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INEQUALITIES FOR THE HARTLEY–FOURIER COSINE POLYCONVOLUTION

PHI THI VAN ANH AND NGUYEN XUAN THAO

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Abstract. In this paper we present some new inequalities of Young's type and Saitoh's type for the Hartley-Fourier cosine polyconvolution. The inequalities of these types hold true for the Fourier convolution, but not for any convolutions or polyconvolutions, which are very few in the literature. We then use these inequalities to estimate the solution of certain integral equations and differential equations.

1. Introduction

The theory of the convolutions for integral transforms has been developed for a long time and is applied in various branches of mathematics [5, 6, 8, 14, 15]. In 1997, Kakichev [6] proposed general definition of a polyconvolution: the polyconvoluton of n functions $f_1, f_2, ..., f_n$ for $n+1$ arbitrary integral transforms $T, T_1, T_2, ..., T_n$ $(n \in \mathbb{N},\ n \geqslant 3)$ with a weight-function $\gamma(x)$ is $\overset{\gamma}{*}(f_1, f_2, ..., f_n)(x)$ such that the following factorization identity holds

$$T[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)](y) = \gamma(y)(T_1 f_1)(y)(T_2 f_2)(y) \cdots (T_n f_n)(y). \tag{1}$$

In recent years, the polyconvolutions for different integral transforms and their applications attracted great attention of many researchers. Several new polyconvolutions for the Fourier, Fourier sine, Fourier cosine, Kontorovich-Lebedev transforms have been introduced (see, e.g. [3, 4, 6, 7, 16, 17] and the references there-in). Their estimations in the various function spaces have been proved, for example in [7, 11, 12, 13].

In this paper, after reviewing the new polyconvolution $[*(f, g, h)](x)$ namely Hartley-Fourier cosine polyconvolution, which was introduced in, we consider about some inequalities such as Young's type inequality, Saitoh's type inequality, and its applications.

The paper consists of four sections and it is organized as follows. In section 2, we briefly introduce the definition of the Hartley-Fourier cosine polyconvolution [18], its factorization identities and two known inequalities. In section 3, we prove some new inequalities of Young's type and Saitoh's type on various function spaces. In the last section, we apply these inequalities to estimate the solution to a class of the integral equations and the differential equations in function spaces.

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2. Preliminaries

First, we recall the definition of the Hartley-Fourier cosine polyconvolution of three functions f,g,h [18]:

$$[*(f,g,h)](x) := \frac{1}{4\pi}\int_{-\infty}^{\infty} f(u)[k_1(x-u)+k_2(x+u)]du,\ x\in\mathbb{R}, \tag{2}$$

where k_1,k_2 have the forms

$$k_1(t) := \int_0^{\infty} g(v)[h(-t+v)+h(t-v)+h(-t-v)+h(t+v)]dv,\ \ t\in\mathbb{R}, \tag{3}$$

$$k_2(t) := \int_0^{\infty} g(v)[-h(t+v)+h(-t-v)-h(t-v)+h(-t+v)]dv,\ \ t\in\mathbb{R}. \tag{4}$$

We see that the kernels in the expression (2) are Toeplitz plus Hankel kernels. Here the Toeplitz kernel k_1 is an even function. Using this polyconvolution, one can solve a class of integral equation of Toeplitz plus Hankel form.

For convenience for the proofs in the next section, we rewrite the polyconvolution (2) in the form

$$[*(f,g,h)](x) := \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(u)g(v)\theta_h(x,u,v)dvdu, \tag{5}$$

where

$$\begin{aligned}\theta_h(x,u,v) =& h(-x+u+v)+h(x-u-v)-h(x+u+v)+h(-x-u-v)\\ &+h(-x+u-v)+h(x-u+v)-h(x+u-v)+h(-x-u+v).\end{aligned} \tag{6}$$

It is shortly written in the form "*H-F_c polyconvolution*".

Further, we recall the Hartley transforms [2], which are defined by

$$(H_1 f)(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\operatorname{cas}(xy)dx\,;\ \ (H_2 f)(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\operatorname{cas}(-xy)dx,$$

where $y\in\mathbb{R}$ and $\operatorname{cas}u = \cos u + \sin u$ and the Fourier cosine transform defined by [9, 10]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(yx)\,dx.$$

The following results have been proved in [18].

LEMMA 1. *Suppose that* $f,h \in L_1(\mathbb{R})$ *and* $g \in L_1(\mathbb{R}_+)$. *Then, the polyconvolution* $[*(f,g,h)](x)$, *belongs to space* $L_1(\mathbb{R})$ *and the following factorization identities hold,*

$$H_1[*(f,g,h)](y) = (H_1 f)(y)(F_c g)(y)(H_2 h)(y), \tag{7}$$
$$H_2[*(f,g,h)](y) = (H_2 f)(y)(F_c g)(y)(H_1 h)(y), \quad \forall y \in \mathbb{R}. \tag{8}$$

Moreover, in case of $h \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, *we obtain the Parseval equalities,*

$$[*(f,g,h)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (H_1 f)(y)(F_c g)(y)(H_2 h)(y) \operatorname{cas}(xy) dy, \tag{9}$$

$$[*(f,g,h)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (H_2 f)(y)(F_c g)(y)(H_1 h)(y) \operatorname{cas}(-xy) dy.$$

The expression (9) can be alternatively rewritten as,

$$[*(f,g,h)](x) = H_1\left[(H_1 f)(y)(F_c g)(y)(H_2 h)(y)\right](x), \tag{10}$$
$$[*(f,g,h)](x) = H_2\left[(H_2 f)(y)(F_c g)(y)(H_1 h)(y)\right](x).$$

Also in [18], two inequalities for the H-F_c polyconvolution have been proved: First, we have the inequality for the H-F_c polyconvolution in the space $L_1(\mathbb{R})$.

