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OPTIMAL EVALUATIONS FOR THE SÁNDOR–YANG MEAN BY POWER MEAN

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Abstract. In this paper, we present the best possible upper and lower bounds for the Sándor-Yang mean in terms of the power mean.

1. Introduction

For $r \in \mathbb{R}$, the Sándor-Yang mean $B(a,b)$ [12] and rth power mean $M_r(a,b)$ of two distinct positive real numbers a and b are respectively defined by

$$B(a,b) = Q(a,b)e^{A(a,b)/T(a,b)-1} \tag{1.1}$$

and

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0, \end{cases} \tag{1.2}$$

where $Q(a,b) = \sqrt{(a^2+b^2)/2}$, $A(a,b) = (a+b)/2$ and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ are respectively the quadratic, arithmetic and Seiffert mean of a and b.

It is well known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many classical means are the special cases of the power mean, for example, $M_{-1}(a,b) = 2ab/(a+b) = H(a,b)$ is the harmonic mean, $M_0(a,b) = \sqrt{ab} = G(a,b)$ is the geometric mean, $M_1(a,b) = (a+b)/2 = A(a,b)$ and $M_2(a,b) = \sqrt{(a^2+b^2)/2} = Q(a,b)$. The main properties for the power mean are given in [3].

Recently, the bounds for certain bivariate means in terms of the power mean have attracted the attention of many mathematicians [1, 2, 4, 5, 6, 7, 8, 16]. Yang et. al. [14] proved that

$$M_1(a,b) < B(a,b) < M_2(a,b) \tag{1.3}$$

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for all $a,b>0$ with $a\neq b$.

Motivated by inequality (1.3), it is natural to ask what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b)<B(a,b)<M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (See [13, Lemma 7]) *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m>0$ and $\sum_{k=m+1}^{\infty}a_k>0$, and*

$$P(t)=\sum_{k=0}^{m}a_kt^k-\sum_{k=m+1}^{\infty}a_kt^k$$

be a convergent power series on the interval $(0,\infty)$. Then there exists $t^*\in(0,\infty)$ such that $P(t^*)=0$, $P(t)>0$ for $t\in(0,t^*)$ and $P(t)<0$ for $t\in(t^*,\infty)$.

LEMMA 2.2. (See [11, Lemma 6]) *The function $r\to 2^{1/r}M_r(a,b)$ is strictly decreasing and log-convex on $(0,\infty)$ for all $a,b>0$ with $a\neq b$.*

LEMMA 2.3. *Let $t>0$, $p\in\mathbb{R}$ and*

$$f_1(t,p)=-\arctan(\tanh(t))+\sinh(t)\cosh(t)-\tanh(pt)\sinh^2(t). \tag{2.1}$$

Then the following statements are true:

(i) if $p\leqslant 1$, then $f_1(t,p)$ is strictly increasing with respect to t on $(0,\infty)$;

(ii) if $p\geqslant 4/3$, then $f_1(t,p)$ is strictly decreasing with respect to t on $(0,\infty)$;

(iii) if $p\in(1,4/3)$, then there exists $t_1\in(0,\infty)$ such that $f_1(t,p)$ is strictly increasing with respect to t on $(0,t_1)$ and strictly decreasing with respect to t on (t_1,∞).

Proof. Let

$$u_n(p)=(2-p)^{2n}-p^{2n}+(1-p)2^{2n}+2p, \tag{2.2}$$

$$f_2(t,p)=4\sinh^2(t)\cosh^2\left(\frac{pt}{2}\right)-4p\cosh(t)\sinh^2\left(\frac{t}{2}\right)-\sinh(2t)\sinh(pt).$$

Then simple computations lead to

$$u_1\left(\frac{4}{3}\right)=0,\ \ u_n\left(\frac{4}{3}\right)=-\frac{4^{2n}-2^{2n}}{3^{2n}}-\frac{2^{2n}-8}{3}<0\ \ (n\geqslant 2), \tag{2.3}$$

$$\begin{aligned}\frac{\partial f_1(t,p)}{\partial t} &= -\frac{1}{\cosh(2t)}+\cosh(2t)-\frac{p\sinh^2(t)}{\cosh^2(pt)}-\tanh(pt)\sinh(2t)\\ &= \frac{f_2(2t,p)}{4\cosh(2t)\cosh^2(pt)}, \qquad (2.4)\\ f_2(t,p) &= \cosh[(p-2)t]-\cosh(pt)+(1-p)\cosh(2t)+2p\cosh(t)-p-1\\ &= \sum_{n=1}^{\infty}\frac{u_n(p)}{(2n)!}t^{2n}, \qquad (2.5)\\ \frac{\partial f_2(t,p)}{\partial p} &= 2\cosh(t)-\cosh(2t)+t\sinh[(p-2)t]-t\sinh(pt)-1\\ &= -2[\cosh(t)-1]\cosh(t)-2t\sinh(t)\cosh[(p-1)t]<0. \qquad (2.6)\end{aligned}$$

