Mathematical
Inequalities
& Applications
Volume 19, Number 3 (2016), 1131–1139 doi:10.7153/mia-19-83

GLOBAL TRUDINGER–MOSER INEQUALITY ON METRIC SPACES

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(*Communicated by J. Pečarić*)

Abstract. We study Trudinger-Moser type inequalities on the entire metric measure spaces. Moreover, we give the necessary and sufficient conditions under which the Trudinger-Moser inequality holds.

1. Introduction

Morrey and Sobolev embedding theorems [1] play a major role in the existence and regularity problems in PDE. Yudovich [16], Pohozaev [14] and Trudinger [15] obtained the imbedding of Sobolev space $W^{1,p}(\Omega)$ into an Orlicz space in a bounded domain $\Omega \subset \mathbb{R}^n$ in the border line case when $p = n$. These type of embeddings, called the Trudinger-Moser inequalities, play a crucial role in solving many problems in Physics and Geometry. Recently there is a lot of interest in improving and extending this inequality in different directions. One such is the extension of this in the entire space (see [13]). Moreover, the Trudinger-Moser inequality was proved on complete noncompact Riemannian manifolds [17], on the hyperbolic space [20] and on the entire Heisenberg group [19]. More recently, this type of results have been extended to the metrizable abelian groups [4] (see [7] and [3] for the definition of Sobolev spaces on abelian groups). Furthermore the Sobolev space are generalized to metric measure space and Trudinger imbedding has been obtained on balls in [10]. The main objective of the paper is to prove the global version of the Trudinger-Moser inequality on metric measure spaces.

The remainder of the paper is structured as follows. In Section 2, we introduce the notations and recall the notion of Sobolev spaces on general metric measure spaces. Our principal assertion, concerning the global Trudinger-Moser inequality is formulated and proven in Section 3. We also give sufficient and necessary conditions for that the Trudinger-Moser inequality holds.

Mathematics subject classification (2010): 46E35, 30L99.
Keywords and phrases: Trudinger-Moser inequality, metric measure space.

2. Preliminaries

Let (X,ρ,μ) be a metric measure space equipped with a metric ρ and the Borel regular measure μ. We assume throughout the paper that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. In the most part of the paper we shall assume that the metric measure space (X,ρ,μ) is Ahlfors s-regular (we also say that (X,ρ,μ) is a metric measure space with s-regular measure μ). It means that there exists a constant b such that

$$\frac{1}{b}r^s \leqslant \mu\left(B(x,r)\right) \leqslant br^s$$

for all balls $B(x,r) \subset X$ with $r \leqslant \operatorname{diam} X$.

In particular, such kind of spaces are doubling, it means that, there exists a constant $C_d > 0$ such that for every ball $B(x,r)$,

$$\mu\left(B(x,2r)\right) \leqslant C_d \mu\left(B(x,r)\right).$$

We are now in a position to recall the notion of Sobolev spaces on metric measure spaces (see [8]). Let (X,ρ,μ) be a metric measure space. We say that a p-integrable function f belongs to the Hajłasz-Sobolev space $M^{1,p}(X)$ if there exists non negative $g \in L^p(X)$, called a generalized gradient or Hajłasz gradient of f, such that

$$|f(x) - f(y)| \leqslant \rho(x,y)\left(g(x) + g(y)\right) \quad \text{a.e. for} \quad x,y \in X.$$

We equip the space $M^{1,p}(X)$ with the norm

$$\|f\|_{M^{1,p}(X)} = \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all the generalized gradients. Then $M^{1,p}$ is a Banach space. For the basic properties of this kind of spaces we refer to [8, 9, 10, 11, 12].

If f is locally integrable and A is a measurable set then by f_A we denote the integral average of the function f over the set A, that is

$$f_A := \fint_A f d\mu = \frac{1}{\mu(A)} \int_A f d\mu.$$

3. Trudinger inequality

In this section, we show the Trudinger-Moser inequality on the entire metric measure spaces. We start with the following lemma.

