)$$

We establish the following sharp result as a counterpart of (1.15).

**THEOREM 3.** For  $x \in (0, \pi)$ , we have

$$2 - \frac{x^4}{\pi^4} < 3 \frac{\sin(x)}{x} - \cos(x) < 2 - \frac{x^3}{\pi^3}.$$

The well-know Jordan's inequality [21] states,

$$\frac{\pi}{2} \leq \frac{\sin(x)}{x}, \quad 0 < x \leq \frac{\pi}{2},$$

equality with  $x = \pi/2$ .

In [12], Debnath and Zhao refined the Jordan's inequality as below,

$$dz_l(x) = \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2) \leq \frac{\sin(x)}{x}, \quad (1.16)$$

$$DZ_l(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin(x)}{x}, \quad (1.17)$$

for  $x \in (0, \pi/2)$ , equality in both inequalities with  $x = \pi/2$ .

The following inequality

$$O_l(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3} \left(x - \frac{\pi}{2}\right)^2 \leq \frac{\sin(x)}{x}, \quad (1.18)$$

for  $x \in (0, \pi/2)$  is due to Özban [27], equality with  $x = \pi/2$ .

In [54], Zhu proved that

$$\frac{\sin(x)}{x} \leq \frac{2}{\pi} + \frac{(\pi - 2)(\pi^2 - 4x^2)}{\pi^3} = Z_u(x), \quad (1.19)$$

for  $x \in (0, \pi/2)$ .

For the following inequalities

$$J_l(x) = \frac{2}{\pi} + \frac{\pi^4 - 16x^4}{2\pi^5} < \frac{\sin(x)}{x} < \frac{2}{\pi} + \frac{(\pi - 2)(\pi^4 - 16x^4)}{\pi^5} = J_u(x), \quad (1.20)$$

$x \in (0, \pi/2)$ , see [14].

In [18, Theorem 1.3], Klén et al. proved that

$$1 - \frac{x^2}{6} < \frac{\sin(x)}{x} < 1 - \frac{2x^2}{3\pi^2}, \quad (1.21)$$

for  $x \in (-\pi/2, \pi/2)$ . The following inequality refines the second inequality in (1.21),

$$\frac{\sin(x)}{x} \leq \frac{1 - x^2/\pi^2}{\sqrt{1 + 3(x/\pi)^4}}, \quad (1.22)$$

for  $x \in (0, \pi)$ , see [19].

Our first main result reads as follows:

**THEOREM 4.** For  $x \in (0, \pi/2)$ , we have

$$C_l(x) < \frac{\sin(x)}{x} < C_u(x),$$

where

$$C_l(x) = \left(1 - \frac{x^2}{\pi^2}\right)^{\pi^2/6} \quad \text{and} \quad C_u(x) = \left(1 - \frac{x^2}{\pi^2}\right)^{3/2}.$$

The right side of the following theorem can not be compared with the corresponding side of Theorem 4, and the second inequality is weaker than the above result.

**THEOREM 5.** For  $x \in (0, \pi)$ , we have

$$D_l(x) < \frac{\sin(x)}{x} < D_u(x),$$

where

$$D_l(x) = 1 - \left(\frac{x}{\pi}\right)^2 \left(2 - \frac{x}{\pi}\right) \quad \text{and} \quad D_u(x) = 1 - \frac{1}{2} \left(\frac{x}{\pi}\right)^2 \left(3 - \left(\frac{x}{\pi}\right)^2\right).$$

It is not difficult to see that Theorem 5 refines the inequalities (1.21) and (1.22). Obviously, one can see that  $dz_l(x) < DZ_l(x) < O_l(x)$ ,  $J_l(x) < DZ_l(x)$ , and  $Z_u(x) < J_u(x)$  for  $x \in (0, \pi/2)$ . Now it is natural to compare our result with (1.18) and (1.19). For this purpose we give the following inequalities by using the Mathematica Software<sup>®</sup> [31],

$$\begin{aligned} O_l(x) &< C_l(x), & x \in (0, 1.19540), \\ O_l(x) &< D_l(x), & x \in (0, 0.92409), \\ C_u(x) &< Z_u(x), & x \in (0, 1.09447), \\ D_u(x) &< Z_u(x), & x \in (0, 0.95784). \end{aligned}$$

We see that our result refines (1.18) and (1.19) in the given interval of  $x$ .