(i) If $p \leqslant 1$, then equations (2.5) and (2.6) lead to

$$f_2(t,p) \geqslant f_2(t,1) = 2[\cosh(t)-1] > 0. \qquad (2.7)$$

Therefore, Lemma 2.3 (i) follows easily from (2.4) and (2.7).

(ii) If $p \geqslant 4/3$, then from (2.3), (2.5) and (2.6) we have

$$f_2(t,p) \leqslant f_2\left(t,\frac{4}{3}\right) = \sum_{n=1}^{\infty}\frac{u_n(4/3)}{(2n)!}t^{2n} < 0. \qquad (2.8)$$

Therefore, Lemma 2.3 (ii) follows easily from (2.4) and (2.8).

(iii) If $p \in (1,4/3)$, then from (2.4) it is enough to prove that there exists $t_1 \in (0,\infty)$ such that $f_2(t,p) > 0$ for $t \in (0,t_1)$ and $f_2(t,p) < 0$ for $t \in (t_1,\infty)$.

It follows from (2.2) that

$$u_1(p) = 2(4-3p) > 0, \quad \lim_{n\to\infty}\frac{u_n(p)}{2^{2n}} = 1-p < 0, \qquad (2.9)$$

$$u_{n+1}(p)-u_n(p) = -(p-1)\left[(3-p)(2-p)^{2n}+3\times 2^{2n}+(p+1)p^{2n}\right] < 0 \qquad (2.10)$$

for all $n \geqslant 1$.

Therefore, the desired result follows from (2.5), (2.9), (2.10) and Lemma 2.1. □

LEMMA 2.4. *Let* $t > 0$, $p \in \mathbb{R}$ *and* $f_1(t,p)$ *be defined by* (2.1). *Then*

(i) $f_1(t,p) > 0$ *for all* $t \in (0,\infty)$ *if and only if* $p \leqslant 1$;

(ii) $f_1(t,p) < 0$ *for all* $t \in (0,\infty)$ *if and only if* $p \geqslant 4/3$;

(iii) *there exists* $t_0 \in (0,\infty)$ *such that* $f_1(t_0,p) = 0$, $f_1(t,p) > 0$ *for* $t \in (0,t_0)$ *and* $f_1(t,p) < 0$ *for* $t \in (t_0,\infty)$ *if* $p \in (1,4/3)$.

Proof. (i) If $p \leqslant 1$, then Lemma 2.3 (i) and (2.1) lead to the conclusion that $f_1(t,p) > f_1(0,p) = 0$ for all $t \in (0,\infty)$.

If $f_1(t,p) > 0$ for all $t \in (0,\infty)$, then $\lim_{t\to\infty} f_1(t,p) \geqslant 0$. We claim that $p \leqslant 1$. Indeed, if $p > 1$, then from (2.1) we have

$$\begin{aligned}\lim_{t\to\infty} f_1(t,p) &= \lim_{t\to\infty}\left[-\arctan(\tanh(t)) + \frac{\sinh(t)\cosh((p-1)t)}{\cosh(pt)}\right]\\ &= \lim_{t\to\infty}\left[-\arctan(\tanh(t)) + \frac{1-e^{-2t}}{2}\frac{1+e^{-2|p-1|t}}{1+e^{-2|p|t}}e^{(1+|p-1|-|p|)t}\right]\\ &= -\frac{\pi}{4}+\frac{1}{2} < 0.\end{aligned}$$

(ii) If $p \geqslant 4/3$, then Lemma 2.3 (ii) and (2.1) imply that $f_1(t,p) < f_1(0,p) = 0$ for all $t \in (0,\infty)$.

