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B–STATISTICAL A–SUMMABILITY IN CONSERVATIVE APPROXIMATION

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Abstract. This paper deals with the approximation of functions by sequences of linear operators via B-statistical A-summability. Quantitative results and asymptotic formulae are stated under a conservative approximation setting. A short discussion is addressed in connection with the rate of statistical convergence. Finally, the applicability of the results is illustrated.

1. Introduction

In the general setting of the approximation of functions by sequences of linear operators, Korovkin-type theory merits special attention, primarily because of its connections with other areas of mathematics (see [1]). This theory originated in the fifties with the classical results of Popoviciu, Bohmann and Korovkin, and since then, it has been extended in many directions. One of them refers to the study of convergence methods that are stronger than the classical notion. Here one finds B-statistical A-summability, that has shown to be quite effective in summing sequences which are not convergent in the classical sense. In this paper, we shall deal with this method that has received a lot of recent attention in different branches of mathematics (such as approximation theory, stochastic processes or fuzzy logic) and has found applications in engineering, artificial intelligence and computational mathematics among others. We begin by recalling its definition.

Let $A = (a_{ij})$, $i, j = 1,2,\dots$ be an infinite summability matrix. For a given sequence of real numbers $z = (z_j)$, the A-transform of z is another sequence, denoted by Az or $A(z_j)$, whose elements are defined by

$$(Az)_i := \sum_{j=1}^{\infty} a_{ij} z_j,$$

provided the series converges for each i. A is said to be regular if Az converges to ℓ (as $i \to +\infty$) whenever z converges to ℓ (as $j \to +\infty$). Obviously, the infinite identity matrix I is regular, since $Iz = z$. As usual, if we use certain letter to represent a sequence, then its elements or terms are denoted by the same letter followed by a subindex.

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Let $B=(b_{nk})$, $n,k=1,2,\dots$ be a non-negative regular summability matrix. For a given set of natural numbers $K\subset\mathbb{N}$, the B-density of K, denoted by $\delta_B(K)$, is defined by

$$\delta_B(K):=\lim_{n\to\infty}\sum_{k\in K}b_{nk}=\lim_{n\to\infty}(B\chi_K)_n=\lim_{n\to\infty}\sum_{k=1}^{\infty}b_{nk}(\chi_K)_k,$$

provided the limit exists, where χ_K denotes the characteristic sequence of K, i.e. $(\chi_K)_k=1$ if $k\in K$ and $(\chi_K)_k=0$ otherwise. Obviously, the regularity of B implies that $\delta_B(\mathbb{N})=1$. This definition was introduced by Freedman and Sember [13] as a generalization of the natural density which appears when B is defined by $b_{nk}=1/n$ if $n\geqslant k$ and $b_{nk}=0$ otherwise, that is to say, when B coincides with $(C,1)$, the Cesàro matrix of order 1. In this particular setting, $\delta_B(K)$ is simply denoted by $\delta(K)$ and obeys the following expression:

$$\delta(K)=\lim_{n\to\infty}\frac{\#K_n}{n},$$

where $K_n=\{k\in K: k\leqslant n\}$ and $\#K_n$ denotes its cardinality.

DEFINITION 1. Let $A=(a_{ij})$ and $B=(b_{nk})$ be two non-negative regular summability matrices. A sequence $z=(z_j)$ is said to be B-statistically A-summable to ℓ, denoted by $\ell=st_B-\lim Az$, if for every $\varepsilon>0$, the B-density of $K_\varepsilon:=\{i:|(Az)_i-\ell|\geqslant\varepsilon\}$ is zero, i.e.

$$\delta_B(K_\varepsilon)=\lim_{n\to\infty}\sum_{k\in K_\varepsilon}b_{nk}=\lim_{n\to\infty}(B\chi_{K_\varepsilon})_n=\lim_{n\to\infty}\sum_{k=1}^{\infty}b_{nk}(\chi_{K_\varepsilon})_k=0.$$

The concept of B-statistical A-summability has been recently presented in [11] as a generalization of a long list of variants and extensions of the classical statistical convergence introduced by Fast in 1951 [12], which, by the way, appears after taking in the definition above $A=I$ and $B=(C,1)$. It is worth mentioning some other outstanding particular cases of this concept: if $A=B=I$, it appears the classical convergence; if only $A=I$, it reduces to B-statistical convergence; if only $B=(C,1)$, then it becomes statistical A-summability; finally, if only $B=I$, the ordinary matrix summability is recovered. It must be remarked that many other particular choices of A and B give rise to well established settings, and that there is an extensive literature dealing with this matter. We suggest the novel interested reader to consult firstly the recent easy to get papers [11, 22] (which study this matter in approximation theory) and then the references therein.

As regards notation, in the definition above, st_B takes the simpler form st when $B=(C,1)$, and it is merely deleted if $B=I$. Obviously, $\lim Az$ is written as $\lim z$ when $A=I$.