The following result is the consequence of Theorem 5.

**THEOREM 6.** For  $y \in (0, 1)$ , we have

$$1. \quad B(x, y) < \frac{1}{xy} \frac{x+y}{1+xy}, \quad 0 < x < 1,$$

$$2. \frac{1}{xy} \frac{x+y}{1+xy} < B(x,y), \quad 1 < x < \infty.$$

It is worth to mention that the part (1) of Theorem 6 recently appeared in [16]. The proof of the claim is based on [16, Lemma 2.5], and the proof of the lemma is invalid.

The following Redheffer inequality [30]

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \quad 0 < x \leq \pi, \tag{1.23}$$

was proved by Williams [47]. Chen et al. [10] obtained the three Redheffer-type inequalities for  $\cos x$ ,  $\cosh x$  and  $(\sinh x)/x$ . Sun and Zhu [19, 45] proved the Redheffer-type two-sided inequalities for trigonometric and hyperbolic functions. The inequalities appeared in [19] were refined in [54], and read as follows:

$$\left( \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}} \right)^{\pi^2/6} < \frac{\sin(x)}{x} < \left( \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}} \right), \quad 0 < x < \pi, \tag{1.24}$$

$$\left( \frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}} \right)^{\pi^2/6} < \cos(x) < \left( \frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}} \right)^{3/4}, \quad 0 < x < \frac{\pi}{2}. \tag{1.25}$$

$$\left( \frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2} \right)^{1/2} < \frac{\tan(x)}{x} < \left( \frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2} \right)^{\pi^2/6}, \quad 0 < x < \frac{\pi}{2}. \tag{1.26}$$

One can see easily that the first inequality in Theorem 5 refines the (1.23). Our following theorem refines (1.24), as well as Cusa-Huygens inequality, and also gives a new upper bound for the right side of (1.9).

**THEOREM 7.** *For  $x \in (0, \pi/2)$  one has*

$$\frac{\sin(x)}{x} < \frac{\pi^2 - x^2}{\pi^2 + \alpha x^2} < \frac{2 + \cos(x)}{3} < \frac{\pi^2 - x^2}{\pi^2 + \beta x^2}, \tag{1.27}$$

with best possible constants  $\alpha = \pi^2/6 - 1 \approx 0.644934$  and  $\beta = 1/2$ .

**THEOREM 8.** *The function*

$$g(t) = \frac{\log((1+t^2)/(1-t^2))}{\log(1/\cos(\pi t/2))}$$

is strictly decreasing from  $(0, 1)$  on  $(\alpha, \beta)$ . In particular, for  $x \in (0, \pi)$

$$\cos(x/2)^\alpha < \frac{\pi^2 - x^2}{\pi^2 + x^2} < \cos(x/2)^\beta,$$

with the best possible constants  $\alpha = 16/\pi^2 \approx 1.62114$  and  $\beta = 1$ .

In the following theorem we refine the inequalities given in (1.25).

THEOREM 9. *The following function*

$$h(t) = \frac{1 - 4t^2}{t^2 \cos(\pi t)} - \frac{1}{t^2}$$

is strictly increasing from  $(0, 1/2)$  onto  $((\pi^2 - 8)/2, 16/\pi - 4)$ . In particular, for  $x \in (0, \pi/2)$

$$\frac{\pi^2 - 4x^2}{\pi^2 + \alpha x^2} < \cos(x) < \frac{\pi^2 - 4x^2}{\pi^2 + \beta x^2},$$

with the best possible constants  $\alpha = 16/\pi - 4 \approx 1.09296$  and  $\beta = (\pi^2 - 8)/2 \approx 0.934802$ .

