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SOME CLASSES OF COMPLETELY MONOTONIC FUNCTIONS RELATED TO q–GAMMA AND q–DIGAMMA FUNCTIONS

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Abstract. In the paper, some classes of completely monotonic functions involving the q-gamma and q-digamma functions are derived. The monotonicity properties of these functions are exploited to establish a double inequality for the ratio of the q-gamma function and a double inequality for the q-digamma function. Moreover, a class of inequalities for the q-polygamma functions is presented.

1. Introduction

A real-valued function f, defined on an interval I, is called completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0, \qquad n \in \mathbb{N}_0 = \{0,1,2,\cdots\}; \;\; x \in I. \tag{1.1}$$

If the inequality (1.1) is strict for all $x \in I$ and for all $n \geqslant 1$, then f is said to be strictly completely monotonic. These functions have numerous applications in various branches, like, for instance, numerical analysis and probability theory.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\log f$ satisfies

$$(-1)^n [\log f(x)]^{(n)} \geqslant 0, \qquad n \in \mathbb{N}; \;\; x \in I. \tag{1.2}$$

If inequality (1.2) is strict for all $x \in I$ and for all $n \geqslant 1$, then f is said to be strictly logarithmically completely monotonic.

The notion of logarithmically completely monotonic functions was recovered by Feng Qi and Bai-Ni Guo [22]. It has been proved once again in [8, 12, 23] that the class of logarithmically completely monotonic functions is a subclass of the completely monotonic functions. For more information, see ([19], p. 134, Section 1.3) and the references given therein.

Anderson et al. [4] proved that the function

$$f_\alpha(x) = x^{\alpha - x} e^x \Gamma(x) \tag{1.3}$$

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is decreasing and logarithmically convex from $(0,\infty)$ onto $(\sqrt{2\pi},\infty)$ when $\alpha=\frac{1}{2}$ and is increasing and logarithmically concave from $(0,\infty)$ onto $(1,\infty)$ when $\alpha=1$ where $\Gamma(x)$ is the Euler-gamma function defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, \qquad x>0 \tag{1.4}$$

The monotonicity and convexity of $f_\alpha(x)$ were slightly extended by Alzer [3] as follows: The function $f_a(x)$ is decreasing on $(0,\infty)$ if and only if $\alpha\leqslant\frac{1}{2}$ and increasing on $(0,\infty)$ if and only if $\alpha\geqslant 1$. The logarithmically complete monotonicity of $f_\alpha(x)$ is equivalent to the complete monotonicity of the function

$$\psi(x)-\log x+\frac{\alpha}{x}$$

and its negative, which was proved in [21], where $\psi(x)$ is the so-called digamma (or psi) function defined as the logarithmic derivative of the gamma function (1.4). Also, the function (1.3) has been studied in a number of references such as the newly published papers [11, 13], the survey article [20] and closely-related references therein.

The main purpose of this paper is to extend the previous results to the q-gamma function for $q>0$ by means of studying the monotonicity properties of the following function

$$F_\alpha(x;q)=[x]_q^{\alpha-x}e^{-\frac{\mathrm{Li}_2(1-q^x)}{\log q}}\Gamma_q(x) \tag{1.5}$$

where $\Gamma_q(x)$ is the q-gamma function and $\mathrm{Li}_2(x)$ is the dilogarithm function. The following section will be devoted to present the definition of the q-gamma function and related functions and formulas.

2. The q-gamma function

The q-gamma function is defined for positive real numbers x and $q\neq 1$ as [17]

$$\Gamma_q(x)=(1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0<q<1 \tag{2.1}$$

and

$$\Gamma_q(x)=(q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{n=0}^{\infty}\frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q>1 \tag{2.2}$$

From the previous definitions, for a positive x and $q\geqslant 1$, we get

$$\Gamma_q(x)=q^{\frac{(x-1)(x-2)}{2}}\Gamma_{q^{-1}}(x). \tag{2.3}$$

The close connection between the ordinary gamma function (1.4) and the q-gamma function is given by the limit relations

$$\lim_{q\to 1^-}\Gamma_q(x)=\lim_{q\to 1^+}\Gamma_q(x)=\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt. \tag{2.4}$$