THEOREM 1. *Assume that* $f,h \in L_1(\mathbb{R})$ *and* $g \in L_1(\mathbb{R}_+)$. *Then the* H-F_c *polyconvolution* $*(f,g,h)(x)$ *belongs to* $L_1(\mathbb{R})$ *and satisfies the following inequality*

$$\| *(f,g,h)\|_{L_1(\mathbb{R})} \leqslant \frac{2}{\pi} \|f\|_{L_1(\mathbb{R})} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_1(\mathbb{R})}. \tag{11}$$

Next, we present the corresponding inequalities in another function space $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$, which consists of all functions $f(x)$ such that

$$\int_{-\infty}^{\infty} |x|^{\alpha} e^{-\beta |x|^{\gamma}} |f(x)|^s dx < \infty,$$

with the parameters $\alpha > -1$, $\beta > 0$, $\gamma > 0$, $s > 1$, and endowed by the norm

$$\|f\|_{L_s^{\alpha,\beta,\gamma}(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |x|^{\alpha} e^{-\beta |x|^{\gamma}} |f(x)|^s dx \right)^{\frac{1}{s}}.$$

THEOREM 2. *Suppose that* $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R}_+)$ *and* $h \in L_r(\mathbb{R})$, *where* $p,q,r > 1$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. *Then the* H-F_c *polyconvolution* $*(f,g,h)$ *is bounded in the space* $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$, *with* $s > 1$, $\alpha > -1$, $\beta > 0$, $\gamma > 0$, *and the following inequality holds*

$$\| *(f,g,h)\|_{L_s^{\alpha,\beta,\gamma}(\mathbb{R})} \leqslant C \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R})}, \tag{12}$$

where

$$C=\frac{2^{1+\frac{1}{s}}}{\pi\gamma^{\frac{1}{s}}}\beta^{-\frac{\alpha+1}{\gamma s}}\Gamma^{\frac{1}{s}}\left(\frac{\alpha+1}{\gamma}\right).$$

In the next section we prove some new inequalities for the H-F_c polyconvolution (5) in various function spaces.

3. Some new inequalities for the H-F_c polyconvolution

Young's inequality was first introduced for the Fourier convolution [1], it gives the estimation about the norm for the Fourier convolution in the space $L_s(\mathbb{R})$, $s>1$. In this section, we shall study the inequality of Young's type for the H-F_c polyconvolution.

THEOREM 3. (Inequality of Young's type) *Assume that* $p,q,r,s>1$ *with* $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}=3$ *and* $f\in L_p(\mathbb{R})$, $g\in L_q(\mathbb{R}_+)$, $h\in L_r(\mathbb{R})$, $k\in L_s(\mathbb{R})$. *Then,*

$$\left|\int_{-\infty}^{\infty}[*(f,g,h)](x)k(x)dx\right|\leqslant\frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}\|k\|_{L_s(\mathbb{R})}. \tag{13}$$

Proof. From the definition of the polyconvolution (5), it follows that

$$\begin{aligned}\left|\int_{-\infty}^{\infty}[*(f,g,h)](x)k(x)dx\right|&=\left|\int_{-\infty}^{\infty}\left(\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(u)g(v)\theta_h(x,u,v)dvdu\right)k(x)dx\right|\\&=\left|\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(u)g(v)\theta_h(x,u,v)k(x)dvdudx\right|\\&\leqslant\sum_{k=1}^{8}I_k,\end{aligned} \tag{14}$$

where I_k, $k=1,2,\ldots,8$, is the integral obtained by expanding $\theta_h(x,u,v)$ into the sum (6). Without loss of generality, we only estimate the integral I_1, the estimate for the remaining integrals I_k is similar. We have

$$I_1=\frac{1}{4\pi}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(u)g(v)h(-x+u+v)k(x)dvdudx\right|. \tag{15}$$

Let p_1,q_1,r_1,s_1, respectively, be the conjugate exponentials of p,q,r,s, it means

$$\frac{1}{p}+\frac{1}{p_1}=\frac{1}{q}+\frac{1}{q_1}=\frac{1}{r}+\frac{1}{r_1}=\frac{1}{s}+\frac{1}{s_1}=1.$$

For simplicity, we denote $\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ and

$$\begin{aligned}
U(x,u,v) &= |g(v)|^{q/p_1}|h(-x+u+v)|^{r/p_1}|k(x)|^{s/p_1} \in L_{p_1}(\Omega),\\
V(x,u,v) &= |h(-x+u+v)|^{r/q_1}|k(x)|^{s/q_1}|f(u)|^{p/q_1} \in L_{q_1}(\Omega),\\
P(x,u,v) &= |k(x)|^{s/r_1}|f(u)|^{p/r_1}|g(v)|^{q/r_1} \in L_{r_1}(\Omega),\\
Q(x,u,v) &= |f(u)|^{p/s_1}|g(v)|^{q/s_1}|h(-x+u+v)|^{r/s_1} \in L_{s_1}(\Omega).
\end{aligned}$$

Then

$$UVPQ = |f(u)||g(v)||h(-x+u+v)||k(x)|. \tag{16}$$

In the function space $L_{p_1}(\Omega)$, we have

$$\begin{aligned}
\|U\|^{p_1}_{L_{p_1}(\Omega)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} |U(x,u,v)|^{p_1} dvdudx\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} |g(v)|^q|h(-x+u+v)|^r|k(x)|^s dvdudx\\
&\leqslant \left(\int_0^{\infty}|g(v)|^q dv\right)\left(\int_{-\infty}^{\infty}|h(t)|^r dt\right)\left(\int_{-\infty}^{\infty}|k(x)|^s dx\right)\\
&= \|g\|^q_{L_q(\mathbb{R}_+)}\|h\|^r_{L_r(\mathbb{R})}\|k\|^s_{L_s(\mathbb{R})}.
\end{aligned} \tag{17}$$

Similarly, we obtain

$$\|V\|^{q_1}_{L_{q_1}(\Omega)} \leqslant \|f\|^p_{L_p(\mathbb{R})}\|h\|^r_{L_r(\mathbb{R})}\|k\|^s_{L_s(\mathbb{R})}, \tag{18}$$

$$\|P\|^{r_1}_{L_{r_1}(\Omega)} \leqslant \|f\|^p_{L_p(\mathbb{R})}\|g\|^q_{L_q(\mathbb{R}_+)}\|k\|^s_{L_s(\mathbb{R})}, \tag{19}$$

$$\|Q\|^{s_1}_{L_{s_1}(\Omega)} \leqslant \|f\|^p_{L_p(\mathbb{R})}\|g\|^q_{L_q(\mathbb{R}_+)}\|h\|^r_{L_r(\mathbb{R})}. \tag{20}$$

From (15)–(16), it follows that

$$\begin{aligned}
I_1 &\leqslant \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} |f(u)||g(v)||h(-x+u+v)||k(x)| dvdudx\\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} UVPQ dvdudx.
\end{aligned}$$

