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NEW OSTROWSKI LIKE INEQUALITIES FOR GG–CONVEX AND GA–CONVEX FUNCTIONS

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Abstract. In this paper, we established some Ostrowski like integral inequalities for functions whose derivatives of absolute values are GG-convex and GA-convex functions via a new integral identity. General results are obtained using the weighted Montgomery identity. Also, particular results for the weight function $w(t) = \frac{1}{t\log b/a}$ are given.

1. Introduction

We will start with the definiton of convexity that has utilization in all branches of mathematics and has several applications in mathematical analysis, optimization and statictics.

The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on an interval I, if the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f'(x)| \leqslant M$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leqslant \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

This inequality is well known in the literature as the Ostrowski inequality.

In [4], Niculescu mentioned the following considerable definitions:

The GG-convex functions (called also multiplicatively convex functions) are those functions $f : I \to J$ (I, J are subintervals of $(0,\infty)$) such that

$$x,y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leqslant f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1}$$

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The class of all GA-convex functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$x,y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leqslant (1-\lambda)f(x)+\lambda f(y). \tag{2}$$

Besides, recall that the condition of GA-convexity is $x^2 f'' + xf' \geqslant 0$ which implies all twice differentiable nondecreasing convex functions are also GA-convex.

For recent results, generalizations, improvements and counterparts see the papers [3], [4], [6], [7], [8], [9] and references therein.

In [2], the authors have mentioned the weighted Montgomery identity as following (see also [1] and [5]):

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt, \tag{3}$$

where $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, $f' : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is some normalized weight function, i.e. an integrable function satisfying $\int_a^b w(t)dt = 1$, $W(t) = \int_a^t w(x)dx$ for $t \in [0,1]$. The weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ W(t)-1, & x < t \leqslant b. \end{cases}$$

For the uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, (3) reduces to the Montgomery identity,

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{4}$$

where

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leqslant t \leqslant x \\ \dfrac{t-b}{b-a}, & x < t \leqslant b. \end{cases}$$

In [2], Aljinović and Pečarić have given a discrete analogue of the weighted Montgomery identity for mappings of two variables and have proved some new discrete Ostrowski type inequalities.

The main aim of this paper is to prove some new integral inequalities for GG-convex and GA-convex functions by using a new integral identity.

2. New inequalities for GG- and GA-convex functions

We will give a new integral identity which is emboided in the following lemma to obtain our results. The proof of identity is based on using the weighted montgomery identity.

LEMMA 1. *Let $I \subset (0,\infty)$ be an interval, $a,b \in I^o$, $a<b$. Let w be a nonnegative integrable function on $[a,b]$ such that $\int_a^b w(x)dx = 1$ and let $W(t) = \int_a^t w(x)dx$. Let $f : I \to \mathbb{R}$ be a function differentiable on I^o. Then for $x \in [a,b]$*

$$f(x) - \int_a^b w(t)f(t)dt = \log\frac{x}{a} I\left(f', a, x; W\right) + \log\frac{b}{x} I\left(f', x, b; W-1\right)$$

holds, where $I(F, v, u; Q) = \int_0^1 Q\left(u^{1-\tau}v^\tau\right) F\left(u^{1-\tau}v^\tau\right) u^{1-\tau}v^\tau d\tau$, provided that all integrals exist.

Proof. Using the weighted Montgomery identity we get

$$\begin{aligned} f(x) - \int_a^b w(t)f(t)dt &= \int_a^b P(w,t)f'(t)dt \\ &= \int_a^x W(t)f'(t)dt + \int_x^b (W(t)-1)f'(t)dt \\ &= \int_0^1 W\left(x^{1-\tau}a^\tau\right) f'\left(x^{1-\tau}a^\tau\right) x^{1-\tau}a^\tau \log\frac{x}{a} d\tau \\ &\quad + \int_0^1 \left(W(b^{1-\tau}x^\tau) - 1\right) f'(b^{1-\tau}x^\tau) b^{1-\tau}x^\tau \log\frac{b}{x} d\tau \\ &= \log\frac{x}{a} I\left(f', a, x; W\right) + \log\frac{b}{x} I\left(f', x, b; W-1\right) \end{aligned}$$

where in the first integral we use substitution $t = x^{1-\tau}a^\tau$, and in the second integral we use substitution $t = b^{1-\tau}x^\tau$.