Many of the classical facts about the ordinary gamma function have been extended to the q-gamma function (see [5, 16, 17, 18] and the references given therein). Numerous papers appeared providing inequalities for the q-gamma function and its logarithmic derivatives. Many properties of the q-gamma function and some developments in this area are given in [2, 6, 7, 10, 24, 25, 26, 27, 28, 29, 31] and the references given therein. The logarithmic derivative of the q-gamma function is the so-called q-digamma (q-psi) function denoted by ψ_q

$$\psi_q(z) = \frac{d}{dz}(\log \Gamma_q(z)) = \frac{\Gamma_q'(z)}{\Gamma_q(z)}. \tag{2.5}$$

From (2.1), we get for $0 < q < 1$ and for all real variable $x > 0$

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k}. \tag{2.6}$$

and from (2.2) we obtain for $q > 1$ and $x > 0$

$$\psi_q(x) = -\log(q-1) + \log q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1-q^{-k}} \right]. \tag{2.7}$$

Krattenthaler and Srivastava [15] proved that $\psi_q(x)$ tends to $\psi(x)$ when letting $q \to 1$. Some properties and expansions associated with the q-digamma function have been derived in [30]. Among these results, we need the recursive formula

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}, \qquad q > 0; \;\; x > 0. \tag{2.8}$$

In the previous section, we mentioned another two concepts without definitions. The first is the symbol $[x]_q$ which is the basic number defined as

$$[x]_q = \frac{1-q^x}{1-q}, \qquad q \neq 1. \tag{2.9}$$

The second is the dilogarithm function $\mathrm{Li}_2(z)$ which is defined for complex argument z as [1]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt, \qquad z \notin (1, \infty), \tag{2.10}$$

An important recursive formula for the dilogarithm function that we need, can be derived as

$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) = -\mathrm{Li}_2(1-z) - \frac{1}{2}\log^2 z. \tag{2.11}$$

Also, we see that

$$\lim_{q \to 1} \frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x. \tag{2.12}$$

3. The main results

In this section, we investigate the monotonicity and complete monotonicity properties for functions involving the q-gamma function and consequently, we establish sharp double inequalities for the q-gamma and the q-polygamma functions.

LEMMA 3.1. *Let $0 < y < 1$. Then the function*

$$g(y) = \frac{1}{1-y} + \frac{1}{\log y} \tag{3.1}$$

is decreasing on $(0,1)$ and $\frac{1}{2} < g(y) < 1$.

Proof. Differentiation gives

$$g'(y) = \frac{y\log^2 y - (1-y)^2}{y\log^2 y(1-y)^2}$$

which can be represented as

$$g'(y) = \frac{-y}{\log^2 y(1-y)^2} \sum_{n=2}^{\infty} \frac{\log^n(1/y)}{n!} \lambda(n)$$

where $\lambda(n) = 2^n - n^2 + n - 2$, $n \geqslant 2$. It is not difficult to see that $\Delta^2\lambda(n) = 2^n - 2 > 0$ for all $n \geqslant 2$ where Δ is the forward shift operator. Since $\Delta\lambda(2) = \lambda(2) = 0$, then $\lambda(n) > 0$ for all $n \geqslant 2$ and so $g(y)$ is decreasing on $(0,1)$. Using l'Hôpital's rule gives the bounds of $g(y)$. □

THEOREM 3.2. *Let x, q and α be real numbers such that $x, q > 0$. The function $F_\alpha(x;q)$ as defined in (1.5) satisfies the following monotonicity properties:*

1. *The function $F_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ if and only if $\alpha \leqslant g(q)$ when $0 < q < 1$ and if and only if $\alpha \leqslant \frac{1}{2}$ when $q \geqslant 1$ where $g(y)$ defined as in (3.1).*
2. *The function $F_\alpha(x;q)$ is increasing and logarithmically concave on $(0,\infty)$ if and only if $\alpha \geqslant 1$ for all $q > 0$.*
3. *The function $[F_\alpha(x;q)]^{-1}$ is completely monotonic on $(0,\infty)$ if and only if $\alpha \geqslant 1$ for all $q > 0$.*

Proof. Logarithmic differentiaion yields

$$\frac{d}{dx}\{\log F_\alpha(x;q)\} = \psi_q(x) - \log[x]_q - \frac{\alpha q^x \log q}{1-q^x} \tag{3.2}$$