The assumption $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}=3$ implies that $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}+\frac{1}{s_1}=1$. Using the Hölder inequality for four functions U,V,P,Q on the corresponding function spaces, we get

$$I_1 \leqslant \frac{1}{4\pi} \cdot \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} |U|^{p_1} dv du dx \right)^{1/p_1} \cdot \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} |V|^{q_1} dv du dx \right)^{1/q_1}$$
$$\times \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} |P|^{r_1} dv du dx \right)^{1/r_1} \cdot \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} |Q|^{s_1} dv du dx \right)^{1/s_1}$$
$$= \frac{1}{4\pi} \|U\|_{L_{p_1}(\Omega)} \|V\|_{L_{q_1}(\Omega)} \|P\|_{L_{r_1}(\Omega)} \|Q\|_{L_{s_1}(\Omega)}.$$

By the given conditions, we obtain

$$p\left(\frac{1}{q_1}+\frac{1}{r_1}+\frac{1}{s_1}\right) = q\left(\frac{1}{p_1}+\frac{1}{r_1}+\frac{1}{s_1}\right) = r\left(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{s_1}\right) = s\left(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}\right) = 1. \tag{21}$$

From the estimations (17)–(21), we get

$$I_1 \leqslant \frac{1}{4\pi} \left(\|g\|^{q/p_1}_{L_q(\mathbb{R}_+)} \|h\|^{r/p_1}_{L_r(\mathbb{R})} \|k\|^{s/p_1}_{L_s(\mathbb{R})} \right) \left(\|f\|^{p/q_1}_{L_p(\mathbb{R})} \|h\|^{r/q_1}_{L_r(\mathbb{R})} \|k\|^{s/q_1}_{L_s(\mathbb{R})} \right)$$
$$\times \left(\|f\|^{p/r_1}_{L_p(\mathbb{R})} \|g\|^{q/r_1}_{L_q(\mathbb{R}_+)} \|k\|^{s/r_1}_{L_s(\mathbb{R})} \right) \left(\|f\|^{p/s_1}_{L_p(\mathbb{R})} \|g\|^{q/s_1}_{L_q(\mathbb{R}_+)} \|h\|^{r/s_1}_{L_r(\mathbb{R})} \right)$$
$$= \frac{1}{4\pi} \|f\|^{p\left(\frac{1}{q_1}+\frac{1}{r_1}+\frac{1}{s_1}\right)}_{L_p(\mathbb{R})} \|g\|^{q\left(\frac{1}{p_1}+\frac{1}{r_1}+\frac{1}{s_1}\right)}_{L_q(\mathbb{R}_+)} \|h\|^{r\left(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{s_1}\right)}_{L_r(\mathbb{R})} \|k\|^{s\left(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}\right)}_{L_s(\mathbb{R})}$$
$$= \frac{1}{4\pi} \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R})} \|k\|_{L_s(\mathbb{R})}.$$

Similarly, we obtain

$$I_k \leqslant \frac{1}{4\pi} \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R})} \|k\|_{L_s(\mathbb{R})}, \quad k = 2, 3, ..., 8. \tag{22}$$

From (14) and (22), it follows that

$$\left| \int_{-\infty}^{\infty} [*(f,g,h)](x) k(x) dx \right| \leqslant \frac{2}{\pi} \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R})} \|k\|_{L_s(\mathbb{R})}.$$

So, the proof is completed. □

Next, in a special case of parameters $p,q,r,s > 1$, we get the Young type inequality for the H-F_c polyconvolution. This inequality generalizes a result of (11) with $p = q = r = 1$.

COROLLARY 1. *Assume that $p,q,r,s > 1$ and satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2 + \frac{1}{s}$. Then, for all functions $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R}_+)$ and $h \in L_r(\mathbb{R})$, the polyconvolution $[*(f,g,h)](x)$ belongs to the space $L_s(\mathbb{R})$ and*

$$\| *(f,g,h)\|_{L_s(\mathbb{R})} \leqslant \frac{2}{\pi} \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R})}. \tag{23}$$

Proof. By the assumptions $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R})$ and Definition 5, it follows that the polyconvolution $[*(f,g,h)](x)$ is determined for $x \in \mathbb{R}$. Let s_1 be the conjugate exponential of s, i.e. $\frac{1}{s}+\frac{1}{s_1}=1$. The condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2+\frac{1}{s}$ follows that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s_1}=3$, so p,q,r,s_1 satisfy the hypothesis of Theorem 3. Now, we choose the function $k(x)=[*(f,g,h)]^{\alpha}(x)$, where α is a some constant such that $k \in L_{s_1}(\mathbb{R})$. Then $*(f,g,h)$ will be in the function space $L_{\alpha s_1}$ and we have

$$\|k\|_{L_{s_1}(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |k(x)|^{s_1} dx\right)^{1/s_1} = \left(\int_{-\infty}^{\infty} |[*(f,g,h)]^{\alpha}(x)|^{s_1} dx\right)^{1/s_1}$$
$$= \left(\int_{-\infty}^{\infty} |[*(f,g,h)](x)|^{\alpha s_1} dx\right)^{\frac{1}{\alpha s_1}\alpha} = \|*(f,g,h)\|^{\alpha}_{L_{\alpha s_1}(\mathbb{R})}.$$

Putting this k in the expression (13), we obtain

$$\left|\int_{-\infty}^{\infty} [*(f,g,h)]^{\alpha+1}(x)dx\right| \leqslant \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}\|k\|_{L_{s_1}(\mathbb{R})}.$$

Hence,

$$\left|\int_{-\infty}^{\infty} [*(f,g,h)]^{(\alpha s_1)\frac{\alpha+1}{\alpha s_1}}(x)dx\right| \leqslant \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}\|*(f,g,h)\|^{\alpha}_{L_{\alpha s_1}(\mathbb{R})}. \tag{24}$$

If we choose α such that $\frac{\alpha+1}{\alpha s_1}=1$, then the left hand side of the above expression has the form

$$\left|\int_{-\infty}^{\infty} [*(f,g,h)]^{(\alpha.s_1)\frac{\alpha+1}{\alpha s_1}}(x)dx\right| = \left|\int_{-\infty}^{\infty} [*(f,g,h)]^{(\alpha s_1)}(x)dx\right| = \|*(f,g,h)\|^{\alpha s_1}_{L_{\alpha s_1}(\mathbb{R})}.$$