The paper is organized into three sections as follows. Section 1, contains the introduction and the statements of our main results. In Section 2, we give some lemmas, which will be used in our proofs. Section 3 is consists of the proofs of the main results and some corollaries.

## 2. Preliminaries

The following result is sometime called the Monotone l'Hôpital rule, which is due to Anderson et al. [4].

LEMMA 1. *For  $-\infty < a < b < \infty$ , let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ , and be differentiable on  $(a, b)$ . Let  $g'(x) \neq 0$  on  $(a, b)$ . If  $f'(x)/g'(x)$  is increasing (decreasing) on  $(a, b)$ , then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If  $f'(x)/g'(x)$  is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2. *For  $a \in (0, 1)$  and  $x > 0$ , the following inequality holds,*

$$\frac{1 - a}{x + a} < \psi(1 + x) - \psi(x + a).$$

*Proof.* It is well-known that the function  $f(x) = x\psi(x)$ ,  $x > 0$  is strictly convex [43, Theorem 6]. This implies that,

$$f(ar + (1 - a)s) < af(r) + (1 - a)f(s), \quad r, s > 0, a \in (0, 1).$$

Setting  $r = 1 + x$  and  $s = x$  in the above inequality, we get

$$(a + x)\psi(a + x) < (a + ax)\psi(1 + x) + (1 - a)x\psi(x),$$

now the proof follows easily if we replace  $\psi(x) = \psi(1 + x) - 1/x$ .  $\square$

LEMMA 3. For  $x \in (1, \infty)$  and  $y \in (0, 1)$ , we have

$$\frac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(1+x+y)} > \frac{1}{1+xy}.$$

*Proof.* Let

$$f_y(x) = \log(\Gamma(1+x)) + \log(\Gamma(1+y)) - \log(\Gamma(1+x+y)) + \log((1+xy)),$$

for  $x \in (1, \infty)$  and  $y \in (0, 1)$ , clearly  $f_y(1) = 0$ . Differentiating  $f$  with respect to  $x$  and using the formula (1.3), we get

$$\begin{aligned} f'_y(x) &= \frac{y}{1+xy} + \psi(1+x) - \psi(1+x+y) \\ &= \frac{y}{1+xy} - \frac{1}{x+y} + \psi(1+x) - \psi(x+y) \\ &> \frac{y}{1+xy} - \frac{1}{x+y} + \frac{1-y}{x+y} = \frac{y(1-y)(x-1)}{(x+y)(1+xy)} > 0, \end{aligned}$$

by Lemma 2. Thus,  $f$  is strictly increasing, and  $f_y(x) > f_y(1) = 0$ , this implies the proof of part (1). For the proof of part (2), see [16, (3.2)].  $\square$

REMARK 1. It is easy to see that the convexity of the function  $x \mapsto \log(x\Gamma(x))$ ,  $x > 0$ , implies the following inequality,

$$\frac{1}{xy} \left( \frac{x+y}{2} \right)^2 < \frac{\Gamma(x)\Gamma(y)}{\Gamma((x+y)/2)^2}. \tag{2.1}$$

Replacing  $x$  by  $1-x/\pi$  and  $y$  by  $1-x/\pi$  in (2.1), and applying (1.5), we get

$$\frac{\sin(x)}{x} < \frac{\pi^2 - x^2}{\pi^2}, \quad 0 < x < \pi. \tag{2.2}$$

This improves the following inequality

$$\frac{\sin(x)}{x} \leq \left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{1/2}, \quad 0 < x \leq \pi,$$

which was proved in [42].

### 3. Proof of the main results

In this section we will give the proofs of the main results highlighted in the first section, as well as some corollaries are being established.