LEMMA 1. *Let (X,ρ,μ) is a connected metric measure space with s-regular measure μ, where $s > 1$. There exist C_1, C_2 depending on s and b, such that for any $u \in M^{1,s}(X)$ the following inequality holds*

$$\fint_B \exp\left(\mathscr{C}\left(\frac{|u|}{\|u\|_{L^s(X)} + \|g\|_{L^s(X)}}\right)^{\frac{s}{s-1}}\right) d\mu \leqslant C_2 \exp\left(\frac{C_1}{diam(B)}\right)^{\frac{s}{s-1}}, \tag{1}$$

where $\mathscr{C} = \left(\frac{C_1}{2(2b)^{\frac{1}{s}}}\right)^{\frac{s}{s-1}}$ *and* g *is a Hajłasz gradient of* u.

Proof. First of all let us recall the local version of the Trudinger-Moser inequality ([10]). Namely, there exist C_1 and C_2 such that

$$\fint_B \exp\left(\frac{C_1\mu(B)^{\frac{1}{s}}|u-u_B|}{\operatorname{diam}(B)\|g\|_{L^s(6B)}}\right)^{\frac{s}{s-1}} d\mu \leqslant C_2, \tag{2}$$

where $6B$ is the ball with the same center as B and with radius six-time that of B. Since the measure μ is s-regular, by the Hölder inequality we have

$$\begin{aligned}
&\frac{C_1}{2b^{\frac{1}{s}}}\frac{|u|}{\|u\|_{L^s(X)}+\|g\|_{L^s(X)}}\\
&\leqslant \frac{C_1\mu(B)^{\frac{1}{s}}}{\operatorname{diam}(B)}\frac{|u-u_B|}{\|u\|_{L^s(X)}+\|g\|_{L^s(X)}}+\frac{C_1\mu(B)^{\frac{1}{s}}}{\operatorname{diam}(B)}\frac{|u_B|}{\|u\|_{L^s(X)}+\|g\|_{L^s(X)}}\\
&\leqslant \frac{C_1\mu(B)^{\frac{1}{s}}}{\operatorname{diam}(B)}\frac{|u-u_B|}{\|g\|_{L^s(6B)}}+\frac{C_1}{\operatorname{diam}(B)}.
\end{aligned}$$

Thus

$$\left(\frac{C_1}{2(2b)^{\frac{1}{s}}}\frac{|u|}{\|u\|_{L^s(X)}+\|g\|_{L^s(X)}}\right)^{\frac{s}{s-1}} \leqslant \left(\frac{C_1\mu(B)^{\frac{1}{s}}}{\operatorname{diam}(B)}\frac{|u-u_B|}{\|g\|_{L^s(6B)}}\right)^{\frac{s}{s-1}}+\left(\frac{C_1}{\operatorname{diam}(B)}\right)^{\frac{s}{s-1}}.$$

Finally, from inequality (2) the proof follows. □

Next, we state and prove our principal assertion.

THEOREM 1. *Suppose* (X,ρ,μ) *is a connected metric measure space with* s*-regular measure* μ, *where* $s>1$.

i) If $\beta < \mathscr{C}$, *then there exists* B *such that*[1]

$$\sup_{\|u\|_{M^{1,p}}\leqslant 1}\int_X\left(e^{\beta|u|^{\frac{s}{s-1}}}-\sum_{k=0}^{\lceil s\rceil-2}\frac{\beta^k u^{k\frac{s}{s-1}}}{k!}\right)d\mu < B. \tag{3}$$

ii) For any $\beta>0$ *and for any* $u\in M^{1,p}(X)$ *we have*

$$\int_X\left(e^{\beta|u|^{\frac{s}{s-1}}}-\sum_{k=0}^{\lceil s\rceil-2}\frac{\beta^k u^{k\frac{s}{s-1}}}{k!}\right)d\mu < \infty.$$

[1] $\lceil\alpha\rceil = \min\{k\in\mathbb{Z} : k\geqslant\alpha\}$

The proof of the theorem relies on methods of Yang (see [17]).