When $0 < q < 1$, the relation (2.6), Taylor series of logarithm function and binomial theorem give

$$\frac{d}{dx}\{\log F_\alpha(x;q)\} = \sum_{k=1}^{\infty} \frac{q^{xk}}{k(1-q^k)} f(\alpha, y), \qquad y = q^k \tag{3.3}$$

where

$$f(\alpha,y)=\log y+(1-y)-\alpha(1-y)\log y$$

It is clear that the function $\alpha\mapsto f(\alpha,y)$ is increasing on $\mathbb{R}$ for each fixed $y\in(0,1)$. Also, we have

$$f(0,y)=\log y+1-y=-y\sum_{n=2}^{\infty}\frac{\log^n(1/y)}{n!}(n-1)<0$$

and

$$f(1,y)=y\log y+1-y=y\sum_{n=2}^{\infty}\frac{\log^n(1/y)}{n!}>0$$

which lead to the function $f(\alpha,y)$ has just one zero depends on y at $\alpha=g(y)$ where $g(y)$ is defined in (3.1). In view of the result obtained by Lemma 3.1 and the previous notes, we have to take $\alpha\leqslant g(q)$ to ensure that $f(\alpha,y)<0$ for all $y=q^k;\ k\in\mathbb{N}$ and to take $\alpha\geqslant 1$ to ensure that $f(\alpha,y)>0$ for all $y=q^k;\ k\in\mathbb{N}$. This reveals that

$$(-1)^n[\log F_\alpha(x;q)]^{(n)}=-\sum_{k=1}^{\infty}\frac{q^{xk}k^{n-2}\log^{n-1}(q^{-1})}{k(1-q^k)}g(q^k)>0,\qquad \alpha\leqslant g(q)$$

which concludes that $F_\alpha(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $\alpha\leqslant g(q)$ and $0<q<1$. Since any (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic, then the function $F_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ if $\alpha\leqslant g(q)$ and $0<q<1$.

When $q\geqslant 1$, (2.3) gives

$$\begin{aligned}F_\alpha(x;q)&=q^{(x-1)(\alpha-x)}[x]_{q^{-1}}^{\alpha-x}e^{-\frac{\mathrm{Li}_2\left(\frac{q^{-x}-1}{q^{-x}}\right)}{\log q}}q^{\frac{(x-1)(x-2)}{2}}\Gamma_{q^{-1}}(x)\\&=q^{x(\alpha-\frac{1}{2})-\alpha+1}[x]_{q^{-1}}^{\alpha-x}e^{-\frac{\mathrm{Li}_2\left(1-q^{-x}\right)}{\log q^{-1}}}\Gamma_{q^{-1}}(x)\\&=q^{x(\alpha-\frac{1}{2})-\alpha+1}F_\alpha(x;q^{-1})\end{aligned}\tag{3.4}$$

Here, we used the identity (2.11). It is easy to see that $q^{x(\alpha-\frac{1}{2})-\alpha+1}$ is completely monotonic on $(0,\infty)$ if $\alpha\leqslant\frac{1}{2}$ for $q\geqslant 1$. Since $F_\alpha(x;q^{-1})$ is completely monotonic if $\alpha\leqslant g(q^{-1})$ for $q\geqslant 1$ and $g(q^{-1})\geqslant\frac{1}{2}$, then $F_\alpha(x;q^{-1})$ is completely monotonic if $\alpha\leqslant\frac{1}{2}$ for $q\geqslant 1$. Since the product of two completely monotonic functions is also completely monotonic, then $F_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ if $\alpha\leqslant\frac{1}{2}$ for $q\geqslant 1$ which concludes that $F_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ if $\alpha\leqslant g(q)$ when $0<q<1$ and if $\alpha\leqslant\frac{1}{2}$ when $q\geqslant 1$.