REMARK 1. If we choose $w(u) = \frac{1}{u\log b/a}$ in Lemma 1, we get the following equality:

$$\begin{aligned} &\log\frac{b}{a} f(x) - \int_a^b \frac{f(u)}{u} du \\ &= \log^2\frac{x}{a} \int_0^1 (1-\tau)x^{1-\tau}a^\tau f'\left(x^{1-\tau}a^\tau\right) d\tau - \log^2\frac{b}{x} \int_0^1 \tau b^{1-\tau}x^\tau f'\left(b^{1-\tau}x^\tau\right) d\tau. \end{aligned}$$

A new inequality for GG-convex functions is given in the following theorem.

THEOREM 1. *Let $I \subset (0,\infty)$ be an interval, $a,b \in I^o$, $a<b$. Let w be a nonnegative integrable function on $[a,b]$ such that $\int_a^b w(x)dx = 1$ and let $W(t) = \int_a^t w(x)dx$. Let $f : I \to \mathbb{R}$ be a function differentiable on I^o. If $|f'|^q$ is GG-convex function on $[a,b]$ for some $q>1$, then for all $x \in [a,b]$, following inequality holds*

$$
\begin{aligned}
&\left|f(x)-\int_a^b w(t)f(t)dt\right| \\
&\leqslant \log^{\frac{1}{q}}\frac{x}{a}\left(\int_a^x W(t)dt\right)^{\frac{1}{p}}\left(\frac{x\left|f'(x)\right|^q}{\log A_{a,x}}\left[-W(x)+\int_a^x w(t)A_{a,x}^{\frac{\log t/x}{\log a/x}}dt\right]\right)^{\frac{1}{q}} \\
&\quad+\log^{\frac{1}{q}}\frac{b}{x}\left(\int_x^b (1-W(t))\,dt\right)^{\frac{1}{p}}\left(\frac{b\left|f'(b)\right|^q}{\log A_{x,b}}\left[(1-W(x))A_{x,b}-\int_x^b w(t)A_{x,b}^{\frac{\log t/b}{\log x/b}}dt\right]\right)^{\frac{1}{q}}
\end{aligned}
$$

where $\frac{1}{p}=1-\frac{1}{q}$ *and* $A_{v,u}=\frac{v|f'|^q(v)}{u|f'|^q(u)}$, *provided that all integrals exist.*

Proof. From Lemma 1 and the Hölder inequality, we get

$$
\begin{aligned}
&\left|f(x)-\int_a^b w(t)f(t)dt\right| \qquad (5)\\
&\leqslant \log\frac{x}{a}I\left(\left|f'\right|,a,x;W\right)+\log\frac{b}{x}I\left(\left|f'\right|,x,b;1-W\right) \\
&\leqslant \log\frac{x}{a}\left(\int_0^1 W\left(x^{1-\tau}a^{\tau}\right)x^{1-\tau}a^{\tau}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 W\left(x^{1-\tau}a^{\tau}\right)\left|f'\right|^q\left(x^{1-\tau}a^{\tau}\right)x^{1-\tau}a^{\tau}d\tau\right)^{\frac{1}{q}} \\
&\quad+\log\frac{b}{x}\left(\int_0^1\left(1-W(b^{1-\tau}x^{\tau})\right)b^{1-\tau}x^{\tau}d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\left(1-W(b^{1-\tau}x^{\tau})\right)\left|f'\right|^q\left(b^{1-\tau}x^{\tau}\right)b^{1-\tau}x^{\tau}d\tau\right)^{\frac{1}{q}} \\
&=\log\frac{x}{a}\left(I\left(1,a,x;W\right)\right)^{\frac{1}{p}}\left(I\left(|f'|^q,a,x;W\right)\right)^{\frac{1}{q}} \\
&\quad+\log\frac{b}{x}\left(I\left(1,x,b;1-W\right)\right)^{\frac{1}{p}}\left(I\left(|f'|^q,x,b;1-W\right)\right)^{\frac{1}{q}},
\end{aligned}
$$

where $I(F,v,u;Q)$ is defined as in Lemma 1. Using the GG-convexity of $|f'|^q$ and integration by parts, we get

$$
I\left(\left|f'\right|^q,v,u;Q\right)\leqslant\frac{u\left|f'\right|^q(u)}{\log A_{v,u}}\left[-Q(u)+Q(v)A_{v,u}+\int_v^u Q'(t)A_{v,u}^{\frac{\log t/u}{\log v/u}}dt\right]. \qquad (6)
$$

Calculating $I(1,v,u;Q)$, we get

$$
I(1,v,u;Q)=\frac{1}{\log u/v}\int_v^u Q(t)dt. \qquad (7)
$$

Using (6) and (7) in (5), we get the desired result.