Proof. i) We shall need the following lemma.

LEMMA 2. *Let $u \in M^{1,s}(X)$ such that $\operatorname{supp} u, \operatorname{supp} g \subset B(x_0,r)$, where $x_0 \in X$ and $r > 0$. If*

$$\int_{B(x_0,r)} |u|^s d\mu + \int_{B(x_0,r)} g^s d\mu \leqslant 1,$$

then

$$\int_{B(x_0,r)} \left(e^{\mathscr{C}|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s \rceil -2} \frac{\mathscr{C}^k u^{k\frac{s}{s-1}}}{k!} \right) d\mu$$
$$\leqslant C_2 \mu(B) \exp\left(\frac{C_1}{diam(B)} \right)^{\frac{s}{s-1}} \left(\int_{B(x_0,r)} |u|^s d\mu + \int_{B(x_0,r)} g^s d\mu \right).$$

Proof. Let us introduce the following quantity

$$F(A,f,h) = \left(\int_A |f|^s d\mu + \int_A h^s d\mu \right)^{\frac{1}{s}}.$$

Since $F(B(x_0,r),u,g) \leqslant 1$ and $\lceil s \rceil \geqslant s$, we have

$$e^{\mathscr{C}|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s \rceil -2} \frac{\mathscr{C}^k u^{k\frac{s}{s-1}}}{k!} = \sum_{k=\lceil s \rceil -1}^{\infty} \frac{\mathscr{C}^k u^{k\frac{s}{s-1}}}{k!}$$
$$\leqslant \left(\int_{B(x_0,r)} |u|^s d\mu + \int_{B(x_0,r)} g^s d\mu \right) \sum_{k=\lceil s \rceil -1}^{\infty} \frac{\mathscr{C}^k \left(\frac{u}{F(B(x_0,r),u,g)} \right)^{k\frac{s}{s-1}}}{k!}$$
$$\leqslant \left(\int_{B(x_0,r)} |u|^s d\mu + \int_{B(x_0,r)} g^s d\mu \right) \exp \mathscr{C} \left(\frac{u}{F(B(x_0,r),u,g)} \right)^{\frac{s}{s-1}}.$$

This together with Lemma 1 completes the proof of Lemma 2. □

We are now in a position to continue the proof of Theorem 1. In order to show the theorem we need to use the following Covering lemma (Lemma 2.1 in [2]).

LEMMA 3. (Covering lemma) *Let (X,ρ,μ) be a metric measure space with doubling measure and $r > 0$. Then, there exists a sequence $\{x_i\} \subset X$ such that for any $\delta > r$:*

a) $X = \bigcup_i B(x_i,r)$;

b) For any $x \in X$, x belongs to at most $C_d^6 \left(\frac{\delta}{r}\right)^{\log_2 C_d}$ balls $B(x_i,\delta)$.

Let $u \in M^{1,p}(X)$ with the generalized gradient g such that

$$\|u\|_{L^s(X)} + \|g\|_{L^s(X)} \leqslant 1.$$

For $x_i \in X$ and $r > 0$ we define the following Lipschitz cut-off function

$$\phi_{x_i,r}(x) := \begin{cases} \frac{1}{r}(2r - \rho(x,x_i)) & \text{if } x \in B(x_i,2r) \setminus B(x_i,r) \\ 1 & \text{if } x \in B(x_i,r) \\ 0 & \text{if } x \in X \setminus B(x_i,2r) \end{cases},$$

where the Lipschitz constant $L_r = \frac{1}{r}$ does not depend on x_i.