Thus, the inequality (24) is equivalent to the following one

$$\|*(f,g,h)\|^{\alpha s_1}_{L_{\alpha s_1}(\mathbb{R})} \leqslant \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}\|*(f,g,h)\|^{\alpha}_{L_{\alpha s_1}(\mathbb{R})}.$$

Thus,

$$\|*(f,g,h)\|^{\alpha s_1-\alpha}_{L_{\alpha s_1}(\mathbb{R})} \leqslant \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}. \tag{25}$$

The condition $\frac{\alpha+1}{\alpha s_1}=1$ follows that $\alpha=\frac{1}{s_1-1}$. Since $\frac{1}{s}+\frac{1}{s_1}=1$, it implies that $s_1=\frac{s}{s-1}$, $\alpha=\frac{1}{\frac{s}{s-1}-1}=s-1$, $\alpha s_1=(s-1)\frac{s}{s-1}=s$, and $\alpha s_1-\alpha=1$. Thus the function space $L_{\alpha s_1}(\mathbb{R})$ is exactly the function space $L_s(\mathbb{R})$. Therefore, $[*(f,g,h)] \in L_s(\mathbb{R})$ and

$$\| *(f,g,h)\|_{L_s(\mathbb{R})} \leqslant \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}.$$

The proof is completed. $\square$

We can see that in the inequality (12), the parameter s only satisfies the inequality $s>1$, but in the inequality (23) the parameter s depends on p,q,r by $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2+\frac{1}{s}$. Another different thing is the constant C. In the inequality (12), C depends on α,β,γ and s, so we can choose the suitable parameters to get the best constant C for our purpose. However in (23), the constant C is fixed by $\frac{2}{\pi}$.

Although Young's inequality for the H-F_c polyconvolution is simple, it is not true in the typical case $p=q=r=2$. In [11, 12, 13], Saitoh and his co-authors introduced a new type inequality for the Fourier convolution in the function space $L_p(\mathbb{R},\rho)$ with non-vanished weight ρ for which the inequality still holds in L_2 space. The space $L_p(\mathbb{R},\rho)$ consists of all p-order integrable functions such that

$$\int_{-\infty}^{\infty} |f(x)|^p\rho(x)dx<\infty,$$

and the norm of a function $f(x)$ in $L_p(\mathbb{R},\rho)$ is defined by

$$\|f\|_{L_p(\mathbb{R},\rho)} := \left(\int_{-\infty}^{\infty} |f(x)|^p\rho(x)dx\right)^{\frac{1}{p}}.$$

THEOREM 4. (Inequality of Saitoh's type) *Assume that* ρ_i, $i=1,2,3$ *are positive, bounded functions such that* $[*(\rho_1,\rho_2,\rho_3)]$ *exists. Then, for all the functions* $F_1\in L_p(\mathbb{R},\rho_1)$, $F_2\in L_p(\mathbb{R}_+,\rho_2)$ *and* $F_3\in L_p(\mathbb{R},\rho_3)$ $(p>1)$, *we have the following inequality for the Hartley-Fourier cosine polyconvolution*

$$\begin{aligned}&\|[*(F_1\rho_1,F_2\rho_2,F_3\rho_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})}\\ &\leqslant \frac{2}{\pi}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R}_+,\rho_2)}\|F_3\|_{L_p(\mathbb{R},\rho_3)}.\end{aligned} \tag{26}$$

Proof. By definition of the norm in the space $L_p(\mathbb{R})$, we have

$$\begin{aligned}&\|[*(F_1\rho_1,F_2\rho_2,F_3\rho_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})}\\ &=\int_{-\infty}^{\infty} |*(F_1\rho_1,F_2\rho_2,F_3\rho_3)(x)|^p\,|*(\rho_1,\rho_2,\rho_3)(x)|^{1-p}\,dx\end{aligned}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_{0}^{\infty} (F_1\rho_1)(u)(F_2\rho_2)(v)\theta_{F_3\rho_3}(x,u,v)dvdu \right|^p \times \left| \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho_1(u)\rho_2(v)\theta_{\rho_3}(x,u,v)dvdu \right|^{1-p} dx. \tag{27}$$

Denote

$$A := \left| \int_{-\infty}^{\infty} \int_{0}^{\infty} (F_1\rho_1)(u)(F_2\rho_2)(v)\theta_{F_3\rho_3}(x,u,v)dvdu \right|,$$

$$\begin{array}{llll} t_1 := -x+u+v & t_2 := x-u-v & t_3 := x+u+v & t_4 := -x-u-v \\ t_5 := -x+u-v & t_6 := x-u+v & t_7 := x+u-v & t_8 := -x-u+v. \end{array}$$

and

$$A_i := \int_{-\infty}^{\infty} \int_{0}^{\infty} |(F_1\rho_1)(u)|\,|(F_2\rho_2)(v)|\,|(F_3\rho_3)(t_i)|\,dvdu, \ \ i = \overline{1,8}.$$

Then,

$$A \leqslant \sum_{i=1}^{8} A_i.$$

Now, we estimate the norm for A_1. Using the Hölder inequality, we have

$$\begin{aligned} A_1 :=& \int_{-\infty}^{\infty} \int_{0}^{\infty} |(F_1\rho_1)(u)|\,|(F_2\rho_2)(v)|\,|(F_3\rho_3)(-x+u+v)|\,dvdu \\ &\leqslant \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} |F_1(u)|^p \rho_1(u) |F_2(v)|^p \rho_2(v) |F_3(-x+u+v)|^p \rho_3(-x+u+v) dvdu \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} \rho_1(u)\rho_2(v)\rho_3(-x+u+v)dvdu \right)^{\frac{1}{q}} \\ &= \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} |F_1(u)|^p \rho_1(u) |F_2(v)|^p \rho_2(v) |F_3(t_1)|^p \rho_3(t_1) dvdu \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} \rho_1(u)\rho_2(v)\rho_3(t_1)dvdu \right)^{\frac{1}{q}}. \end{aligned} \tag{28}$$

Similarly, we obtain expressions for A_i, $i=2,\dots,8$,

$$A_i \leqslant \left(\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)|F_3(t_i)|^p\rho_3(t_i)dvdu\right)^{\frac{1}{p}} \times\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\rho_3(t_i)dvdu\right)^{\frac{1}{q}}. \tag{29}$$

Since (28), (29) we get the following estimation for A

$$A \leqslant \sum_{i=1}^{8}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u).|F_2(v)|^p\rho_2(v).|F_3(t_i)|^p\rho_3(t_i)dvdu\right)^{\frac{1}{p}} \times\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\rho_3(t_i)dvdu\right)^{\frac{1}{q}}.$$