If $f_1(t,p) < 0$ for all $t \in (0,\infty)$, then we clearly see that

$$\lim_{t\to 0}\frac{f_1(t,p)}{t^3} \leqslant 0. \tag{2.11}$$

It follows from (2.1), (2.2), (2.4) and (2.5) that

$$\begin{aligned}\lim_{t\to 0}\frac{f_1(t,p)}{t^3} &= \lim_{t\to 0}\frac{\partial f_1(t,p)/\partial t}{3t^2} = \lim_{t\to 0}\frac{1}{3\cosh(2t)\cosh^2(pt)} \times \lim_{t\to 0}\frac{f_2(2t,p)}{(2t)^2}\\ &= \frac{1}{3}\times\frac{1}{2}u_1(p) = -\left(p-\frac{4}{3}\right).\end{aligned} \tag{2.12}$$

Inequality (2.11) and equation (2.12) lead to the conclusion that $p \geqslant 4/3$.

(ii) If $p \in (1,4/3)$, then from Lemma 2.3 (iii) and the facts that $f_1(0,p) = 0$ and $\lim_{t\to\infty} f_1(t,p) = -\pi/4 + 1/2 < 0$ we clearly see that there exists $t_0 \in (0,\infty)$ such that $f_1(t_0,p) = 0$, $f_1(t,p) > 0$ for $t \in (0,t_0)$ and $f_1(t,p) < 0$ for $t \in (t_0,\infty)$. □

LEMMA 2.5. *Let* $t > 0$, $p \in (-\infty,0)\cup(0,\infty)$ *and*

$$F(t,p) = \frac{1}{2}\log[\cosh(2t)] + \frac{\arctan(\tanh(t))}{\tanh(t)} - \frac{1}{p}\log[\cosh(pt)] - 1. \tag{2.13}$$

Then

(i) $F(t,p)$ *is strictly increasing with respect to* t *on* $(0,\infty)$ *if and only if* $p \leqslant 1$;

(ii) $F(t,p)$ *is strictly decreasing with respect to* t *on* $(0,\infty)$ *if and only if* $p \geqslant 4/3$;

(iii) *there exists* $t_0 \in (0,\infty)$ *such that* $f_1(t_0,p) = 0$, $F(t,p)$ *is strictly increasing with respect to* t *on* $(0,t_0)$ *and strictly decreasing with respect to* t *on* (t_0,∞), *where* $f_1(t,p)$ *is defined by* (2.1).

Proof. It follows from (2.13) that

$$\frac{\partial F(t,p)}{\partial t} = \frac{-\arctan(\tanh(t)) + \sinh(t)\cosh(t) - \tanh(pt)\sinh^2(t)}{\sinh^2(t)} = \frac{f_1(t,p)}{\sinh^2(t)}. \tag{2.14}$$

Therefore, Lemma 2.5 follows from Lemma 2.4 and (2.14). □

3. Main results

THEOREM 3.1. *The inequality*

$$B(a,b) < M_p(a,b) \tag{3.1}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $p \geqslant 4/3$. *Moreover, the inequality*

$$B(a,b) > \lambda_p M_p(a,b) \tag{3.2}$$

holds for all $a,b>0$ *and* $a \neq b$ *with the best possible parameter* $\lambda_p = e^{\pi/4-1}2^{1/p-1/2}$ *if* $p \geqslant 4/3$.

Proof. Since $B(a,b)$ and $M(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b>a>0$. Let $t = \log\sqrt{b/a} > 0$, $p \in \mathbb{R}$ and $p \neq 0$, $f_1(t,p)$ and $F(t,p)$ be defined by (2.1) and (2.13), respectively. Then (1.1), (1.2), (2.1), (2.12), (2.13) and (2.14) lead to

$$M_p(a,b) = \sqrt{ab}\cosh^{1/p}(pt), \quad T(a,b) = \sqrt{ab}\frac{\sinh(t)}{\arctan[\tanh(t)]},$$

$$B(a,b) = \sqrt{ab}\cosh^{1/2}(2t)e^{\arctan(\tanh(t))/\tanh(t)-1},$$

$$\log[B(a,b)] - \log[M_p(a,b)] = F(t,p), \tag{3.3}$$

$$F(0^+,p) = 0, \tag{3.4}$$

$$\lim_{t\to 0^+}\frac{F(t,p)}{t^2} = \lim_{t\to 0^+}\frac{\partial F(t,p)/\partial t}{2t} = \lim_{t\to 0^+}\frac{f_1(t,p)}{2t\sinh^2(t)} = -\frac{1}{2}\left(p-\frac{4}{3}\right), \tag{3.5}$$

$$\begin{aligned}
&\lim_{t\to\infty}F(t,p)\\
=&\lim_{t\to\infty}\left[\left(1-\frac{|p|}{p}\right)t+\frac{1}{2}\log\left(\frac{1+e^{-4t}}{2}\right)+\frac{\arctan(\tanh(t))}{\tanh(t)}-\frac{1}{p}\log\left(\frac{1+e^{-2|p|t}}{2}\right)-1\right]\\
=&\frac{1}{4}\pi-\frac{1}{2}\log 2+\frac{1}{p}\log 2-1 = \log(\lambda_p) \quad (p>0).
\end{aligned} \tag{3.6}$$