Next, we assume that $F_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ for $q>0$. Then we have for all $x>0$ and $q>0$

$$\frac{F'_\alpha(x;q)}{F_\alpha(x;q)}=\psi_q(x)-\log[x]_q-\frac{\alpha q^x\log q}{1-q^x}<0$$

or equivalently

$$\alpha < \frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q)$$

When $0 < q < 1$, (2.6) gives

$$\lim_{x\to\infty} \frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q) = \lim_{x\to\infty} \frac{1-q^x}{\log q} \sum_{k=1}^{\infty} \frac{q^{x(k-1)}(\log q^k + 1 - q^k)}{k(1-q^k)}$$
$$= \frac{1}{1-q} + \frac{1}{\log q} = g(q)$$

When $q \geqslant 1$, (2.7) gives

$$\lim_{x\to\infty} \frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q) = \lim_{x\to\infty} \left[\frac{1-q^{-x}}{2} + \sum_{k=1}^{\infty} \frac{q^{-xk}(q^{-x}-1)(\log q^{-k} + 1 - q^{-k})}{k(1-q^{-k})} \right]$$
$$= \frac{1}{2}$$

These were proven the first statement.

In order to prove the second statement, we have $g(y) > 0$ when $\alpha \geqslant 1$ and so $F'_\alpha(x;q) > 0$ if $\alpha \geqslant 1$ for $0 < q < 1$. This yields that $F_\alpha(x;q)$ is increasing on $(0,\infty)$ for $0 < q < 1$. By differentiating (3.3) yields the function $F_\alpha(x;q)$ is logarithmically concave on $(0,\infty)$ if $\alpha \geqslant 1$ for $0 < q < 1$. In fact, we have

$$\frac{d}{dx}(q^{x(\alpha-\frac{1}{2})-\alpha+1}) = (\alpha - \frac{1}{2})(q^{x(\alpha-\frac{1}{2})-\alpha+1}) \log q \geqslant 0, \qquad q \geqslant 1; \;\; \alpha \geqslant \frac{1}{2}$$

Therefore, by applying (3.3), the function $F_\alpha(x;q)$ is increasing and logarithmically concave on $(0,\infty)$ for $q \geqslant 1$ if $\alpha \geqslant 1$ and consequently the function $F_\alpha(x;q)$ is increasing and logarithmically concave on $(0,\infty)$ for $q > 0$ if $\alpha \geqslant 1$.

Conversely, let the function $F_\alpha(x;q)$ is increasing on $(0,\infty)$ for $q > 0$. Then we have for all $x > 0$

$$\frac{F'_\alpha(x;q)}{F_\alpha(x;q)} = \psi_q(x) - \log[x]_q - \frac{\alpha q^x \log q}{1-q^x} > 0$$

or equivalently

$$\alpha > \frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q). \tag{3.5}$$

Using the recursive formula (2.8) and L'Hospital's rule would yield

$$\lim_{x\to 0} \frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q) = 1$$

which reveals that $\alpha \geqslant 1$ for $q > 0$. The proof of second statement is completed.

Similarly, by using (3.3), (3.4) and (3.5), we can easily prove the third statement. □

COROLLARY 3.3. *For all $q>0$, the double inequality*

$$\frac{[b]_q^{b-1}}{[a]_q^{a-1}}e^{\frac{Li_2(1-q^b)-Li_2(1-q^a)}{\log q}}<\frac{\Gamma_q(b)}{\Gamma_q(a)}<\frac{[b]_q^{b-\frac{1}{2}}}{[a]_q^{a-\frac{1}{2}}}e^{\frac{Li_2(1-q^b)-Li_2(1-q^a)}{\log q}} \tag{3.6}$$

is valid for all real numbers a and b where $b>a>0$.

Proof. From the monotonicity of the functions $F_{\frac{1}{2}}(x;q)$ and $F_1(x;q)$, we conclude that

$$F_{\frac{1}{2}}(a;q)>F_{\frac{1}{2}}(b;q), \qquad b>a>0$$

and

$$F_1(a;q)<F_1(b;q), \qquad b>a>0$$

which give (3.6). This ends the proof. □

REMARK 3.4. When letting $q\to 1$ to (3.6), we obtain the double inequality

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b}<\frac{\Gamma(b)}{\Gamma(a)}<\frac{b^{b-\frac{1}{2}}}{a^{a-\frac{1}{2}}}e^{a-b} \tag{3.7}$$

which was established at the first time by J. D. Kečkić and P. M. Vasić [14] for $b>a\geqslant 1$ and extended to $b>a>0$ by C.-P. Chen and F. Qi [9].