*Proof of Theorem 1.* The proof of the first inequality is trivial. For the proof of the second and the third inequality, we define

$$f(x) = \frac{2 + \cos(x) - 3 \sin(x)/x}{x^4},$$



for  $x \in (0, \pi)$ . We will prove that  $f$  is strictly decreasing from  $(0, \pi)$  onto  $(1/\pi^4, 1/60)$ . One has,

$$x^6 f(x) = -x^2 \sin(x) - 7x \cos(x) + 15 \sin(x) - 8x = f_1(x),$$

$$f_1'(x) = 5x \sin(x) + 8 \cos(x) - x^2 \cos(x) - 8,$$

$$f_1''(x) = -3 \sin(x) + 3x \cos(x) + x^2 \sin(x), f_1'''(x) = x(x \cos(x) - \sin(x)) < 0.$$

Thus  $f_1''(x) < f_1''(0) = 0$ ,  $f_1'(x) < f_1'(0) = 0$ , and  $f_1(x) < f_1(0) = 0$ . The limiting values can be achieved by l'Hôpital rule.  $\square$

Similarly, for the proof of the first inequality we will prove that the function

$$g(x) = \frac{2 + \cos(x) - 3 \sin(x)/x}{x^3}$$

is strictly increasing from  $(0, \pi)$  onto  $(0, 1/\pi^3)$ . One has

$$x^5 g'(x) = -(x^2) \sin(x) - 6x \cos(x) + 12 \sin(x) - 6x = h(x),$$

$$h'(x) = 4x \sin(x) - x^2 \cos(x) + 6 \cos(x) - 6,$$

$$h''(x) = 2x \cos(x) + x^2 \sin(x) - 2 \sin(x), h'''(x) = x^2 \cos(x).$$

Thus  $h'''(x)$  is positive in  $x \in (0, \pi/2)$  and negative in  $x \in (\pi/2, \pi)$ . Since  $h''(\pi/2) = \pi^2/4 - 2 > 0$  and  $h''(\pi) = -2\pi < 0$ , and  $h''(x)$  is strictly increasing (resp. decreasing) in  $x \in (0, \pi/2)$  (resp.  $x \in (\pi/2, \pi)$ ). There exists a unique  $x_0$  in  $(\pi/2, \pi)$  such that  $h''(x_0) = 0$ . Thus  $h''(x) > 0$ ,  $x \in (0, x_0)$  and  $h''(x) < 0$ ,  $x \in (x_0, \pi)$ . This implies that  $h'(x)$  is strictly increasing in  $x \in (0, x_0)$  and strictly decreasing in  $x \in (x_0, \pi)$ . As  $h'(0) = 0$ , one has  $h'(x_0) > 0$ . By  $h'(\pi) = \pi^2 - 12 < 0$ , we get that there exists a unique  $x_1$  in  $(x_0, \pi)$  such that  $h'(x_1) = 0$ . We get that  $h(x)$  is strictly increasing in  $(0, x_1)$ , and decreasing in  $(x_1, \pi)$ , with  $h(0) = 0$  and  $h(\pi) = 0$ . Thus  $h(x) > h(0) = 0$ ,  $x \in (0, x_1)$  and  $h(x) > h(\pi) = 0$ ,  $x \in (x_1, \pi)$ . Hence, in all cases, one has  $h(x) > 0$ . This completes the proof of the second inequality.

It follows from the proof of the second inequality of Theorem 1 that  $f < 1/60$  and  $f_1 < 0$ . This implies the following result.

**COROLLARY 1.** For  $x \in (0, \pi)$ , we have

$$1. M_1 = \frac{2 - x^4/60 + \cos(x)}{3} < \frac{\sin(x)}{x} < \frac{8 + 7 \cos(x)}{15 - x^2} = M_2,$$

$$2. 2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} < 3 + \frac{x^4}{60} \frac{x}{\sin(x)},$$

$$3. (8/7) \frac{x}{\sin(x)} + \frac{x}{\tan(x)} > \frac{15 - x^2}{7}.$$

Recently, the following inequalities appeared in [51],

$$Y_l = \cos\left(\frac{x}{\sqrt{x}}\right)^a < \frac{\sin(x)}{x} < \cos\left(\frac{x}{\sqrt{x}}\right)^{5/3} = Y_u \tag{3.1}$$

for  $x \in (0, \pi/2)$  with  $a = \log(2/\pi)/\log(\cos(\sqrt{5}\pi/10)) \approx 1.67141$ . By using Mathematica Software<sup>®</sup> [31], one can see that

$$Y_l < M_1, x \in (0, 1.06580) \quad \text{and} \quad Y_u - M_2 \in (0, -0.0009), x \in (0, \pi/2).$$