Now, using the inequality of the form $\sum\limits_{i=1}^{m} a_i^{\frac{1}{p}}.b_i^{\frac{1}{q}} \leqslant \left(\sum\limits_{i=1}^{m} a_i\right)^{\frac{1}{p}}.\left(\sum\limits_{i=1}^{m} b_i\right)^{\frac{1}{q}}$, where a_i, b_i, $i=1,2,...,m$, $(m\in\mathbb{N})$ are the nonnegative numbers, $p>1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Then, we have

$$\begin{aligned} A &\leqslant \left(\sum_{i=1}^{8}\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)|F_3(t_i)|^p\rho_3(t_i)dvdu\right)^{\frac{1}{p}} \\ &\quad\times\left(\sum_{i=1}^{8}\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\rho_3(t_i)dvdu\right)^{\frac{1}{q}} \\ &=\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)\left[\sum_{i=1}^{8}|F_3(t_i)|^p\rho_3(t_i)\right]dvdu\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\left[\sum_{i=1}^{8}\rho_3(t_i)\right]dvdu\right)^{\frac{1}{q}}. \end{aligned} \tag{30}$$

Putting (30) to (27), and noticing $\sum_{i=1}^{8}\rho_3(t_i)=\theta_{\rho_3}(x,u,v)$, we get

$$
\begin{aligned}
&\|[*(F_1\rho_1,F_2\rho_2,F_3\rho_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})}\\
&\leqslant \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)\left[\sum_{i=1}^{8}|F_3(t_i)|^p\rho_3(t_i)\right]dvdu\right)\\
&\quad\times\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\left[\sum_{i=1}^{8}\rho_3(t_i)\right]dvdu\right)^{\frac{p}{q}}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)\theta_{\rho_3}(x,u,v)dvdu\right)^{1-p}dx\\
&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)\left[\sum_{i=1}^{8}|F_3(t_i)|^p\rho_3(t_i)\right]dvdu\right)dx.
\end{aligned}
$$

Expanding the integral into the sum of 8 integrals, using Fubini's theorem to change the order of the integration and the differentiation, and changing variables $t:=t_i,\ i=\overline{1,8}$ to compute dx by dt, we obtain

$$
\begin{aligned}
&\|[*(F_1\rho_1,F_2\rho_2,F_3\rho_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})}\\
&\leqslant \frac{1}{4\pi}.8\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}|F_1(u)|^p\rho_1(u)|F_2(v)|^p\rho_2(v)|F_3(t)|^p\rho_3(t)dtdvdu\\
&=\frac{2}{\pi}\left(\int_{-\infty}^{\infty}|F_1(u)|^p\rho_1(u)du\right)\left(\int_{0}^{\infty}|F_2(v)|^p\rho_2(v)dv\right)\left(\int_{-\infty}^{\infty}|F_3(t)|^p\rho_3(t)dt\right)\\
&=\frac{2}{\pi}\|F_1\|^p_{L_p(\mathbb{R},\rho_1)}\|F_2\|^p_{L_p(\mathbb{R}_+,\rho_2)}\|F_3\|^p_{L_p(\mathbb{R},\rho_3)}. \qquad (31)
\end{aligned}
$$

So, the proof is completed. □

Below we specify this inequality for some concrete cases of the functions ρ_1,ρ_2 and ρ_3.

COROLLARY 2. *Suppose that* $\rho_1\in L_1(\mathbb{R}),\ \rho_2\in L_1(\mathbb{R}_+)$ *are positive functions* $\rho_3=1$, *then the formula* (26) *becomes*

$$
\begin{aligned}
&\|[*(F_1\rho_1,F_2\rho_2,F_3)]\|_{L_p(\mathbb{R})}\\
&\leqslant \frac{2}{\pi^{2-\frac{1}{p}}}\|\rho_1\|^{1-\frac{1}{p}}_{L_1(\mathbb{R})}\|\rho_2\|^{1-\frac{1}{p}}_{L_1(\mathbb{R}_+)}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R}_+,\rho_2)}\|F_3\|_{L_p(\mathbb{R})}. \qquad (32)
\end{aligned}
$$

Proof. Under the hypothesis on the functions ρ_i, $i=1,2,3$, we have $\theta_{\rho_3}(x,u,v)=4$ and

$$[*(\rho_1,\rho_2,\rho_3)](x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho_1(u)\rho_2(v)4dvdu=\frac{1}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_2\|_{L_1(\mathbb{R}_+)}.$$

Then the left hand side of the inequality (26) becomes

$$
\begin{aligned}
&\|[*(F_1\rho_1,F_2\rho_2,F_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})}\\
&=\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2\rho_2,F_3)](x)|^p\,|[*(\rho_1,\rho_2,\rho_3)](x)|^{1-p}\,dx\right)^{\frac{1}{p}}\\
&=\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2\rho_2,F_3)](x)|^p\left(\frac{1}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_2\|_{L_1(\mathbb{R}_+)}\right)^{1-p}dx\right)^{\frac{1}{p}}\\
&=\left(\frac{1}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_2\|_{L_1(\mathbb{R}_+)}\right)^{\frac{1}{p}-1}\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2\rho_2,F_3)](x)|^p\,dx\right)^{\frac{1}{p}}\\
&=\left(\frac{1}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_2\|_{L_1(\mathbb{R}_+)}\right)^{\frac{1}{p}-1}\|[*(F_1\rho_1,F_2\rho_2,F_3)]\|_{L_p(\mathbb{R})}.
\end{aligned}
$$

Returning to the inequality (26), we obtain

$$
\begin{aligned}
&\left(\frac{1}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}.\|\rho_2\|_{L_1(\mathbb{R}_+)}\right)^{\frac{1}{p}-1}\|[*(F_1\rho_1,F_2\rho_2,F_3)]\|_{L_p(\mathbb{R})}\\
&\leqslant\frac{2}{\pi}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R}_+,\rho_2)}\|F_3\|_{L_p(\mathbb{R})},
\end{aligned}
$$

or

$$
\|[*(F_1\rho_1,F_2\rho_2,F_3)]\|_{L_p(\mathbb{R})}\leqslant C\|\rho_1\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}}\|\rho_2\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R}_+,\rho_2)}\|F_3\|_{L_p(\mathbb{R})},
$$

where the constant $C=\dfrac{2}{\pi^{2-\frac{1}{p}}}=\dfrac{2}{\pi^{1+\frac{1}{q}}}<1$. The proof is completed. □