If $B(a,b) < M_p(a,b)$, then (3.3) and (3.5) lead to $p \geqslant 4/3$.

If $p \geqslant 4/3$, then from (3.4) and (3.6) together with Lemma 2.5 (ii) we clearly see that

$$\log(\lambda_p) = \lim_{t\to\infty}F(t,p) < F(t,p) < F(0^+,p) = 0 \tag{3.7}$$

for all $t>0$ with the best possible parameter λ_p.

Therefore, the double inequality

$$\lambda_p M_p(a,b) < B(a,b) < M_p(a,b)$$

holds for all $a,b>0$ and $a \neq b$ with the best possible parameter λ_p follows from (3.3) and (3.7). □

Note that

$$\lambda_p M_p(a,b) = \frac{\sqrt{2}}{2} e^{\pi/4-1} \left(2^{1/p} M_p(a,b)\right), \quad \lim_{p\to\infty} M_p(a,b) = \max\{a,b\}. \tag{3.8}$$

Let $p = 4/3,\ 3/2,\ 2, 3, \cdots, \infty$. Then from Lemma 2.2, (3.1), (3.2) and (3.8) together with the monotonicity of the function $p \to M_p(a,b)$ we get Corollary 3.1.

COROLLARY 3.1. *The inequalities*

$$\lambda_\infty \max\{a,b\} < \cdots < \lambda_2 M_2(a,b) < \lambda_{3/2} M_{3/2}(a,b) < \lambda_{4/3} M_{4/3}(a,b)$$
$$< B(a,b) < M_{4/3}(a,b) < M_{3/2}(a,b) < M_2(a,b) < \cdots < \max\{a,b\}$$

hold for all $a,b > 0$ and $a \neq b$ with the best possible parameters $\lambda_\infty = \frac{\sqrt{2}}{2} e^{\pi/4-1} = 0.5705\cdots$, $\lambda_2 = e^{\pi/4-1} = 0.8068\cdots$, $\lambda_{3/2} = 2^{1/6} e^{\pi/4-1} = 0.9056\cdots$ and $\lambda_{4/3} = 2^{1/4} e^{\pi/4-1} = 0.9595\cdots$.

THEOREM 3.2. *Let $p_0 = 4\log 2/(4 + 2\log 2 - \pi) = 1.2351\cdots$. Then the inequality*

$$B(a,b) > M_p(a,b) \tag{3.9}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leqslant p_0$.

Proof. If $B(a,b) > M_p(a,b)$, then (3.3) and (3.6) lead to $p \leqslant p_0$.

If $p = p_0$, then (3.4), (3.6) and Lemma 2.5 (iii) lead to the conclusion that

$$F(0^+, p_0) = \lim_{t\to\infty} F(t, p_0) = 0 \tag{3.10}$$

and there exists $t_0 \in (0,\infty)$ such that the function $t \to F(t,p_0)$ is strictly increasing on $(0,t_0)$ and strictly decreasing on (t_0,∞).

Therefore,

$$B(a,b) > M_{p_0}(a,b) > M_p(a,b)$$

for all $p \leqslant p_0$ follows easily from (3.3) and (3.10) together with the piecewise monotonicity of the function $t \to F(t,p_0)$ and the monotonicity of the function $p \to M_p(a,b)$. □

COROLLARY 3.2. *Let $f_1(t,p)$, $F(t,p)$ and λ_p be respectively defined by (2.1), (2.13) and Theorem 3.1, and $p_0 = 4\log 2/(4 + 2\log 2 - \pi) = 1.2351\cdots$. Then the inequality*

$$B(a,b) < \lambda_p M_p(a,b) \tag{3.11}$$

holds for all $a,b > 0$ and $a \neq b$ with the best possible parameter λ_p if $p \in (0,1]$, and the inequality

$$B(a,b) \leqslant e^{F(t_0,p)} M_p(a,b) \tag{3.12}$$

holds for all $a,b > 0$ and $a \neq b$ with the best possible parameter $e^{F(t_0,p)}$ if $p \in (1,p_0]$, where t_0 is the unique solution of the equation $f_1(t,p) = 0$ on the interval $(0,\infty)$. In particular, Numerical computations show that $e^{F(t_0,p_0)} = 1.012\cdots$.