Once introduced the notion of convergence that we are going to consider in this work, we shall roughly describe our subject of interest within approximation theory. We are concerned with Korovkin-type results that deal with the approximation of functions of the form $\mathscr{D}f$, $\mathscr{D}$ being certain differential operator to be defined, by using sequences of functions built from $\mathscr{D}$ and from L_jf, where $L=(L_j)$ is a sequence of linear operators. Here $\mathscr{D}f$ and L_jf denote the images of f by $\mathscr{D}$ and L_j respectively.

The idea of examining different notions of generalized convergence in approximation theory turns to be a pattern that has been followed by a long list of researchers for many years. One of the archetypes within Korovkin-type theory is represented by the results of King and Swetits [16], who proved qualitative results via the almost convergence introduced by Lorentz in 1948 [17]. We notice, however, that the almost convergence is not a particular case of B-statistical A-summability. As far as the notion of the classical statistical convergence is concerned, Gadjiev and Orhan in 2002 [14] were the first who examined it in approximation theory. Specifically, they proved some basic Korovkin-type results related to the approximation of functions by sequences of linear positive operators. In this line and very recently, Mursaleen and Kiliçman [22] have done some work under the more general notion of B-statistical A-summability. Chronologically, between these two last aforementioned papers, one finds in the literature many others about this subject through an important number of variants and generalizations of the classical statistical convergence. For example, statistical A-summability and the particular case of statistical $(C,1)$-summability have been studied in [7] and [18] respectively, and B-statistical convergence has been investigated in [2, 8, 9, 10]. Further results appear in [19, 20, 21].

To be more specific with our aim, in this paper, under the assumption that the operators L_j fulfill certain shape preserving property related to $\mathscr{D}$, we first prove a quantitative result about the pointwise B-statistical A-summability of the sequence of functions $(\mathscr{D}L_j f)$ towards the function $\mathscr{D}f$, and then we analyze the rate of approximation by stating an asymptotic formula of Voronovskaya type. The inequality that yields our quantitative result is more general than the one stated in [22], and more importantly, it is also more appropriate, as it does allow to recover properly the corresponding qualitative version. As regards asymptotic expressions under statistical convergence, up to our knowledge, only the particular case of the well-known Szász-Mirakjan-Kantorovich operators has been studied in [2], whereas the result we are stating here is general. In this respect we refer the reader to the recent related papers [3, 23], that deal with different notions of statistical convergence.

We also pursue a further aim with this work, namely, to bring a sort of unification to the notation, the results and their tools in some of the aforementioned papers that deal with this subject. In this respect, a short discussion is addressed in connection with the appropriate notion of rate of B-statistical A-summability to be considered.

To finish this introduction and just to fix ideas about the aforesaid conservative setting, we suggest the reader to assume that $\mathscr{D}$ is the classical differential operator of order m, namely $\mathscr{D} = D^m$; this takes us to the framework of the so-called simultaneous approximation, where it is natural to assume that the operators are m-convex, i.e. they map m-convex functions onto m-convex functions (recall that a m-times continuously differentiable function is said to be m-convex if its mth derivative is non-negative). In this case, the interest would be focused on the pointwise B-statistical A-summability of the sequence of functions $(D^m L_j f)$ towards the function $D^m f$.

2. General setting, specific objectives and preliminaries

Let $J=[0,1]\subset\mathbb{R}$, $J^\circ=(0,1)$ and let $i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We denote by $C^i(J)$ the space of all real valued i-times continuously differentiable functions defined on J and, as pointed out in the introduction, we denote by D^i the classical ith differential operator. Obviously, $C^0(J)=C(J)$ is the space of all continuous functions on J, $D^0=\mathbb{I}$ is the identity operator and $C^\infty(J)=\cap_{i\in\mathbb{N}}C^i(J)$. For low order derivatives of a function f we keep on using the classical notation f', f'', ... As usual, for $f\in C(J)$, $\omega(f,\xi)$ denotes its classical modulus of continuity with argument ξ.

Let $\tau\in C^\infty(J)$ such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(t)>0$ for $t\in(0,1)$. As a generalization of the usual notation for the monomials $e_i(t)=t^i$ and $e_i^x(t)=(t-x)^i$, we denote by $e_{\tau,i}$ and $e_{\tau,i}^x$ the functions

$$e_{\tau,i}(t)=\tau(t)^i,\quad e_{\tau,i}^x(t)=(\tau(t)-\tau(x))^i.$$

Now, we detail the differential operator referred as $\mathscr{D}$ in the previous section by considering D_τ^i, already used by the authors in [5] and defined by

$$D_\tau^i f(t):=D^i\left(f\circ\tau^{-1}\right)(\tau(t)). \tag{1}$$

Obviously, $D_\tau^0=\mathbb{I}$, and if $\tau=e_1$, then $D_\tau^i=D^i$. Moreover, we notice that the operator D_τ^i coincides with the ith iterate of the operator $\frac{1}{\tau'}D^1$, denoted by $D^{i,\tau'}$ and defined recursively as follows:

$$D^{0,\tau'}=\mathbb{I},\quad D^{1,\tau'}=\frac{1}{\tau'}D^1,\quad D^{i+1,\tau'}=D^{1,\tau'}\circ D^{i,\tau'},\ i\in\mathbb{N}.$$