Then $\phi_{x_i,r}u \in M^{1,p}(X)$. Indeed, taking

$$g_{x_i}(x) = (|u(x)|L_r + g(x))\chi_{B(x_i,2r)},$$

one can easily check that g_{x_i} is the Hajłasz gradient of $\phi_{x_i,r}u$. Furthermore, the Minkowski inequality together with the elementary inequality $(a+b)^\delta \leqslant a^\delta + b^\delta$ with $0 < \delta < 1$ lead to

$$\begin{aligned} &\left(\int_{B(x_i,2r)} |\phi_{x_i,r}u|^s d\mu + \int_{B(x_i,2r)} g_{x_i}^s d\mu\right)^{\frac{1}{s}} \\ &\leqslant \left(\int_{B(x_i,2r)} |u|^s d\mu\right)^{\frac{1}{s}} + \left(\int_{B(x_i,2r)} \left(\frac{1}{r}|u| + g\right)^s d\mu\right)^{\frac{1}{s}} \\ &\leqslant \left(1 + \frac{1}{r}\right)\left(\|u\|_{L^s(X)} + \|g\|_{L^s(X)}\right). \end{aligned}$$

Thus, taking

$$\phi_{x_i,r}^b = \frac{\phi_{x_i,r}}{\left(1+\frac{1}{r}\right)}$$

we have that $\phi_{x_i,r}^b u \in M^{1,p}(X)$ and

$$\int_{B(x_i,2r)} |\phi_{x_i,r}^b u|^s d\mu + \int_{B(x_i,2r)} \left(g_{x_i}^b\right)^s d\mu \leqslant 1.$$

Subsequently, by the Covering Lemma we get

$$\begin{aligned} &\int_X \left(e^{\beta|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\beta^k u^{k\frac{s}{s-1}}}{k!}\right) d\mu \leqslant \sum_{i=1}^{\infty} \int_{B(x_i,r)} \left(e^{\beta|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\beta^k u^{k\frac{s}{s-1}}}{k!}\right) d\mu \\ &= \sum_{i=1}^{\infty} \int_{B(x_i,r)} \left(e^{\beta\left(1+\frac{1}{r}\right)^{\frac{s}{s-1}} |\phi_{x_i,r}^b u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\left(\beta\left(1+\frac{1}{r}\right)^{\frac{s}{s-1}}\right)^k |\phi_{x_i,r}^b u|^{k\frac{s}{s-1}}}{k!}\right) d\mu. \end{aligned}$$

Next, since $\beta < \mathscr{C}$, we can take r large such that $\beta\left(1+\frac{1}{r}\right)^{\frac{s}{s-1}} \leqslant \mathscr{C}$ and using Lemma 2, we have

$$\begin{aligned}
&\int_X \left(e^{\beta|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\beta^k |u|^{k\frac{s}{s-1}}}{k!} \right) d\mu \\
&\leqslant C_2 b r^s \exp\left(\frac{C_1}{2r}\right)^{\frac{s}{s-1}} \sum_{i=1}^{\infty} \left(\int_{B(x_i,2r)} |\phi^b_{x_i,r} u|^s d\mu + \int_{B(x_i,2r)} \left(g^b_{x_i}\right)^s d\mu \right) \\
&\leqslant C_2 b r^s \exp\left(\frac{C_1}{2r}\right)^{\frac{s}{s-1}} \left(1+\frac{2^{s-1}}{r^s}\right) \sum_{i=1}^{\infty} \left(\int_{B(x_i,2r)} |u|^s d\mu + \int_{B(x_i,2r)} g^s d\mu \right) \\
&\leqslant C_2 b r^s \exp\left(\frac{C_1}{2r}\right)^{\frac{s}{s-1}} \left(1+\frac{2^{s-1}}{r^s}\right) (b^2 2^s)^7 \left(\int_X |u|^s d\mu + \int_X g^s d\mu \right),
\end{aligned}$$

where in the last line we applied the Covering lemma. This completes the proof of i).

ii) First of all, let us define the subset of Lipschitz function on X with support on bunded sets

$$\mathrm{Lip}_0(X) := \{\phi \in \mathrm{Lip}(X) : \mathrm{supp}\phi \text{ is bounded}\}.$$

We shall need the following density lemma.