REMARK 2. In Theorem 1, if we choose $q \to 1,$ we get the following inequality:

$$\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t) dt \right| \\
&\leqslant \log \frac{x}{a} \left(\frac{x|f'(x)|}{\log A_{a,x}} \left[-W(x) + \int_a^x w(t) A_{a,x}^{\frac{\log t/x}{\log a/x}} dt \right] \right) \\
&\quad + \log \frac{b}{x} \left(\frac{b|f'(b)|}{\log A_{x,b}} \left[(1-W(x)) A_{x,b} - \int_x^b w(t) A_{x,b}^{\frac{\log t/b}{\log x/b}} dt \right] \right).
\end{aligned}$$

We can consider inequalities for different weights $w,$ but here we give only result for a particular weight $w(t) = \frac{1}{t \log b/a}.$

COROLLARY 1. *If assumptions of Theorem 1 are satisfied with* $w(t) = \frac{1}{t \log b/a},$ *then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\log b - \log a} \int_a^b \frac{f(t)}{t} dt \right| \\
&\leqslant \frac{\log^{1+\frac{1}{q}} \frac{x}{a}}{\log \frac{b}{a}} (x - L(a,x))^{\frac{1}{p}} \left[\frac{x|f'|^q(x) - L(a|f'|^q(a), x|f'|^q(x))}{\log \frac{x|f'|^q(x)}{a|f'|^q(a)}} \right]^{\frac{1}{q}} \\
&\quad + \frac{\log^{1+\frac{1}{q}} \frac{b}{x}}{\log \frac{b}{a}} (L(x,b) - x)^{\frac{1}{p}} \left[\frac{L(x|f'|^q(x), b|f'|^q(b)) - x|f'|^q(x)}{\log \frac{b|f'|^q(b)}{x|f'|^q(x)}} \right]^{\frac{1}{q}}
\end{aligned}$$

where $L(x,y) = \frac{y-x}{\log y - \log x}.$

THEOREM 2. *Under the assumptions of Theorem 1, we get the following inequality*

$$\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t) dt \right| \\
&\leqslant x|f'(x)| \log^{\frac{1}{q}} \frac{x}{a} \left(\int_a^x \frac{W^p(t)}{t} dt \right)^{\frac{1}{p}} \left(\frac{1}{\log C_{a,x}} [C_{a,x} - 1] \right)^{\frac{1}{q}} \\
&\quad + b|f'(b)| \log^{\frac{1}{q}} \frac{b}{x} \left(\int_x^b \frac{(1-W(t))^p}{t} dt \right)^{\frac{1}{p}} \left(\frac{1}{\log C_{x,b}} [C_{x,b} - 1] \right)^{\frac{1}{q}}
\end{aligned}$$

where $C_{v,u} = \frac{v^q |f'|^q(v)}{u^q |f'|^q(u)}.$

Proof. From Lemma 1, using the Hölder inequality and GG-convexity of $|f'|^q$ we get

$$\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t) dt \right| \qquad (8)\\
&\leqslant \log\frac{x}{a} \left(\int_0^1 W^p\left(x^{1-\tau}a^{\tau}\right) d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^q \left(x^{1-\tau}a^{\tau}\right) x^{(1-\tau)q} a^{\tau q} d\tau \right)^{\frac{1}{q}} \\
&\quad + \log\frac{b}{x} \left(\int_0^1 \left(1 - W(b^{1-\tau}x^{\tau})\right)^p d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^q \left(b^{1-\tau}x^{\tau}\right) b^{(1-\tau)q} x^{\tau q} d\tau \right)^{\frac{1}{q}} \\
&\leqslant \log\frac{x}{a} \left(\int_0^1 W^p\left(x^{1-\tau}a^{\tau}\right) d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^{(1-\tau)q}(x) |f'|^{\tau q}(a) x^{(1-\tau)q} a^{\tau q} d\tau \right)^{\frac{1}{q}} \\
&\quad + \log\frac{b}{x} \left(\int_0^1 \left(1 - W(b^{1-\tau}x^{\tau})\right)^p d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^{(1-\tau)q}(b) |f'|^{\tau q}(x) b^{(1-\tau)q} x^{\tau q} d\tau \right)^{\frac{1}{q}}.
\end{aligned}$$

By a simple computation, we get

$$\int_0^1 Q^p\left(u^{1-\tau}v^{\tau}\right) d\tau = \frac{1}{\log\frac{u}{v}} \int_v^u \frac{Q^p(t)}{t} dt \qquad (9)$$

and

$$\int_0^1 C_{v,u}^{\tau} d\tau = \frac{C_{v,u} - 1}{\log C_{v,u}}, \qquad (10)$$

where Q is a function and $C_{v,u}$ is defined in theorem. Using (9) and (10) in (8) we get the desired result.