Besides, it can be easily checked that for $x\in J$,

$$D_\tau^i e_{\tau,j}^x=\begin{cases}\frac{j!}{(j-i)!}e_{\tau,j-i}^x, & \text{if } j\geqslant i;\\ 0, & \text{if } j<i.\end{cases} \tag{2}$$

Finally, let $m\in\mathbb{N}$, let $A=(a_{ij})$ and $B=(b_{nk})$ be two non-negative regular summability matrices, and let us consider a sequence of linear operators $L=(L_j)$, $L_j:C^m(J)\longrightarrow C^m(J)$.

Then, given $f\in C^m(J)$ and $x\in J$, we are interested in the B-statistical A-summability of the sequence $(D_\tau^m L_j f(x))$ towards $D_\tau^m f(x)$. Equivalently, if we use the notation

$$\mathscr{A}_{L,i}^{\tau,m}f(x):=\sum_{j=1}^{\infty}a_{ij}D_\tau^m L_j f(x),$$

our interest is focused on the B-statistical convergence of $(\mathscr{A}_{L,i}^{\tau,m}f(x))$ towards $D_\tau^m f(x)$.

With that purpose, in Section 3, with the use of the modulus of continuity of the function $D^m f\circ\tau^{-1}$, we first estimate the quantity

$$|\mathscr{A}_{L,i}^{\tau,m}f(x)-D_\tau^m f(x)|,\quad x\in J.$$

Then we study the rate of B-statistical A-summability. Section 4 deals with an asymptotic formula for this approximation process, while the last section contains some applications. Now, we end this one by stating a lemma that we shall use in the rest of the paper. We omit its proof as it can be derived directly from the definitions above and follow standard patterns.

LEMMA 1. *Let $\alpha = (\alpha_j)$, $\beta = (\beta_j)$ and $\gamma = (\gamma_j)$ be three sequences of real numbers. Assume that there exists $K \subset \mathbb{N}$ with $\delta_B(K) = 1$ such that*

$$\alpha_j \leqslant \beta_j \leqslant \gamma_j, \quad \forall j \in K.$$

If $st_B - \lim\ \alpha = st_B - \lim\ \gamma = \ell$, then $st_B - \lim\ \beta = \ell$.

3. A quantitative Korovkin-type result

We assume the conditions of the previous section.

THEOREM 1. *Let us suppose that the following (shape preserving) property is satisfied:*

$$D_\tau^m f(t) \geqslant 0,\ \forall t \in J \ \Rightarrow\ \mathscr{A}_{L,i}^{\tau,m} f(t) \geqslant 0,\ \forall t \in J,\ i = 1,2,\ldots. \tag{3}$$

Then, for $f \in C^m(J)$ and $x \in J$,

$$\begin{aligned}\left|\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x)\right| \leqslant{}& \frac{|D_\tau^m f(x)|}{m!} \cdot \left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)\right| \\ &+ \frac{1}{m!}\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x) + D_\tau^m e_{\tau,m}(x)\right| \omega\left(D_\tau^m f \circ \tau^{-1}, \eta_{i,\tau,m}(x)\right),\end{aligned}$$

where

$$\eta_{i,\tau,m}^2(x) = \frac{2}{(m+2)!}\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x)\right|.$$

Proof. Let $f \in C^m(J)$, $x \in J$ and $\xi > 0$. Then, for $t \in J$ such that $|\tau(t) - \tau(x)| > \xi$,

$$\begin{aligned}\left|(D_\tau^m f \circ \tau^{-1})(\tau(t)) - (D_\tau^m f \circ \tau^{-1})(\tau(x))\right| &\leqslant \omega\left(D_\tau^m f \circ \tau^{-1}, |\tau(t) - \tau(x)|\right) \\ &\leqslant \left(1 + \frac{|\tau(t) - \tau(x)|}{\xi}\right) \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right) \\ &\leqslant \left(1 + \frac{(\tau(t) - \tau(x))^2}{\xi^2}\right) \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right).\end{aligned}$$

Obviously, if $|\tau(t) - \tau(x)| \leqslant \xi$, then

$$\left|(D_\tau^m f \circ \tau^{-1})(\tau(t)) - (D_\tau^m f \circ \tau^{-1})(\tau(x))\right| \leqslant \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right).$$