*Proof of Theorem 2 & 3.* The proof of both theorems follow immediately from Theorem 1.  $\square$

*Proof of Theorem 4.* Let us consider the application

$$f(x) = \log\left(\frac{x}{\sin(x)}\right) - c \log(1/(1 - x^2/\pi^2)),$$

for  $x \in (0, \pi/2)$ . A simple computation gives

$$x \sin(x)(\pi^2 - x^2)f'(x) = (\sin x - x \cos x)(\pi^2 - x^2) - 2cx^2 \sin(x) = g(x).$$

One has

$$g'(x)/x = (\pi^2 - 2 - 4c) \sin(x) + (2 - 2c)x \cos(x) - x^2 \sin(x) = h(x).$$

Finally,

$$h'(x) = (\pi^2 - 6c) \cos(x) - (4 - 2c)x \sin(x) - x^2 \cos(x).$$

Now, if we select  $c = \pi^2/6$ , then, as  $\pi^2 - 6c = 0$  and  $4 - 2c = 4 - \pi^2/3 > 0$ , we get  $h'(x) < 0$ , so this finally leads to  $f(x) < 0$ . Hence, the first inequality follows.

For the proof of the second inequality, let  $c = 3/2$ , then one has

$$g'(x)/x = (\pi^2 - 8) \sin(x) - x \cos(x) - (x^2) \sin(x) = h(x).$$

Since  $h(\pi/2) < 0, h(\pi/4) > 0$ , there exists an  $x'$  in  $(\pi/4, \pi/2)$  such that  $h(x') = 0$ . We'll show that  $x'$  is unique. One has  $h(x)/\sin(x) = \pi^2 - 8 - s(x)$ , where  $s(x) = x^2 + x/\tan(x)$ . Now,

$$s'(x) \sin(x)^2 = 2x \sin(x)^2 + \cos(x) \sin(x) - x = p(x).$$

Here  $p'(x) = 4x \sin(x) \cos(x) > 0$ , which shows that  $p(x) > p(0) = 0$ , so  $s'(x) > 0$ , finally:  $s(x)$  is a strictly increasing function. Thus the equation  $s(x) = \pi^2 - 8$  has a single root (which is  $x'$ ), so  $h(x) > 0$  for  $x \in (0, x')$  and  $h(x) < 0$  for  $x \in (x', \pi/2)$ . Thus  $g$  is increasing, resp. decreasing in the above intervals, and  $g(0) = g(\pi/2) = 0$ , so  $g(x) > 0$  for  $x \in (0, \pi/2)$ . This completes the proof of the second inequality.  $\square$

COROLLARY 2. For  $x \in (0, \pi/2)$  one has

$$\frac{\pi^2 - x^2 - \pi^2 x^2/3}{\pi^2 - x^2} < \frac{x}{\tan(x)} < \frac{\pi^2 - 4x^2}{\pi^2 - x^2}.$$

*Proof.* After simplification the derivative of the function

$$f_c(x) = \log\left(\frac{x}{\sin(x)}\right) - c \log(1/(1 - x^2/\pi^2)),$$

can be written as

$$f'_c(x) = \frac{1}{x} - 2cx\pi^2 - x^2 - \frac{1}{\tan(x)}.$$

By the proof of Theorem 4,  $f'_{\pi^2/6}(x) < 0$  and  $f'_{3/2}(x) > 0$ . Clearly,  $f_{\pi^2/6} = 0 = f_{3/2}$ . Now the proof of the inequalities is obvious.  $\square$

The second inequality in Corollary 2 improves the first inequality in (1.26).