COROLLARY 2. *In Theorem 2, if we choose* $w(t) = \frac{1}{t \log b/a}$*, then we get the following inequality:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\log b - \log a} \int_a^b \frac{f(t)}{t} dt \right| \\
&\leqslant \frac{1}{\log b - \log a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\log^2\frac{x}{a} \left(L\left(a^q |f'|^q(a), x^q |f'|^q(x)\right)\right)^{\frac{1}{q}} \right. \\
&\quad \left. + \log^2\frac{b}{x} \left(L\left(x^q |f'|^q(x), b^q |f'|^q(b)\right)\right)^{\frac{1}{q}} \right].
\end{aligned}$$

THEOREM 3. *Under the assumptions of Theorem 1, we get the following inequality*

$$
\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t)dt \right| \\
\leqslant & \left|f'(x)\right| \log^{\frac{1}{q}} \frac{x}{a} \left(\int_a^x W(t) t^{p-1} dt \right)^{\frac{1}{p}} \left(\frac{1}{\log B_{a,x}} \left[-W(x) + \int_a^x B_{a,x}^{\frac{\log t/x}{\log a/x}} w(t) dt \right] \right)^{\frac{1}{q}} \\
&+ \left|f'(b)\right| \log^{\frac{1}{q}} \frac{b}{x} \left(\int_x^b (1 - W(t)) t^{p-1} dt \right)^{\frac{1}{p}} \\
&\times \left(\frac{1}{\log B_{x,b}} \left[(1 - W(x)) B_{x,b} - \int_x^b B_{x,b}^{\frac{\log t/b}{\log x/b}} w(t) dt \right] \right)^{\frac{1}{q}}
\end{aligned}
$$

where $B_{v,u} = \frac{|f'|^q(v)}{|f'|^q(u)}$.

Proof. From Lemma 1, using the Hölder inequality and GG-convexity of $|f'|^q$ we get

$$
\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t)dt \right| \qquad (11) \\
\leqslant & \log \frac{x}{a} \left(\int_0^1 W\left(x^{1-\tau} a^{\tau}\right) \left(x^{1-\tau} a^{\tau}\right)^p d\tau \right)^{\frac{1}{p}} \left(\int_0^1 W\left(x^{1-\tau} a^{\tau}\right) \left|f'\right|^q \left(x^{1-\tau} a^{\tau}\right) d\tau \right)^{\frac{1}{q}} \\
&+ \log \frac{b}{x} \left(\int_0^1 \left(1 - W(b^{1-\tau} x^{\tau})\right) \left(b^{1-\tau} x^{\tau}\right)^p d\tau \right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left(1 - W(b^{1-\tau} x^{\tau})\right) \left|f'\right|^q \left(b^{1-\tau} x^{\tau}\right) d\tau \right)^{\frac{1}{q}} \\
\leqslant & \log \frac{x}{a} \left(\int_0^1 W\left(x^{1-\tau} a^{\tau}\right) \left(x^{1-\tau} a^{\tau}\right)^p d\tau \right)^{\frac{1}{p}} \left(\int_0^1 W\left(x^{1-\tau} a^{\tau}\right) \left|f'\right|^{(1-\tau)q}(x) \left|f'\right|^{\tau q}(a) d\tau \right)^{\frac{1}{q}} \\
&+ \log \frac{b}{x} \left(\int_0^1 \left(1 - W(b^{1-\tau} x^{\tau})\right) \left(b^{1-\tau} x^{\tau}\right)^p d\tau \right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left(1 - W(b^{1-\tau} x^{\tau})\right) \left|f'\right|^{(1-\tau)q}(b) \left|f'\right|^{\tau q}(x) d\tau \right)^{\frac{1}{q}}.
\end{aligned}
$$

By a simple computation, we get

$$
\int_0^1 Q\left(u^{1-\tau} v^{\tau}\right) \left(u^{1-\tau} v^{\tau}\right)^p d\tau = \frac{1}{\log \frac{u}{v}} \int_v^u Q(t) t^{p-1} dt \qquad (12)
$$

and

$$
\int_0^1 Q\left(u^{1-\tau} v^{\tau}\right) B_{v,u}^{\tau} d\tau = \frac{1}{\log B_{v,u}} \left[Q(v) B_{v,u} - Q(u) + \int_v^u B_{v,u}^{\frac{\log t/u}{\log v/u}} Q'(t) dt \right]. \qquad (13)
$$

Using (12) and (13) in (11) we get the desired result.