Hence, for all $t \in J$,

$$\left|(D_\tau^m f \circ \tau^{-1})(\tau(t)) - (D_\tau^m f \circ \tau^{-1})(\tau(x))\right| \leqslant \left(1 + \frac{(\tau(t) - \tau(x))^2}{\xi^2}\right) \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right),$$

which, by using (2), amounts to following functional inequalities:

$$\begin{aligned}&-\omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right) D_\tau^m \left(\frac{e_{\tau,m}}{m!} + \frac{2e_{\tau,m+2}^x}{(m+2)! \cdot \xi^2}\right) \\ &\leqslant D_\tau^m\left(f - D_\tau^m f(x) \frac{e_{\tau,m}}{m!}\right) \leqslant \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right) D_\tau^m \left(\frac{e_{\tau,m}}{m!} + \frac{2e_{\tau,m+2}^x}{(m+2)! \cdot \xi^2}\right).\end{aligned}$$

Directly from hypothesis (3), using linearity arguments and evaluating at x, we obtain

$$\left|\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x) \frac{\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x)}{m!}\right| \leqslant \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right)\left(\frac{\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x)}{m!} + \frac{2\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x)}{(m+2)!\cdot\xi^2}\right).$$

Then, the triangular inequality allows to write

$$\begin{aligned}\left|\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x)\right| &\leqslant \left|\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x) \frac{\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x)}{m!}\right| \\ &\quad + \left|D_\tau^m f(x) \frac{\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x)}{m!} - D_\tau^m f(x) \frac{D_\tau^m e_{\tau,m}(x)}{m!}\right| \\ &\leqslant \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right)\left(\frac{\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x)}{m!} + \frac{2\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x)}{(m+2)!\cdot\xi^2}\right) \\ &\quad + \frac{|D_\tau^m f(x)|}{m!}\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)\right|,\end{aligned}$$

from where the proof is over after taking $\xi = \eta_{i,\tau,m}(x)$. □

4. Rate of B-statistical A-summability

Directly from Theorem 1, the corresponding qualitative version is easily derived. We detail it in the following result.

COROLLARY 1. *Assume the conditions of Theorem 1. If $(D_\tau^m L_j e_{s,m}(x))$ is B-statistically A-summable to $D_\tau^m e_{s,m}(x)$ for $s = m, m+1, m+2$, then $(D_\tau^m L_j f(x))$ is B-statistically A-summable to $D_\tau^m f(x)$.*

Proof. Firstly, we notice that the linearity of the operators L_j and equation (2), together with the shape preserving assumption (3), yield that

$$\mathscr{A}_{L,i}^{\tau,m} e_{\tau,s}(x) = 0, \quad s = 0, 1, \ldots, m-1.$$

As a consequence, the equality

$$e_{\tau,m+2}^x(t) = \sum_{r=0}^{m+2} \binom{m+2}{r} (-1)^r e_{\tau,m+2-r}(t) e_{\tau,r}(x),$$

and the triangular inequality, allows us to write,

$$\begin{aligned}\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x)\right| &\leqslant \left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}(x) - D_\tau^m e_{\tau,m+2}(x)\right| \\ &\quad + (m+2)\tau(x)\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+1}(x) - D_\tau^m e_{\tau,m+1}(x)\right| \\ &\quad + \frac{(m+1)(m+2)}{2}\tau(x)^2\left|\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)\right|.\end{aligned} \tag{4}$$

On the other hand, Fast [12] pointed out that a sequence of real numbers η is statistically convergent to 0 if and only if there exists a sequence of natural numbers σ_n such that $\delta(\{\sigma_n : n \in \mathbb{N}\}) = 1$ and $\eta_{\sigma_n} \to 0$ in the classical sense. With the obvious modifications, this statement holds true if statistic convergence is extended to B-statistic convergence, and directly from it, we deduce that $st_B - \lim\ \omega(D_\tau^m f \circ \tau^{-1}, \eta) = 0$ whenever $st_B - \lim\ \eta = 0$. Obviously, the uniform continuity of $D_\tau^m f \circ \tau^{-1}$ must be taken into account.

Finally, we apply this last statement to $\eta = \eta_{i,\tau,m}(x)$ and finish the proof by using (4), the thesis of Theorem 1 and Lemma 1. □

Now we pay attention to the rate of B-statistical A-summability of $(D_\tau^m L_j f(x))$ towards $D_\tau^m f(x)$, $x \in J$, or equivalently, to the rate of B-statistical convergence of $(\mathscr{A}_{L,i}^{\tau,m} f(x))$ towards $D_\tau^m f(x)$, $x \in J$.

First of all, a definition is required, and a good reference to be consulted in this respect is the paper [9] by Duman, Khan and Orhan.

We follow it to consider below two notions of rate of B-statistical A-summability with respect to a positive non-increasing sequence α.

DEFINITION 2. A sequence z is B-statistically A-summable to the number ℓ with the rate of $o(\alpha)$, denoted by

$$Az - \ell = st_B - o(\alpha),$$

if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\alpha_n} \sum_{k \in K_\varepsilon} b_{nk} = 0,$$

where $K_\varepsilon = \{i : |(Az)_i - \ell| \geqslant \varepsilon\}$ is given in Definition 1.