*Proof of Theorem 5.* For  $t \in (0, 1)$ , let

$$f(t) = \frac{\pi t(t^3 + 1) - \sin(\pi t)}{\pi t^3}, \quad g(t) = \frac{\pi t(2 + t^4) - 2 \sin(\pi t)}{\pi t^3}.$$

By Theorem 1, we get

$$\begin{aligned} f'(x) &= \frac{3\pi t^3 + \pi(t^3 + 1) - \pi \cos(\pi t)}{\pi t^3} - \frac{3(\pi t(t^3 + 1) - \sin(\pi t))}{\pi t^4} \\ &= \frac{3}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - \frac{2 - t^3 + \cos(\pi t)}{3} \right) > 0, \end{aligned}$$

and

$$\begin{aligned} g'(x) &= 2t - \frac{2}{t^2} \left( \frac{\cos(\pi t)}{t} - \frac{\sin(\pi t)}{\pi t^2} \right) + \frac{4}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - 1 \right) \\ &= \frac{2}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - \frac{2 - t^4 + \cos(\pi t)}{3} \right) < 0. \end{aligned}$$

Thus, the functions  $f$  and  $g$  are strictly increasing and decreasing in  $t \in (0, 1)$ , respectively. Hence, the proof follows easily if we use the inequalities,

$$f(t) < \lim_{t \rightarrow 1} = 2, \quad g(t) > \lim_{t \rightarrow 1} g(t) = 3,$$

and replace  $t$  by  $x/\pi$ .  $\square$

*Proof of Theorem 6.* Utilizing (1.5), the first inequality in Theorem 5 is equivalent to

$$\frac{1 + x/\pi - x/\pi}{\Gamma(1 + x/\pi)\Gamma(1 - x/\pi)} > \frac{(1 - x/\pi)(1 + x/\pi - (x/\pi)^2)}{\Gamma(1 + x/\pi - x/\pi)}.$$

Replacing  $x$  by  $x/\pi$  and  $y$  by  $1 - x/\pi$  we get (1). Similarly, the second inequality in Theorem 5 can be written as,

$$\frac{1 + x/\pi + 1 - x/\pi}{(1 - (x/\pi)^2)(2 + (x/\pi)^2)} < \frac{\Gamma(1 + x/\pi)\Gamma(1 - x/\pi)}{\Gamma(1 + x/\pi - x/\pi)}.$$

If we replace  $x$  by  $1 + x/\pi$  and  $y$  by  $1 - x/\pi$  then we get the proof of part (2) for  $1 < x < 2$ . The rest of proof follows from Lemma 3.  $\square$

*Proof of Theorem 7.* Let  $f(t) = f_1(t)/f_2(t), t \in (0, 1/2)$ , where

$$f_1(t) = 1 - 3t^2 - \cos(\pi t) \quad \text{and} \quad f_2(t) = t^2(2 + \cos(\pi t)).$$

A simple calculation gives

$$f_1'(t) = -6t - \pi \sin(\pi t) < 0, \quad \text{and} \quad f_2'(t) = t(4 + 2\cos(\pi t) - \pi t \sin(\pi t)) > 0.$$

Thus,  $f$  is the product of two positive strictly decreasing functions, this implies that  $f$  is strictly decreasing in  $t \in (0, 1/2)$ . Applying l'Hôpital rule, we get  $\lim_{t \rightarrow 1/2} f(t) = 1/2 < f(t) < \alpha = \lim_{t \rightarrow 0} f(t) = \pi^2/6 - 1$ . Here the first inequality implies

$$\frac{3(1 - t^2)}{2 + \cos(\pi t)} - 1 < \alpha t^2,$$

which is equivalent to

$$\frac{2 + \cos(\pi t)}{3} > \frac{1 - t^2}{1 + \alpha t^2}.$$

Letting  $\pi t = x \in (0, \pi/2)$ , we get the second inequality of (1.27), and the third inequality of (1.27) follows similarly. For the proof of the first inequality, see [2, (2.5)].  $\square$

**COROLLARY 3.** For  $x \in (0, 1)$ , we have

$$\frac{4}{\pi} \frac{t}{1 - t^2} < \tan\left(\frac{\pi t}{2}\right) < \frac{\pi}{2} \frac{t}{1 - t^2}. \tag{3.2}$$