LEMMA 4. *Let $1 < p < \infty$, then the set $Lip_0(X)$ is dense in $M^{1,p}(X)$.*

Proof. Since the set $\mathrm{Lip}(X) \cap M^{1,p}(X)$ is dense in $M^{1,p}(X)$ (see Theorem 5 in [8]), it is enough to show that $\mathrm{Lip}_0(X)$ is dense in $\mathrm{Lip}(X) \cap M^{1,p}(X)$. Thus, let us fix $u \in \mathrm{Lip}(X) \cap M^{1,p}(X)$ and denote by $g \in L^p(X)$ a Hajłasz gradient of u. For $\varepsilon > 0$ there exists $R_\varepsilon > 1$ such that

$$\int_{X\setminus B(x_0,R_\varepsilon)} (|u|^p + g^p)\, d\mu \leqslant \varepsilon.$$

Next, let us define

$$\phi_\varepsilon(x) := \begin{cases} \frac{1}{R_\varepsilon}(2R_\varepsilon - \rho(x,x_0)) & \text{if } x \in B(x_0,2R_\varepsilon)\setminus B(x_0,R_\varepsilon) \\ 1 & \text{if } x \in B(x_0,R_\varepsilon) \\ 0 & \text{if } x \in X \setminus B(x_0,2R_\varepsilon)\,, \end{cases}$$

and $u_\varepsilon = u\phi_\varepsilon \in \mathrm{Lip}_0(X)$. Next, we show that $u_\varepsilon \to u$ in $M^{1,p}(X)$

$$\int_X |u_\varepsilon - u|^p d\mu = \int_{X\setminus B(x_0,R_\varepsilon)} |u|^p |1-\phi_\varepsilon|^p d\mu \leqslant \int_{X\setminus B(x_0,R_\varepsilon)} |u|^p d\mu \leqslant \varepsilon,$$

this yields $\|u - u_\varepsilon\|_{L^p(X)} \to 0$ as $\varepsilon \to 0$. Moreover, by a direct calculations we get that the function

$$g_\varepsilon = (|u| + g)\chi_{X\setminus B(x_0,R_\varepsilon)}$$

satisfies

$$|(u_\varepsilon - u)(x) - (u_\varepsilon - u)(y)| \leqslant \rho(x,y)(g_\varepsilon(x) + g_\varepsilon(y))$$

for almost every $x,y \in X$. We have

$$\int_X g_\varepsilon^p d\mu \leqslant 2^{p-1} \int_{X \setminus B(x_0,R_\varepsilon)} (|u|^p + g^p)\, d\mu \leqslant 2^p \varepsilon,$$

and thus we get that $\|u_\varepsilon - u\|_{M^{1,p}(X)} \to 0$ as $\varepsilon \to 0$. This completes the proof. □

We are now in a position to continue the proof of the theorem. Let us take $u \in M^{1,p}(X)$, then by virtue of the above lemma for any $\varepsilon > 0$ there exists $u_\varepsilon \in \mathrm{Lip}_0(X)$ such that $\|u - u_\varepsilon\|_{M^{1,p}(X)} \leqslant \varepsilon$. Thus taking into account Lemma 2.2 form [18] we get

$$\begin{aligned}
&e^{\beta|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\beta^k u^{k\frac{s}{s-1}}}{k!} = \sum_{\lceil s\rceil-1}^{\infty} \frac{\beta^k |u|^{k\frac{s}{s-1}}}{k!} \\
&\leqslant \sum_{\lceil s\rceil-1}^{\infty} \frac{\beta^k}{k!} \left(2^{\frac{1}{s-1}} |u-u_\varepsilon|^{\frac{s}{s-1}} + 2^{\frac{1}{s-1}} |u_\varepsilon|^{\frac{s}{s-1}}\right)^k \\
&\leqslant \frac{1}{2} \sum_{\lceil s\rceil-1}^{\infty} \frac{\beta^k}{k!} \left(2^{\frac{1}{s-1}} \varepsilon^{\frac{s}{s-1}} \left(\frac{|u-u_\varepsilon|}{\|u-u_\varepsilon\|_{M^{1,p}(X)}}\right)^{\frac{s}{s-1}}\right)^k + \frac{1}{2} \sum_{\lceil s\rceil-1}^{\infty} \frac{\beta^k}{k!} \left(2^{\frac{1}{s-1}} |u_\varepsilon|^{\frac{s}{s-1}}\right)^k \\
&= I_1 + I_2.
\end{aligned}$$