DEFINITION 3. A sequence z is B-statistically A-summable to the number ℓ with the rate of $o_\mu(\alpha)$, denoted by

$$Az - \ell = st_B - o_\mu(\alpha),$$

if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k \in K_{\varepsilon,\alpha}} b_{nk} = 0,$$

where $K_{\varepsilon,\alpha} := \{i : |(Az)_i - \ell| \geqslant \varepsilon \alpha_i\}$.

Notice the difference between $o(\alpha)$ in Definition 2 and $o_\mu(\alpha)$ in Definition 3. The reason why we have used the letter μ, is that it usually denotes a measure in measure theory and, as it was pointed out in [9], Definition 3 comes from the concept of convergence in measure.

Definition 2 has been recently used in [22]. It has also been used in [7] under the particular setting of statistical A-summability. Nevertheless, it is worth mentioning that in the specific case of B-statistical convergence, both previous definitions had been considered some years earlier in [9] and, although the authors of this last paper achieved analogous results in approximation theory using both definitions, the comparison stated there showed that Definition 3 was more suitable that Definition 2 because, in this last case, there is more dependence on the entries of B than on the sequence itself. Some further arguments were wielded in [9] to endorse this choice. Here we add the following two simple comments: firstly, notice that if a sequence is A-summable to ℓ, then it is statistically A-summable to ℓ with the rate $o((n^{-q}))$ for every $q>0$; secondly, the following particular sequences z and y are not convergent but statistically convergent, and both of them have the same rate of statistical convergence in the sense of Definition 2:

$$z_j = \begin{cases} j, & \text{if } \sqrt{j}\in\mathbb{N}; \\ 1/j, & \text{otherwise.} \end{cases} \qquad y_j = \begin{cases} j, & \text{if } \sqrt{j}\in\mathbb{N}; \\ 1/j^2, & \text{otherwise.} \end{cases}$$

The following corollary, whose proof we omit, comes directly from Lemma 1, [22, Lemma 5] and, obviously, from Theorem 1.

COROLLARY 2. *Let α, β be two positive non-increasing sequences. Let $f\in C^m(J)$ and $x\in J$ and let us assume that*

$$A\left(D_\tau^m L_j e_{\tau,m}(x)\right) - D_\tau^m e_{\tau,m}(x) = st_B - o_\mu(\alpha)$$

and

$$\left(\omega(D^m f\circ\tau^{-1}, \eta_{i,\tau,m}(x))\right) = st_B - o_\mu(\beta).$$

Then

$$A\left(D_\tau^m L_j f(x)\right) - D_\tau^m f(x) = st_B - o_\mu(\gamma),$$

where $\gamma=(\gamma_n)$ is given by $\gamma_n=\max\{\alpha_i,\beta_i\}$.

It would be quite interesting to study the relationship, if any, between the rate of B-statistical A-summability towards zero of the argument of the modulus of continuity, and the smoothness of the function $D^m f\circ\tau^{-1}$. Up to our knowledge, this matter has not been explored yet.

5. A result on asymptotic formulae

Here we study the optimal rate of B-statistical A-summability of $(D_\tau^m L_j f(x))$ towards $D_\tau^m f(x)$. As it is usual in approximation theory by linear operators, this is carried out with the aid of an asymptotic formula. We shall only make use of Definition 3 because it is the most appropriate notion to compare rates of B-statistical A-summability. In this respect, and to make it easier to understand the result that follows, it is important to point out that a sequence z is B-statistically convergente to 0 with the rate $o_\mu(\alpha)$ if and only if the sequence (z_j/α_j) is B-statistically convergente to 0, i. e.

$$z = st_B - o_\mu(\alpha) \Leftrightarrow st_B - \lim\,(z_j/\alpha_j) = 0.$$

THEOREM 2. *Under the general setting of Section 2, let us suppose that the operators L_j satisfy (3), let $x\in J^\circ$ and let us assume that there exist a sequence of real* *positive numbers* $\lambda_i \to +\infty$ *and three strictly positive functions* w_0, w_1 *and* w_2 *defined on* J° *with* $w_i \in C^{2-i}(J^\circ)$ *such that, for* $s \in \{m, m+1, m+2, m+4\}$,

$$st_B - \lim \left(\lambda_i \left(\mathscr{A}_{L,i}^{\tau,m} e_{\tau,s}^x(x) - D_\tau^m e_{\tau,s}^x(x)\right)\right) = w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D_\tau^m e_{\tau,s}^x))(x). \quad (5)$$

Then, for $f \in C^m(J)$, $m+2$ *times differentiable in some neighborhood of* x,

$$st_B - \lim \left(\lambda_i \left(\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x)\right)\right) = w_2^{-1} D^1 \left(w_1^{-1} D^1 \left(w_0^{-1} D_\tau^m f\right)\right)(x).$$

Proof. We follow a classical pattern based on the Taylor's expansion of the function $D_\tau^m f \circ \tau^{-1}$ at the point $\tau(x)$. This gives the following expression, after using the equality $D_\tau^j D_\tau^m f(t) = D^j \left(D_\tau^m f \circ \tau^{-1}\right)(\tau(t))$ and evaluating at $\tau(t)$ with $t \in J$:

$$\begin{aligned} D_\tau^m f(t) &= D_\tau^0 (D_\tau^m f)(x) e_{\tau,0}^x(t) + D_\tau^1 (D_\tau^m f)(x) e_{\tau,1}^x(t) \\ &\quad + \frac{1}{2} D_\tau^2 (D_\tau^m f)(x) e_{\tau,2}^x(t) + h(\tau(t) - \tau(x)) e_{\tau,2}^x(t), \end{aligned}$$

where h is a continuous function that vanishes at zero.