EXAMPLE 1. Choose $\rho_1=e^{-|x|}$, $\rho_2=e^{-2|x|}$, $\rho_3=1$. Then

$$
\|\rho_1\|_{L_1(\mathbb{R})}=\int_{-\infty}^{\infty}|e^{-|u|}|du=2;\qquad \|\rho_2\|_{L_1(\mathbb{R}_+)}=\int_{0}^{\infty}e^{-2v}dv=\frac{1}{2}.
$$

Also

$$
\begin{aligned}
[*(\rho_1,\rho_2,\rho_3)](x)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-|u|}e^{-2|v|}dvdu\\
&=\frac{1}{\pi}\left(\int_{-\infty}^{\infty}e^{-|u|}du\right)\left(\int_{0}^{\infty}e^{-2|v|}dv\right)=\frac{1}{\pi}\cdot 2\cdot\frac{1}{2}=\frac{1}{\pi}.
\end{aligned}
$$

If we choose $p=2$ then the inequality (32) can be written

$$\|[*(F_1e^{-|u|},F_2e^{-2|v|},F_3)]\|_{L_2(\mathbb{R})} \leqslant \frac{2}{\pi^{\frac{3}{2}}}\|F_1\|_{L_2(\mathbb{R},e^{-|x|})}\|F_2\|_{L_2(\mathbb{R}_+,e^{-2|x|})}\|F_3\|_{L_2(\mathbb{R})}. \quad (33)$$

COROLLARY 3. *If the functions* $\rho_1,\rho_3\in L_1(\mathbb{R})$ *are positive functions and* $\rho_2=1$*, then the inequality* (26) *has the form*

$$\begin{aligned}
&\|[*(F_1\rho_1,F_2,F_3\rho_3)]\|_{L_p(\mathbb{R})}\\
&\leqslant \left(\frac{2}{\pi}\right)^{2-\frac{1}{p}}\|\rho_1\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}}\|\rho_3\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R})}\|F_3\|_{L_p(\mathbb{R},\rho_3)}. \quad (34)
\end{aligned}$$

Proof. Under the assumption on ρ_i, $i=1,2,3$, we have

$$\begin{aligned}
&|[*(\rho_1,\rho_2,\rho_3)](x)|\\
&=\Big|\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\rho_1(u)\,[\rho_3(-x+u+v)+\rho_3(x-u-v)-\rho_3(x+u+v)\\
&\quad+\rho_3(-x-u-v)+\rho_3(-x+u-v)+\rho_3(x-u+v)-\rho_3(x+u-v)\\
&\quad+\rho_3(-x-u+v)]\,dvdu\Big|\\
&\leqslant\frac{2}{\pi}\left(\int_{-\infty}^{\infty}|\rho_1(u)|du\right)\left(\int_0^{\infty}|\rho_3(v)|dv\right)\leqslant\frac{2}{\pi}\left(\int_{-\infty}^{\infty}\rho_1(u)du\right)\left(\int_{-\infty}^{\infty}\rho_3(v)dv\right)\\
&=\frac{2}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}.\|\rho_3\|_{L_1(\mathbb{R})}.
\end{aligned}$$

Then the left hand side of (26) is

$$\begin{aligned}
&\|[*(F_1\rho_1,F_2,F_3\rho_3)][*(\rho_1,\rho_2,\rho_3)]^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})}\\
&=\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2,F_3\rho_3)](x)|^p\,|[*(\rho_1,\rho_2,\rho_3)](x)|^{1-p}\,dx\right)^{\frac{1}{p}}\\
&\leqslant\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2\rho_2,F_3)](x)|^p\left(\frac{2}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_3\|_{L_1(\mathbb{R})}\right)^{1-p}dx\right)^{\frac{1}{p}}\\
&=\left(\frac{2}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_3\|_{L_1(\mathbb{R})}\right)^{\frac{1}{p}-1}.\left(\int_{-\infty}^{\infty}|[*(F_1\rho_1,F_2,F_3\rho_3)](x)|^p\,dx\right)^{\frac{1}{p}}\\
&=\left(\frac{2}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_3\|_{L_1(\mathbb{R})}\right)^{\frac{1}{p}-1}\|[*(F_1\rho_1,F_2,F_3\rho_3)]\|_{L_p(\mathbb{R})}.
\end{aligned}$$

Putting it into (26), we get

$$\left(\frac{2}{\pi}\|\rho_1\|_{L_1(\mathbb{R})}\|\rho_3\|_{L_1(\mathbb{R})}\right)^{\frac{1}{p}-1}\|[*(F_1\rho_1,F_2,F_3\rho_3)]\|_{L_p(\mathbb{R})}$$
$$\leqslant\frac{2}{\pi}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R}_+)}\|F_3\|_{L_p(\mathbb{R},\rho_3)},$$

or

$$\|[*(F_1\rho_1,F_2,F_3\rho_3)]\|_{L_p(\mathbb{R})}$$
$$\leqslant\left(\frac{2}{\pi}\right)^{2-\frac{1}{p}}\|\rho_1\|^{1-\frac{1}{p}}_{L_1(\mathbb{R})}\|\rho_3\|^{1-\frac{1}{p}}_{L_1(\mathbb{R})}\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R})}\|F_3\|_{L_p(\mathbb{R},\rho_3)}.$$

The proof is completed. □

EXAMPLE 2. Choose $\rho_1=e^{-x^2},\ \rho_2=1,\ \rho_3=e^{-|x|}$. Then their norms in the corresponding function spaces are

$$\|\rho_1\|_{L_1(\mathbb{R})}=\int_{-\infty}^{\infty}|e^{-u^2}|du=\sqrt{\pi};\qquad \|\rho_3\|_{L_1(\mathbb{R})}=\int_{-\infty}^{\infty}e^{-|v|}dv=2.$$

If $p=2$, then the inequality (32) has the form

$$\|[*(F_1e^{-u^2},F_2,F_3e^{-|t|})]\|_{L_2(\mathbb{R})}\leqslant\frac{2}{\pi^{\frac{3}{2}}}\|F_1\|_{L_2(\mathbb{R},e^{-x^2})}\|F_2\|_{L_2(\mathbb{R}_+)}\|F_3\|_{L_2(\mathbb{R},e^{-|x|})}. \tag{35}$$