COROLLARY 3. *In Theorem 3, if we choose* $w(t) = \frac{1}{t \log b/a}$, *we get the following inequality:*

$$\left| f(x) - \frac{1}{\log b - \log a} \int_a^b \frac{f(t)}{t} dt \right|$$
$$\leqslant \frac{1}{p^{\frac{1}{p}} \log b/a} \left\{ \log^{1+\frac{1}{q}} \frac{x}{a} \left(x^p - L(a^p, x^p)\right)^{\frac{1}{p}} \left(\frac{L(|f'|^q(a), |f'|^q(x)) - |f'|^q(x)}{\log \frac{|f'|^q(a)}{|f'|^q(x)}} \right)^{\frac{1}{q}} \right.$$
$$\left. + \log^{1+\frac{1}{q}} \frac{x}{b} \left(L(x^p, b^p) - x^p\right)^{\frac{1}{p}} \left(\frac{|f'|^q(x) - L(|f'|^q(x), |f'|^q(b))}{\log \frac{|f'|^q(x)}{|f'|^q(b)}} \right)^{\frac{1}{q}} \right\}.$$

THEOREM 4. *Under the assumptions of Theorem 1, we get the following inequality*

$$\left| f(x) - \int_a^b w(t) f(t) dt \right|$$
$$\leqslant \log^{\frac{1}{q}} \frac{x}{a} \left(\int_a^x W^p(t) dt \right)^{\frac{1}{p}} \left(\frac{x |f'|^q(x)}{\log A_{a,x}} (A_{a,x} - 1) \right)^{\frac{1}{q}}$$
$$+ \log^{\frac{1}{q}} \frac{b}{x} \left(\int_x^b (1 - W(t))^p dt \right)^{\frac{1}{p}} \left(\frac{b |f'|^q(b)}{\log A_{x,b}} (A_{x,b} - 1) \right)^{\frac{1}{q}},$$

where $A_{v,u}$ *defined as in Theorem 1.*

Proof. By a similar argument to the proof of previous theorems, since $|f'|^q$ is GG-convex function on $[a,b]$, from Lemma 1 and the Hölder integral inequality, we get

$$\left| f(x) - \int_a^b w(t) f(t) dt \right| \tag{14}$$
$$\leqslant \log \frac{x}{a} \left(\int_0^1 W^p\left(x^{1-\tau} a^\tau\right) x^{1-\tau} a^\tau d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^q \left(x^{1-\tau} a^\tau\right) x^{1-\tau} a^\tau d\tau \right)^{\frac{1}{q}}$$
$$+ \log \frac{b}{x} \left(\int_0^1 \left(1 - W(b^{1-\tau} x^\tau)\right)^p b^{1-\tau} x^\tau d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^q \left(b^{1-\tau} x^\tau\right) b^{1-\tau} x^\tau d\tau \right)^{\frac{1}{q}}$$
$$\leqslant \log \frac{x}{a} \left(\int_0^1 W^p(x^{1-\tau} a^\tau) x^{1-\tau} a^\tau d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^{(1-\tau)q}(x) \, |f'|^{\tau q}(a) x^{1-\tau} a^\tau d\tau \right)^{\frac{1}{q}}$$
$$+ \log \frac{b}{x} \left(\int_0^1 (1 - W(b^{1-\tau} x^\tau))^p b^{1-\tau} x^\tau d\tau \right)^{\frac{1}{p}} \left(\int_0^1 |f'|^{(1-\tau)q}(b) \, |f'|^{\tau q}(x) b^{1-\tau} x^\tau d\tau \right)^{\frac{1}{q}}.$$

By a simple computation, we get

$$\int_0^1 Q^p\left(u^{1-\tau}v^\tau\right)u^{1-\tau}v^\tau d\tau = \frac{1}{\log\frac{u}{v}}\int_v^u Q^p(t)dt \tag{15}$$

and

$$\int_0^1 A_{v,u}^\tau d\tau = \frac{1}{\log A_{v,u}}\left[A_{v,u}-1\right]. \tag{16}$$

If we use (15) and (16) in (14) we get the desired result.