Equivalently, using (2), we can write, for $t \in J$,

$$D_\tau^m f(t) = D_\tau^m \left(\sum_{s=0}^{2} \frac{1}{(m+s)!} D_\tau^s (D_\tau^m f)(x) e_{\tau,m+s}^x + H_x\right)(t)$$

with $H_x \in C^m(J)$ and $D_\tau^m H_x(t) = h(\tau(t) - \tau(x)) e_{\tau,2}^x(t)$.

Linearity arguments and (3) allow to write, after evaluating at x,

$$\mathscr{A}_{L,i}^{\tau,m} f(x) = \sum_{s=0}^{2} \frac{D_\tau^s (D_\tau^m f)(x)}{(m+s)!} \mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+s}^x(x) + \mathscr{A}_{L,i}^{\tau,m} H_x(x).$$

Now, we introduce the term

$$D_\tau^m f(x) = \sum_{s=0}^{2} \frac{1}{(m+s)!} D_\tau^s (D_\tau^m f)(x) D_\tau^m e_{\tau,m+s}^x(x),$$

and multiply by λ_i to obtain

$$\begin{aligned} &\lambda_i \left(\mathscr{A}_{L,i}^{\tau,m} f(x) - D_\tau^m f(x)\right) \\ &= \lambda_i \mathscr{A}_{L,i}^{\tau,m} H_x(x) + \sum_{s=0}^{2} \frac{D_\tau^s (D_\tau^m f)(x)}{(m+s)!} \lambda_i \left(\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+s}^x(x) - D_\tau^m e_{\tau,m+s}^x(x)\right). \end{aligned} \quad (6)$$

After some cumbersome but easy calculations, the hypothesis of the theorem for $s = m, m+1, m+2$ yield that

$$\begin{aligned} st_B - \lim &\left(\sum_{s=0}^{2} \frac{D_\tau^s (D_\tau^m f)(x)}{(m+s)!} \lambda_i \left(\mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+s}^x(x) - D_\tau^m e_{\tau,m+s}^x(x)\right)\right) \\ &= w_2^{-1} D^1 \left(w_1^{-1} D^1 \left(w_0^{-1} D_\tau^m f\right)\right)(x), \end{aligned}$$

and consequently, if we take into account (6), we end this proof if we check that

$$st_B - \lim \left(\lambda_i \mathscr{A}_{L,i}^{\tau,m} H_x(x)\right) = 0. \quad (7)$$

To do this, we use continuity arguments on the function h to guarantee the existence of a neighborhood of x, say θ_x, for a given $\varepsilon > 0$, such that for $t \in \theta_x$,

$$|h(\tau(t) - \tau(x))| < \varepsilon.$$

Then, for all $t \in J$,

$$|D_\tau^m H_x(t)| = |h(\tau(t) - \tau(x))| e_{\tau,2}^x(t) \leqslant \varepsilon e_{\tau,2}^x(t) + \max\{0, |h(\tau(t) - \tau(x))| - \varepsilon\} e_{\tau,2}^x(t).$$

Let us consider a function $W \in C^m(J)$ such that $D_\tau^m W(t) = \max\{0, |h(\tau(t) - \tau(x))| - \varepsilon\} e_{\tau,2}^x(t)$. As $D_\tau^m W$ vanishes in θ_x, then, for a sufficiently large constant M, one has $|D_\tau^m W(t)| \leqslant M D_\tau^m e_{\tau,m+4}^x(t)$. Thus, gathering the last inequalities we obtain that

$$|D_\tau^m H_x(t)| \leqslant \frac{2\varepsilon}{(m+2)!} D_\tau^m e_{\tau,m+2}^x(t) + M D_\tau^m e_{\tau,m+4}^x(t).$$

By using (3) and the fact that $\lambda_i > 0$, we obtain, after evaluating at the point x,

$$\left| \lambda_i \mathscr{A}_{L,i}^{\tau,m} H_x(x) \right| \leqslant \frac{2\varepsilon}{(m+2)!} \lambda_i \mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x) + M \lambda_i \mathscr{A}_{L,i}^{\tau,m} M e_{\tau,m+4}^x(x).$$

As regards the hypothesis of the result for $s = m+2$ and $s = m+4$, after some calculations using (2), we can write respectively

$$st_B - \lim \left(\lambda_i \mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+2}^x(x) \right) = \frac{2\tau'(x)^2}{w_2(x) w_1(x) w_0(x)} > 0$$

and

$$st_B - \lim \left(\lambda_i \mathscr{A}_{L,i}^{\tau,m} e_{\tau,m+4}^x(x) \right) = 0.$$

Finally, Lemma 1 and the fact that $\varepsilon > 0$ was arbitrary allow to finish the proof. □

6. Applications

Firstly, we mention that Theorem 2 extends previous results stated in [4], [15] and [6] under different settings. Then, in this section, we apply Theorem 2 to a particular sequence of operators that cannot be studied with any of the previous results we have just made reference to.