Proof. If $p\in(0,1]$, then inequality (3.11) holds for all $a,b>0$ and $a\neq b$ with the best possible parameter λ_p follows from (3.3) and (3.6) together with Lemma 2.5 (i).

If $p\in(1,p_0]$, then inequality (3.12) holds for all $a,b>0$ and $a\neq b$ with the best possible parameter $e^{F(t_0,p)}$ follows from (3.3) and Lemma 2.5 (iii). □

Let $p\in\mathbb{R}$, $b>a>0$, $L_p(a,b)=\left(a^{p+1}+b^{p+1}\right)/\left(a^p+b^p\right)$ be the pth Lehmer mean of a and b, $f_1(t,p)$ be defined by (2.1), and $t=\log\sqrt{b/a}>0$. Then $f_1(t,p)$ can be rewritten as

$$\begin{aligned} f_1(t,p) &= -\arctan(\tanh(t))+\sinh(t)\frac{\cosh((p-1)t)}{\cosh(pt)} \\ &= \frac{\arctan(\tanh(t))\cosh((p-1)t)}{\cosh(pt)}\left(\frac{\sinh(t)}{\arctan(\tanh(t))}-\frac{\cosh(pt)}{\cosh((p-1)t)}\right) \\ &= \frac{\arctan(\tanh(t))\cosh((p-1)t)}{\sqrt{ab}\cosh(pt)}\left(T(a,b)-L_{p-1}(a,b)\right). \end{aligned} \tag{3.13}$$

Lemma 2.4 and (3.13) lead to Corollary 3.3 immediately.

COROLLARY 3.3. *(See [9, Theorem 2.2]) The double inequality*

$$L_p(a,b)<T(a,b)<L_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leqslant 0$ and $q\geqslant 1/3$.

COROLLARY 3.4. *The double inequality*

$$\lambda L_{1/3}(a,b)<T(a,b)<\mu L_0(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leqslant 2/\pi$ and $\mu\geqslant 4/\pi$.

Proof. Without loss of generality, we assume that $b>a>0$. Let $t=\log\sqrt{b/a}>0$. Then simple computations lead to

$$\frac{T(a,b)}{L_{1/3}(a,b)}=\frac{\sinh(t)\cosh\left(\frac{t}{3}\right)}{\cosh\left(\frac{4t}{3}\right)\arctan(\tanh(t))},\quad \frac{T(a,b)}{L_0(a,b)}=\frac{\sinh(t)}{\cosh(t)\arctan(\tanh(t))},\tag{3.14}$$

$$\lim_{t\to\infty}\frac{\sinh(t)\cosh\left(\frac{t}{3}\right)}{\cosh\left(\frac{4t}{3}\right)\arctan(\tanh(t))}=\frac{2}{\pi},\quad \lim_{t\to\infty}\frac{\sinh(t)}{\cosh(t)\arctan(\tanh(t))}=\frac{4}{\pi}.\tag{3.15}$$

The log-convexity of the function $r\to 2^{1/r}M_r(a,b)$ given by Lemma 2.2 implies that

$$\left(2^{3/5}M_{5/3}(a,b)\right)^{3/4}\left(2^3M_{1/3}(a,b)\right)^{1/4}>2^{3/4}M_{4/3}(a,b),$$

which can be rewritten as

$$\frac{2^{8/5}}{\pi}M_{5/3}(a,b)>\frac{2}{\pi}\frac{M_{4/3}^{4/3}(a,b)}{M_{1/3}^{1/3}(a,b)}=\frac{2}{\pi}L_{1/3}(a,b).\tag{3.16}$$

Yang et. al. [15] and Witkowski [10] proved that

$$\frac{2^{8/5}}{\pi} M_{5/3}(a,b) < T(a,b) < \frac{4}{\pi} A(a,b) = \frac{4}{\pi} L_0(a,b). \tag{3.17}$$

Therefore, Corollary 3.4 follows from (3.14)–(3.17). □

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