If $p=3$, then the inequality (32) has the form

$$\|[*(F_1e^{-u^2},F_2,F_3e^{-|t|})]\|_{L_3(\mathbb{R})}\leqslant\frac{2}{\pi^{\frac{5}{3}}}\|F_1\|_{L_3(\mathbb{R},e^{-x^2})}\|F_2\|_{L_3(\mathbb{R}_+)}\|F_3\|_{L_3(\mathbb{R},e^{-|x|})}. \tag{36}$$

4. Applications

In this section, we look for the solution to a class of integral equations and a class of differential equations, after that we use the above inequalities to estimate their solution. In doing so, we need the notion of a generalized convolution $(\cdot \underset{HF_c}{*} \cdot)$ related to the Hartley, Fourier cosine transforms [19]:

$$(p\underset{HF_c}{*}q)(x):=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}p(u)[q(x+u)+q(x-u)]du,$$

with its factorization identities

$$H_k(p\underset{HF_c}{*}q)=(F_cp)(|y|)(H_kg)(y),\ k=1,2,y\in\mathbb{R}.$$

4.1. A class of integral equations

Consider the equation

$$f(x)+[*(f,\varphi,\psi)](x)=(p \underset{HF_c}{*} q)(x),\ \ x\in\mathbb{R}, \tag{37}$$

where f is an unknown function, $\psi,q\in L_1(\mathbb{R})$, $\varphi,p\in L_1(\mathbb{R}_+)$, and $(\cdot \underset{HF_c}{*} \cdot)$ is a generalized convolution for the Hartley, Fourier cosine transforms.

THEOREM 5. *The condition* $1+(F_c\varphi)(|y|)(H_2\psi)(y)\neq 0$ *is the sufficient condition for the equation* (37) *has a unique solution in the space* $L_1(\mathbb{R})$. *The solution has the form*

$$f=(p \underset{HF_c}{*} q)(x)-[*(q,p,l)](x), \tag{38}$$

and satisfies the estimate

$$\|f\|_{L_1(\mathbb{R})}\leqslant\|p\|_{L_1(\mathbb{R}_+)}\|q\|_{L_1(\mathbb{R})}[1+\frac{2}{\pi}\|l\|_{L_1(\mathbb{R})}], \tag{39}$$

where $l\in L_1(\mathbb{R})$ *such that*

$$(H_2l)(y)=\frac{H_2(\varphi \underset{HF_c}{*} \psi)(y)}{1+H_2(\varphi \underset{HF_c}{*} \psi)(y)}.$$

Proof. Applying the Hartley H_1 transform to both sides of the equation (37), then respectively using the factorization identity for the convolution and polyconvolution, we obtain

$$(H_1f)(y)+(H_1f)(y)(F_c\varphi)(|y|)(H_2\psi)(y)=(F_cp)(|y|)(H_1q)(y).$$

Hence,

$$(H_1f)(y)[1+(F_c\varphi)(|y|)(H_2\psi)(y)]=(F_cp)(|y|)(H_1q)(y).$$

Since $1+(F_c\varphi)(|y|)(H_2\psi)(y)=1+H_2(\varphi \underset{HF_c}{*} \psi)(y)\neq 0$, we get

$$\begin{aligned}(H_1f)(y)&=\frac{(F_cp)(|y|)(H_1q)(y)}{1+(F_c\varphi)(|y|)(H_2\psi)(y)}=(F_cp)(|y|)(H_1q)(y)\left[1-\frac{(F_c\varphi)(|y|)(H_2\psi)(y)}{1+(F_c\varphi)(|y|)(H_2\psi)(y)}\right]\\&=(F_cp)(|y|)(H_1q)(y)\left[1-\frac{H_2(\varphi \underset{HF_c}{*} \psi)(y)}{1+H_2(\varphi \underset{HF_c}{\psi})(y)}\right].\end{aligned}$$

The hypothesis of the theorem implies that $(\varphi \underset{HF_c}{*} \psi)(x) \in L_1(\mathbb{R})$. By using the Wiener-Lévy theorem for the Hartley transform as in the reference [19], we see that the condition $1 + H_2(\varphi \underset{HF_c}{*} \psi)(y) \neq 0$ is the necessary and sufficient condition for the existence of a function $l \in L_1(\mathbb{R})$ such that

$$(H_2 l)(y) = \frac{H_2(\varphi \underset{HF_c}{*} \psi)(y)}{1 + H_2(\varphi \underset{HF_c}{*} \psi)(y)}.$$

Thus,

$$\begin{aligned}(H_1 f)(y) &= (F_c p)(|y|)(H_1 q)(y)[1 - (H_2 l)(y)] \\ &= (F_c p)(|y|)(H_1 q)(y) - (H_1 q)(y)(F_c p)(|y|)(H_2 l)(y) \\ &= H_1(p \underset{HF_c}{*} q)(y) - H_1[*(q,p,l)](y) \\ &= H_1((p \underset{HF_c}{*} q)(x) - [*(q,p,l)](x))(y), \quad \forall y \in \mathbb{R}.\end{aligned}$$

It implies that

$$f = (p \underset{HF_c}{*} q)(x) - [*(q,p,l)](x).$$

Using the norm estimation in the space $L_1(\mathbb{R})$, we have

$$\begin{aligned}\|f\|_{L_1(\mathbb{R})} &\leqslant \|(p \underset{HF_c}{*} q)\|_{L_1(\mathbb{R})} + \|*(q,p,l)\|_{L_1(\mathbb{R})} \\ &\leqslant \|p\|_{L_1(\mathbb{R}_+)} \|q\|_{L_1(\mathbb{R})} + \frac{2}{\pi} \|q\|_{L_1(\mathbb{R})} \|p\|_{L_1(\mathbb{R}_+)} \|l\|_{L_1(\mathbb{R})} \\ &\leqslant \|p\|_{L_1(\mathbb{R}_+)} \|q\|_{L_1(\mathbb{R})} [1 + \frac{2}{\pi} \|l\|_{L_1(\mathbb{R})}].\end{aligned}$$