Let $A = (a_{ij})$ and $B = (b_{nk})$ be the regular matrixes defined by

$$a_{ij} = \begin{cases} 1, & \text{if } j = 2i; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$b_{nk} = \begin{cases} 0, & \text{if } k > n \vee k/4 \in \mathbb{N}; \\ 2/n, & \text{if } k < n \wedge (k+1)/4 \in \mathbb{N}; \\ 1/n, & \text{otherwise.} \end{cases}$$

Let us also consider the sequence $h = (h_j)$ defined as

$$h_j = \begin{cases} j, & \text{if } (j+1)/2 \in \mathbb{N} \vee j/8 \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that the sequence h does not converge to 0 in the classical sense and it is not A-summable to 0. Moreover, it is not B-statistically convergent to 0 either, because if this were so, then for each $\varepsilon > 0$ (sufficiently small)

$$\delta_B(\{j : |h_j| \geqslant \varepsilon\}) = \delta_B(\{j : h_j \neq 0\}) = 0,$$

or equivalently $\delta_B(K^\varepsilon) = 1$, where

$$K^\varepsilon = \{j : |h_j| \leqslant \varepsilon\} = \{2,4,6,10,12,14,18,20,22,26,\ldots\}.$$

But this is not true since for each $n \in \mathbb{N}$

$$\sum_{k=1}^{\infty} b_{nk}\,(\chi_{K^\varepsilon})_k \leqslant \frac{1}{2}.$$

On the other hand, it takes just a while to check that the sequence h is B-statistically A-summable to 0.

Now we fix τ in the same conditions as in the paper and consider the sequence of operators

$$\mathscr{B}^\tau = (\mathscr{B}_j^\tau) := ((1+h_j)B_j^\tau),$$

where (B_j^τ) is the sequence studied in [5] and defined for $f \in C(J)$ by

$$B_j^\tau f(t) = \sum_{\nu=1}^{j} \binom{j}{\nu} \tau(t)^\nu (1-\tau(t))^{1-\nu} (f \circ \tau^{-1})(\nu/j).$$

Given $f \in C^3(J)$ and $x \in J^\circ$, our interest is focused on the B-statistical A-summability of the sequence $\left(D_\tau^3 \mathscr{B}_j^\tau f(x)\right)$ towards $D_\tau^3 f(x)$ or, equivalently, on the B-statistical convergence of $\left(\mathscr{A}_{\mathscr{B}^\tau,i}^{\tau,3} f(x)\right)$ towards $D_\tau^3 f(x)$.

The choice of the third order in the differential operator and of the operators B_j^τ obeys to illustrative purposes and intend to give a general idea of the applicability of the results of the paper.

COROLLARY 3. *Under the previous conditions, assume that f is 5-times differentiable in a neighborhood of x. Then*

$$st_B - \lim \left(2i\left(\mathscr{A}_{\mathscr{B}^\tau,i}^{\tau,3} f(x) - D_\tau^3 f(x)\right)\right) = D_\tau^3\left(\frac{\tau(1-\tau)}{2} D_\tau^2 f\right)(x).$$

Proof. We shall apply Theorem 2 with $L_j = \mathscr{B}_j^\tau$, $m = 3$, $\lambda_i = 2i$ and

$$w_0 = \frac{1}{\tau^2}, \;\; w_1 = \frac{2\tau\tau'}{(1-\tau)^3}, \;\; w_2 = (1-\tau)^2\tau'.$$

Now we show that all the hypotheses are satisfied. Firstly, it was observed in [5] that the operators B_j^τ are positive and τ-convex of order k for every $k \in \mathbb{N}$, i.e. $D_\tau^k f \geqslant 0$ implies that $D_\tau^k B_j^\tau f \geqslant 0$. Obviously, this is also the case for $\mathscr{B}_j^\tau$. As a consequence, if $D_\tau^3 f(t) \geqslant 0$ for all $t \in J$ one deduces that for $i = 1,2,\ldots$

$$\mathscr{A}_{\mathscr{B}^\tau,i}^{\tau,3} f(t) = \sum_{j=1}^{\infty} a_{ij} D_\tau^3 \mathscr{B}_j^\tau f(t) = \sum_{j=1}^{\infty} a_{ij}(1+h_j) D_\tau^3 B_j^\tau f(t) = (1+h_{2i}) D_\tau^3 B_{2i}^\tau f(t) \geqslant 0.$$

Hence, hypothesis (3) is fulfill.