COROLLARY 3.5. *For all $q>0$, the inequality*

$$\Gamma_q(x)>[x]_q^{x-1}\exp\left(\frac{Li_2(1-q^x)}{\log q}\right) \tag{3.8}$$

holds for all $x>0$.

Proof. Using the recursive formula $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$ and the definition (1.5) yields $\lim_{x\to 0}F_1(x;q)=1$. From the increasing monotone of the function $F_1(x;q)$, the inequality (3.8) is obtained easily. □

COROLLARY 3.6. *The function*

$$G_\alpha(x;q)=\log[x]_q-\psi_q(x)+\frac{\alpha q^x\log q}{1-q^x} \tag{3.9}$$

is completely monotonic on $(0,\infty)$ for $0<q<1$ if and only if $\alpha\leqslant g(q)$ and for $q\geqslant 1$ if and only if $\alpha\leqslant\frac{1}{2}$ where $g(q)$ defined as in (3.1); and the function $-G'_\alpha(x;q)$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\leqslant g(\hat{q})$ for $q>0$ where $\hat{q}=q$ if $0<q<1$ and $\hat{q}=q^{-1}$ if $q\geqslant 1$. Also, the function

$$H_\beta(x;q)=\psi_q(x)-\log[x]_q-\frac{\beta q^x\log q}{1-q^x} \tag{3.10}$$

is completely monotonic on $(0,\infty)$ for $q>0$ if and only if $\beta\geqslant 1$.

Proof. The proof comes immediately from the proof of Theorem 3.2 and noting that $G'_\alpha(x;q^{-1})=G'_\alpha(x;q)$ for all $q\geqslant 1$. □

COROLLARY 3.7. *Suppose that x is positive real number. Then the two-sided inequalities*

$$\log[x]_q + \frac{\beta q^x \log q}{1-q^x} < \psi_q(x) < \log[x]_q + \frac{\alpha q^x \log q}{1-q^x} \tag{3.11}$$

hold true for all $\beta \geqslant 1$ and $\begin{cases} \alpha < g(q) & \text{if} \quad 0<q<1, \\ \alpha < \frac{1}{2} & \text{if} \quad q>1 \end{cases}$ with the best possible constants $\beta = 1$ and $\begin{cases} \alpha = g(q) & \text{if} \quad 0<q<1, \\ \alpha = \frac{1}{2} & \text{if} \quad q>1 \end{cases}$ where $g(q)$ is defined as in (3.1). Moreover, the two-sided inequalities

$$\begin{aligned} &(-1)^{r+1}\left(\frac{\log q}{1-q^x}\right)^r q^x P_{r-2}(q^x) + (-1)^r \beta \left(\frac{\log q}{1-q^x}\right)^{r+1} q^x P_{r-1}(q^x) \\ &< (-1)^r \psi_q^{(r)}(x) \\ &< (-1)^{r+1}\left(\frac{\log q}{1-q^x}\right)^r q^x P_{r-2}(q^x) + (-1)^r \alpha \left(\frac{\log q}{1-q^x}\right)^{r+1} q^x P_{r-1}(q^x) \end{aligned} \tag{3.12}$$

hold for all $\beta \geqslant 1$, $\alpha \leqslant g(\hat{q})$ and $r \in \mathbb{N}$ for all $q > 0$ with the best possible constants $\beta = 1$ and $\alpha = g(\hat{q})$ for for $q > 0$.

Proof. From the complete monotonicity of the functions in Corollary 3.6, we get $-H_1(x;q) < 0 < G_{g(q)}(x;q)$ if $0<q<1$ and $-H_1(x;q) < 0 < G_{\frac{1}{2}}(x;q)$ if $q \geqslant 1$ which are equivalent to (3.11). Also, we have

$$(-1)^{r+1} G_{g(\hat{q})}^{(r)}(x;q) < 0 < (-1)^r H_1(x;q)^{(r)}, \qquad r \in \mathbb{N}$$

which is equivalent to (3.12) with using the relation

$$\frac{d^n}{dx^n}\left[\frac{q^x \log q}{1-q^x}\right] = \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x), \qquad n \in \mathbb{N}$$

which was proved by Moak ([16] Theorem 1), where P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2) P'_{k-1}(z) + (kz+1) P_{k-1}(z), \qquad P_0(z) = 1, \quad P_k(1) = (k+1)!$$

for all $k \in \mathbb{N}$. $\square$

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