The proof is completed. □

4.2. A class of differential equations

Consider the following differential equation:

$$\left(\sum_{k=0}^{n} (-1)^k a_k \frac{d^{2k}}{dx^{2k}} \right) f(x) = [(g\rho_1) \underset{HF_c}{*} (h\rho_2)](x), \tag{40}$$

where g, h, ρ_1, ρ_2 are given functions such that $g \in L_1(\mathbb{R}, \rho_1) \cap L_p(\mathbb{R}, \rho_1)$, $h \in L_1(\mathbb{R}, \rho_2)$ $\cap L_p(\mathbb{R}, \rho_2)$, $p > 1$ and $\rho_1, \rho_2 \in L_1(\mathbb{R}_+)$, and the remaining function f is an unknown function. Also the coefficients a_k satisfy $a_0 = 1$, $a_k \in \mathbb{R}$ $(k = \overline{1,n})$ such that there exists a function $Q \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$ determined by

$$(H_2 Q)(y) = \frac{1}{\sum\limits_{k=0}^{n} a_k y^{2k}}, \quad y > 0. \tag{41}$$

We need to find the solution of the equation (40) which satisfies the following boundary condition

$$\frac{d^k}{dx^k}f(x) \to 0 \text{ as } x \to \infty,\ k = 0,1,...,2n-1.$$

To deal this problem under the given hypothesis, we apply the Hartley transform H_1 to both sides of the equation (40)

$$H_1\left[\left(\sum_{k=0}^{n}(-1)^k a_k \frac{d^{2k}}{dx^{2k}}\right) f(x)\right](y) = H_1[(g\rho_1) \underset{HF_c}{*} (h\rho_2)](y)$$
$$\sum_{k=0}^{n} a_k y^{2k}(H_1 f)(y) = (F_c g\rho_1)(y)(H_1 h\rho_2)(y)$$

From (41),

$$(H_1 f)(y) = \frac{(F_c g\rho_1)(y)(H_1 h\rho_2)(y)}{\sum\limits_{k=0}^{n} a_k y^{2k}} = (F_c g\rho_1)(y)(H_1 h\rho_2)(y)(H_2 Q)(y)$$
$$= H_1[*(h\rho_2, g\rho_1, Q)](y).$$

This expression holds for all $y \in \mathbb{R}$, so we obtain the solution $f \in L_1(\mathbb{R})$ in the form

$$f(x) = [*(h\rho_2, g\rho_1, Q)](x),\ \ x \in \mathbb{R}.$$

Using the inequality (32), we get the estimate

$$\|f\|_{L_p(\mathbb{R})} = \|[*(h\rho_2, g\rho_1, Q)]\|_{L_p(\mathbb{R})}$$
$$\leqslant \frac{2^{\frac{1}{p}}.3^{1-\frac{1}{p}}}{\pi^{2-\frac{1}{p}}}.\|\rho_1\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}}\|\rho_2\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}}\|h\|_{L_p(\mathbb{R},\rho_1)}\|g\|_{L_p(\mathbb{R}_+,\rho_2)}\|Q\|_{L_p(\mathbb{R})}.$$

REFERENCES

[1] R. A. ADAMS AND J. J. F. FOURIER, *Sobolev Spaces*, 2nd ed, Academic Press, 300 p, 2003.

[2] R. N. BRACEWELL, *The Hartley transform*, Inc, 200 Madison Avenue, New York, 1986.

[3] L. E. BRITVINA, *Polyconvolutions for the Hankel transform and differential operators*, Doklady Mathematics, **65**, 1 (2002), 32–34.

[4] L. E. BRITVINA, *On polyconvolutions generated by Hankel transforms*, Mathematical Notes, **76**, 1 (2004), 18–24.

[5] F. D. GAKHOV AND YU. I. CERSKII, *Equations of Convolution Type*, Nauka, Moscow, 1978.

[6] V. A. KAKICHEV, *Polyconvolution*, Taganrog, TPTU, 54p (in Russian), 1997.

[7] N. M. KHOA AND T. TUAN, *On the polyconvolution with weight function for Fourier cosine, Fourier and Fourier sine transforms*, Advances in Nonlinear Variational Inequalities, **14**, 1 (2011), 17–30.

[8] V. V. NAPALKOV, *Convolution Equations in Multidimensional Spaces*, Nauka, Moscow, 1982.

[9] R. E. A. C. PALEY AND N. WIENER, *Fourier Transforms in the Complex Domain*, AMS, New York, 1934.

[10] I. N. SNEDDON, *The Use of Integral Transforms*, Mc Gray-Hill. New York, 1972.
[11] S. SAITOH, V. K. TUAN AND M. YAMAMOTO, *Reverse weighted L_p-norm inequalities in convolutions and stability in inverse problems*, J. of Ineq. in Pure & App. Math., **1**, 1 (2000), 1–7.
[12] S. SAITOH, V. K. TUAN AND M. YAMAMOTO, *Reverse convolution inequalities and applications to inverse heat source problems*, J. of Ineq. in Pure & App. Math., **3**, 5 (2002), 1–11.
[13] S. SAITOH, V. K. TUAN AND M. YAMAMOTO, *Convolution inequalities and applications*, J. of Ineq. in Pure & App. Math., **4**, 3 (2003), 1–8.
[14] I. N. SNEDDON, *Fourier Transforms*, McGray – Hill, New York, 1951.
[15] E. C. TITCHMARCH, *Introduction to the Theory of Fourier integrals*, Third Edition, Chelsea publishing Co., NewYork, 1986.
[16] N. X. THAO AND N. A. VIRCHENKO, *On the Polyconvolution for the Fourier Cosine, Fourier Sine, and the Kontorovich-Lebedev Integral Transforms*, Ukrai. Math. Jour., **62**, 10 (2010), 1388–1399.
[17] N. X. THAO, N. M. KHOA AND P. T. VAN ANH, *On the polyconvolution for Hartley, Fourier cosine and Fourier sine transforms*, Integral Transforms and Special Functions, **24**, 7 (2013), 517–531.
[18] N. X. THAO, N. M. KHOA AND P. T. VAN ANH, *Polyconvolution and the Toeplitz plus Hankel integral equation*, Electronic Journal of Differential Equations, **2014**, 110 (2014), 1–14.
[19] N. X. THAO, V. K. TUAN AND H. T. VAN ANH, *On the Toeplitz plus Hankel integral equation II*, Integral Transforms and Special Functions, **25**, 1 (2014), 75–84.

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Phi Thi Van Anh
Department of Mathematics
University of Transport and Communications
Lang Thuong, Dong Da, Hanoi, Vietnam
e-mail: vananh.utcmath@gmail.com

Nguyen Xuan Thao
School of Applied Mathematics and Informatics
Hanoi University of Science and Technology
No. 1, Dai Co Viet, Hanoi, Vietnam
e-mail: thaonxbmai@yahoo.com