Let us fix $\varepsilon > 0$ such that $\frac{1}{2^{s-1}} \varepsilon^{\frac{s}{s-1}} < \mathscr{C}$, then from $i)$ we get that

$$\int_X I_1 d\mu < 2B.$$

Furthermore, since $u_\varepsilon \in \mathrm{Lip}_0(X)$, we have

$$\int_X I_2 d\mu < \infty.$$

The proof is complete. □

Finally, we give the following characterization of metric spaces on which the Trudinger-Moser inequality holds.

THEOREM 2. *Let (X,ρ,μ) be a connected metric measure space with doubling measure μ. Let $C_d > 2$ be a doubling constant and $s = \log_2 C_d$, then the following conditions are equivalent*

i) There exist β and B such that

$$\sup_{\|u\|_{M^{1,s}} \leqslant 1} \int_X \left(e^{\beta|u|^{\frac{s}{s-1}}} - \sum_{k=0}^{\lceil s\rceil-2} \frac{\beta^k u^{k\frac{s}{s-1}}}{k!}\right) d\mu < B. \tag{4}$$

ii) There exists $\theta > 0$ such that for any $x \in X$, the following inequality holds

$$\mu(B(x,1)) \geqslant \theta.$$

Proof. Let us assume that inequality (4) holds. Hence, for $u \in M^{1,s}(X)$ from (4) we get the following inequality

$$\int_X \sum_{k=\lceil s \rceil -1}^{\infty} \frac{\beta^k \left(\frac{u}{\|u\|_{M^{1,p}}} \right)^{k \frac{s}{s-1}}}{k!} d\mu < B.$$

Thus, for any $k \geqslant \lceil s \rceil - 1$ we have

$$\left(\int_X |u|^{\frac{s}{s-1}k} d\mu \right)^{\frac{s-1}{sk}} \leqslant \left(\frac{k!}{\beta^k} B \right)^{\frac{s-1}{sk}} \|u\|_{M^{1,s}(X)}.$$

Hence, by the interpolation inequality we get that for any $q \geqslant s$ we get

$$M^{1,s}(X) \hookrightarrow L^q(X).$$

In view of Theorem 3.2 from [2], we conclude

$$\mu(B(x,r)) \geqslant b(\delta) r^{\delta},$$

where $\frac{1}{\delta} = \frac{1}{s} - \frac{1}{q}$ and $r \leqslant 1$. In particular, we have that there exists $\theta > 0$ such that for any $x \in X$, the following estimate holds $\mu(B(x,1)) \geqslant \theta$.

Now, if we assume that the measure is doubling such that $\mu(B(x,1)) \geqslant \theta$, we get for any $r \leqslant 1$

$$\mu(B(x,r)) \geqslant \theta C_d^{-2} r^s.$$

Hence, we get a version of inequality (1) for balls with $\operatorname{diam}(B) \leqslant \frac{1}{2}$. Next, we can proceed analogously to the proof of Theorem 1 and we obtain inequality (4). The details are left to the reader. Thus, the whole proof of Theorem 2 is complete. □

REMARK 1. To get necessary conditions under which the Trudinger-Moser inequality holds we do not need assume that the metric space is connected.

REMARK 2. Condition *ii*) appears in natural way in the characterization of relatively compact sets in Lebesgue space (see [5, 6]).

Acknowledgements. Most part of the paper has been performed during the visit of the author to the TIFR Centre for Applicable Mathematics in Bangalore. P.G. wish to thank for its hospitality. Special thanks are also given to the referee for valuable comments and kind advice.

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(Received February 10, 2016)

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