*Proof.* Let  $f(t) = t/(\tan(\pi t/2)(1 - t^2)), t \in (0, 1/2)$ . We get

$$f'(t) = \frac{(1 + t^2)\sin(\pi t) - \pi t(1 - t^2)}{2(1 - t^2)^2 \sin(\pi t/2)^2},$$

which is positive by (1.23). By l'Hôpital rule, we get  $\lim_{t \rightarrow 0} f(t) = 2/\pi < f(t) < \pi/4 = \lim_{t \rightarrow 1/2} f(t)$ . This completes the proof.  $\square$

For  $0 < t < 1$ , letting  $x = (\pi t)/2$  in (3.2), we get

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan(x)}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \tag{3.3}$$

which is so-called Becker-Stark inequality [8]. Thus, Corollary 3 gives a simple proof of Becker-Stark inequality. It is easy to see that the second inequality in Corollary 2 improves the first inequality in (3.3).

*Proof of Theorem 8.* Write  $g(t) = g_1(t)/g_2(t)$ ,  $0 < t < 1$ , where

$$g_1(t) = \log\left(\frac{1+t^2}{1-t^2}\right), g_2(t) = \log\left(\frac{1}{\cos(\pi t/2)}\right).$$

We get,

$$\frac{g'_1(t)}{g'_2(t)} = \frac{8}{\pi} \frac{t}{(1-t^4)\tan(\pi t/2)} = \frac{8}{\pi} g_3(t).$$

One has,

$$g'_3(t) = \frac{\sin(\pi t)(1+3t^4) - \pi t(1-t^4)}{2\sin(\pi t/2)^2(1-t^4)^2},$$

which is negative by (1.24). Clearly  $g_1(0) = 0 = g_2(0)$ , thus  $g$  is strictly decreasing by Lemma 1. Using l'Hôpital rule, we get  $\lim_{t \rightarrow 0} g(t) = 16/\pi^2 > g(t) > 1 = \lim_{t \rightarrow 1} g(t)$ . Replacing  $\pi t$  by  $x$ , we get the desired inequalities.  $\square$

**COROLLARY 4.** For  $x \in (0, \pi)$ , one has

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{4/(3\beta)} < \cos\left(\frac{x}{2}\right)^{4/3} < \frac{\sin(x)}{x} < \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{4/(3\alpha)},$$

where  $\alpha$  and  $\beta$  are as in Theorem 8.

*Proof.* The proof of the first inequality follows from Theorem 8, and the second inequality is also well known [25]. The third inequality is just (1.24).  $\square$

**THEOREM 10.** For  $x \in (0, \pi/2)$ , we have

$$\frac{2}{\pi} + \frac{\pi - 2}{\pi} \cos(x) < \frac{\sin(x)}{x}, \tag{3.4}$$

$$\left(\frac{1 + \cos}{2}\right)^{2/3} < \frac{\sin(x)}{x} < \frac{2 \cdot 2^{2/3}}{\pi} \left(\frac{1 + \cos}{2}\right)^{2/3} < \frac{4}{\pi} \frac{1 + \cos(x)}{2}. \tag{3.5}$$

*Proof.* The inequality (3.4) may be rewritten as

$$f(x) = \pi \sin(x) - (\pi - 2)x \cos(x) - 2x > 0,$$

for  $x \in (0, \pi/2)$ . Clearly,  $f(0) = f(\pi/2) = 0$ . On the other hand,

$$f(x) = 2(\cos(x) - 1) + (\pi - 2)x \sin(x) = -4 \sin(x/2)^2 + 2(\pi - 2)x \sin(x/2) \cos(x/2).$$

This implies that

$$f'(x)/(4\sin(x/2)\cos(x/2)) = (\pi - 2)/2 - \tan(x/2) = g(x)$$

for  $x > 0$ . As  $\tan(x/2)$  is strictly increasing, there is a unique  $x'$  in  $(0, \pi/2)$  such that  $g(x') = 0$ . Also,  $g(x) > 0$  for  $x \in (0, x')$  and  $g(x) < 0$  for  $x \in (x', \pi/2)$ ; i.e.  $x'$  is a maximum point of  $f(x)$ . This gives  $f(x) > f(0) = 0$  for  $x \in (0, x')$  and  $f(x) > f(\pi/2) = 0$  for  $x \in (x', \pi/2)$ . This completes the proof of (3.4).