Lastly, we will give a new result for GA-convex functions as following:

THEOREM 5. *Let $I\subset(0,\infty)$ be an interval, $a,b\in I^o$, $a<b$. Let w be a nonnegative integrable function on $[a,b]$ such that $\int_a^b w(x)dx=1$ and let $W(t)=\int_a^t w(x)dx$. Let $f:I\to\mathbb{R}$ be a function differentiable on I^o. If $|f'|^q$ is GA-convex function on $[a,b]$ for some $q>1$, then for $x\in[a,b]$, following inequality holds*

$$\begin{aligned}
&\left|f(x)-\int_a^b w(t)f(t)dt\right|\\
&\leqslant \log^{\frac{1}{q}}\frac{x}{a}\left(\int_a^x W(t)t^{p-1}dt\right)^{\frac{1}{p}}\left(\left|f'(x)\right|^q\kappa_1+\left|f'(a)\right|^q\kappa_2\right)^{\frac{1}{q}}\\
&\quad+\log^{\frac{1}{q}}\frac{b}{x}\left(\int_x^b (1-W(t))t^{p-1}dt\right)^{\frac{1}{p}}\left(\left|f'(b)\right|^q\kappa_3+\left|f'(x)\right|^q\kappa_4\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}=1-\frac{1}{q}$ and

$$\begin{aligned}
\kappa_1 &= \int_a^x\left[\frac{\log\frac{t}{x}}{\log\frac{a}{x}}\left(1-\frac{\log\frac{t}{x}}{2\log\frac{a}{x}}\right)\right]w(t)dt\\
\kappa_2 &= \frac{1}{2}\int_a^x\left(\frac{\log\frac{t}{x}}{\log\frac{a}{x}}\right)^2 w(t)dt\\
\kappa_3 &= (1-W(x))-\int_x^b\left[\frac{\log\frac{t}{b}}{\log\frac{x}{b}}\left(1-\frac{\log\frac{t}{b}}{2\log\frac{x}{b}}\right)\right]w(t)dt\\
\kappa_4 &= (1-W(x))-\frac{1}{2}\int_x^b\left(\frac{\log\frac{t}{b}}{\log\frac{x}{b}}\right)^2 w(t)dt,
\end{aligned}$$

provided that all integrals exist.

Proof. Similar to the proof of the previous theorems, by using Lemma 1, Hölder integral inequality and GA-convexity of $|f'|^q$, one can immediately get the result. We omit the details.

REMARK 3. Results similar to Theorems 2, 3, 4 can also be obtained for GA-convex functions and some applications for special means can be given. It is left to the interested readers.

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, I. PERIĆ, *Estimates of the difference between two weighted integral means via weighted Montgomery identity*, Math. Inequal. Appl. **7** (2004), 315–336.

[2] A. AGLIĆ ALJINOVIĆ AND J. PEČARIĆ, *A discrete weighted Montgomery identity and Ostrowski type inequalities for functions of two variables*, Sarajevo Journal of Mathematics, Vol. **9** (21) (2013), 47–56.

[3] S. S. DRAGOMIR, *Inequalities of Jensen Type for GA-Convex Functions*, RGMIA Research Report Collection, **18** (2015), Article 35.

[4] C. P. NICULESCU, *Convexity according to the geometric mean*, Math. Inequal. Appl., **3** (2) (2000), 155–167.

[5] J. PEČARIĆ, *On the Čebyšev inequality*, Bul. Inst. Politehn. Timisoara, **25** (39) (1980), 5–9.

[6] R. A. SATNOIANU, *Improved GA-convexity inequalities*, Journal of Inequalities in Pure and Applied Mathematics, Volume 3, Issue 5, Article 82, 2002.

[7] X.-M. ZHANG, Y.-M. CHU, AND X.-H. ZHANG, *The Hermite-Hadamard type inequality of GA-convex functions and its application*, Journal of Inequalities and Applications, Volume 2010, Article ID 507560, 11 pages.

[8] T.-Y. ZHANG, A.-P. JI AND F. QI, *Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means*, Le Matematiche, Vol. LXVIII (2013) – Fasc. I, pp. 229–239.

[9] Z. ZHANG, W. WEI AND J.-R. WANG, *Generalization of Hermite-Hadamard Inequalities Involving Hadamard Fractional Integral Inequalities*, Filomat **29**: 7 (2015), 1515–1524.

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