As regards hypothesis (5) we first notice the following equality, that can easily checked with the use of some mathematical software:

$$D_\tau^3\left(\frac{\tau(1-\tau)}{2}D_\tau^2 f\right)=w_2^{-1}D^1(w_1^{-1}D^1(w_0^{-1}D_\tau^3 f)).$$

If f is replaced by $e_{\tau,s}^x$ for $s=3,4,5,7$, we have the following expressions for the right-hand side of (5):

$$D_\tau^3\left(\frac{\tau(1-\tau)}{2}D_\tau^2 e_{\tau,s}^x\right)=\begin{cases}-18, & s=3;\\ 36(1-2\tau), & s=4;\\ 60\tau(1-\tau), & s=5;\\ 0, & s=7.\end{cases}\tag{8}$$

Finally, we have to compute the left-hand side of (5) for $s=3,4,5,7$, and check that the results coincide with the previous expressions (8).

Notice that

$$2i\left(\mathscr{A}_{\mathscr{B}^\tau,i}^{\tau,3}e_{\tau,s}^x(x)-D_\tau^3 e_{\tau,s}^x(x)\right)=2i\left(D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x)-D_\tau^3 e_{\tau,s}^x(x)\right)+2i(h_{2i})D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x),$$

and as

$$2i(h_{2i})D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x)=\begin{cases}(2i)^2\cdot D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x), & \text{if } \frac{i}{4}\in\mathbb{N};\\ 0, & \text{otherwise,}\end{cases}$$

then $2i(h_{2i})D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x)$ is B-statistically convergent to 0, and consequently we can rewrite the left-hand side of (5) as follows:

$$st_B-\lim\left(2i\left(\mathscr{A}_{\mathscr{B}^\tau,i}^{\tau,3}e_{\tau,s}^x(x)-D_\tau^3 e_{\tau,s}^x(x)\right)\right)=st_B-\lim\left(2i\left(D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x)-D_\tau^3 e_{\tau,s}^x(x)\right)\right).$$

In order to calculate $D_\tau^3B_{2i}^\tau e_{\tau,s}^x(x)$ for $s=3,4,5,7$, we can consult [4], where the authors obtained expressions for $D^3B_je_s^x(x)$, B_j being the classical Bernstein polynomials. With the obvious modifications, namely replacing x by $\tau(x)$, they remain valid for $D_\tau^3B_j^\tau e_{\tau,s}^x(x)$. The following identities, that appear immediately, allow to complete the proof:

$$D^3_\tau B^\tau_{2i} e^x_{\tau,3}(x) = 6 - \frac{18}{(2i)} + \frac{12}{(2i)^2}$$

$$D^3_\tau B^\tau_{2i} e^x_{\tau,4}(x) = \frac{36(1-2\tau(x))}{(2i)} - \frac{108(1-2\tau(x))}{(2i)^2} + \frac{72(1-2\tau(x))}{(2i)^3}$$

$$\begin{aligned} D^3_\tau B^\tau_{2i} e^x_{\tau,5}(x) &= \frac{60\tau(x)(1-\tau(x))}{(2i)} + \frac{30(5-30\tau(x)+30\tau(x)^2)}{(2i)^2} \\ &\quad - \frac{30(15-76\tau(x)+76\tau(x)^2)}{(2i)^3} + \frac{30(10-48\tau(x)+48\tau(x)^2)}{(2i)^4} \end{aligned}$$

$$\begin{aligned} D^3_\tau B^\tau_{2i} e^x_{\tau,7}(x) &= \frac{630(1-\tau(x))^2\tau(x)^2}{(2i)^2} - \frac{210\tau(x)(-22+133\tau(x)-222\tau(x)^2+111\tau(x)^3)}{(2i)^3} \\ &\quad + \frac{42(43-930\tau(x)+4290\tau(x)^2-6720\tau(x)^3+3360\tau(x)^4)}{(2i)^4} \\ &\quad + \frac{42(-129+2020\tau(x)-8440\tau(x)^2+12840\tau(x)^3-6420\tau(x)^4)}{(2i)^5} \\ &\quad + \frac{84(43-600\tau(x)+2400\tau(x)^2-3600\tau(x)^3+1800\tau(x)^4)}{(2i)^6}. \qquad \square \end{aligned}$$

7. Conclusions

The main achievement of the paper is Theorem 2. It is a step forward in the study of B-statistical A-summability of sequences of linear operators that fulfill shape preserving properties more general than positivity. Beyond the classical qualitative and quantitative results, that theorem offers a procedure to obtain asymptotic formulae that requires a proper choice of a notion of rate of convergence. The topic of saturation, still to be investigated, turns to be classically the next step.

On the other hand, the paper revisits the aforesaid qualitative and quantitative results. A novelty here is that the qualitative one is obtain from the quantitative one. This had not been done before properly in the general framework represented by the B-statistical A-summability and the simultaneous setting.

The final example illustrates the wide applicability of the results of the paper.

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