For the proof of (3.5), let

$$j(x) = \log\left(\frac{x}{\sin(x)} - \frac{2}{3} \frac{2}{1 + \cos(x)}\right),$$

$x \in (0, \pi/2)$ . One has

$$\begin{aligned} j'(x) &= \frac{1 - x \cot(x)}{x} - \frac{2}{3} \frac{\sin(x)}{1 + \cos(x)} \\ &= \frac{1}{x} - \cot(x) - \frac{2}{3} \frac{1 - \cos(x)}{\sin(x)} = j_1(x), \end{aligned}$$

which is negative, because the inequality  $j_1(x) < 0$  can be written as  $\sin(x)/x < (2 + \cos(x))/3$ , which is so-called Cusa-Huygens inequality [21]. Thus,  $j(x)$  is strictly decreasing, and

$$\lim_{x \rightarrow 0} j(x) = 0 > j(x) > \log(\pi) - (5/3)\log(2) = \lim_{x \rightarrow \pi/2} j(x) \approx -0.010515.$$

By these inequalities we get

$$\cos(x/2)^{4/3} < \frac{\sin(x)}{x} < \exp((5/3)\log(2) - \log(\pi)) \cos(x/2)^{4/3}.$$

The proof of the first and second inequality is completed, and the proof of the third inequality is trivial.  $\square$

The inequality (3.4) improves the following one

$$1 - 2 \frac{\pi - 2}{\pi^2} x < \frac{\sin(x)}{x}, \quad 0 < x < \pi, \quad (3.6)$$

which was proved in [37] as an application of the concavity of  $\sin(x)/x$ . Indeed, the inequality

$$1 - 2 \frac{\pi - 2}{\pi^2} x < \frac{2}{\pi} + \frac{\pi - 2}{\pi} \cos(x),$$

is equivalent to

$$\cos(x) > 1 - \frac{2}{\pi} x, \quad 0 < x < \frac{\pi}{2}, \quad (3.7)$$

which is Kober's inequality, see [21, 32].

*Proof of Theorem 9.* For  $x \in (0, \pi/2)$ , let

$$f(x) = \frac{x}{\sin(x)} \frac{x(\pi-x)}{(2x-\pi)^2} - \frac{4x^2}{(2x-\pi)^2}.$$

We get,

$$f'(x) = -\frac{16\pi^2 \cos(x)}{(2x-\pi)^3 \sin(x)^2} g(x),$$

where

$$g(x) = \tan(x) - \frac{\sin(x)^2}{\cos(x)} - x.$$

One has,

$$g'(x) = \frac{\sin(x)^2}{\cos(x)} (\sin(x)^2 - \sin(x) - 2) < 0,$$

as  $0 < \sin(x) < 1$ , and  $g(x) = 0$ . Thus,  $g < 0$ , and in result  $f'(x) < 0$ , this implies that  $f$  is strictly decreasing. By l'Hôpital rule we get

$$\lim_{x \rightarrow \pi/2} f(x) = \frac{\pi^2 - 8}{2} < f(x) < \lim_{x \rightarrow 0} f(x) = \frac{16}{\pi} - 4,$$

this implies the proof.  $\square$

Replacing  $x$  by  $\pi/2 - x$  in the inequalities of Theorem 9, we get the following corollary as a result.

**COROLLARY 5.** For  $x \in (0, \pi/2)$ , we have

$$\frac{16(\pi-x)}{4\pi^2 + \alpha(2x-\pi)^2} < \frac{\sin(x)}{x} < \frac{16(\pi-x)}{4\pi^2 + \beta(2x-\pi)^2},$$

where  $\alpha$  and  $\beta$  are as in Theorem 9.

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