Mathematical
Inequalities
& Applications
Volume 19, Number 4 (2016), 1169–1180 doi:10.7153/mia-19-86

HERMITE INTERPOLATION AND INEQUALITIES INVOLVING WEIGHTED AVERAGES OF n–CONVEX FUNCTIONS

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(*Communicated by C. P. Niculescu*)

Abstract. By using Hermite interpolation we obtain Popoviciu-type inequalities containing sums $\sum_{i=1}^m p_i f(x_i)$, where f is an n-convex function. We also give integral analogues of the results, as well as bounds for integral remainders of identities associated with the obtained inequalities.

1. Introduction

Pečarić [5] proved the following result (see also [6, p. 262]):

PROPOSITION 1.1. *The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant 0 \tag{1}$$

holds for all convex functions f if and only if the m-tuples $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{R}^m$ satisfy

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geqslant 0 \text{ for } k \in \{1, \dots, m\}. \tag{2}$$

Since

$$\sum_{i=1}^{m} p_i |x_i - x_k| = 2\sum_{i=1}^{m} p_i (x_i - x_k)_+ - \sum_{i=1}^{m} p_i (x_i - x_k),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (2) is equivalent to

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i (x_i - x_k)_+ \geqslant 0 \text{ for } k \in \{1, \dots, m-1\}. \tag{3}$$

Mathematics subject classification (2010): 26D15, 26D10.
Keywords and phrases: n-convex functions, Hermite interpolation, Čebyšev functional.
This work has been fully supported by Croatian Science Foundation under the project 5435.

Let A denote the linear operator $A(f) = \sum_{i=1}^m p_i f(x_i)$, let $w(x,t) = (x-t)_+$ and $x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(m)}$ be the sequence $\mathbf{x}$ in ascending order. Notice that $A(w(\cdot,x_k)) = \sum_{i=1}^m p_i(x_i - x_k)_+$. For $t \in [x_{(k)}, x_{(k+1)}]$ we have

$$A(w(\cdot,t)) = A(w(\cdot,x_{(k)})) + (x_{(k)} - t) \sum_{\{i: x_i > x_{(k)}\}} p_i,$$

so the mapping $t \mapsto A(w(\cdot,t))$ is linear on $[x_{(k)}, x_{(k+1)}]$. Furthermore, $A(w(\cdot,x_{(m)}) = 0$, so condition (3) is equivalent to

$$\sum_{i=1}^m p_i = 0, \quad \sum_{i=1}^m p_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^m p_i(x_i - t)_+ \geqslant 0 \text{ for every } t \in [x_{(1)}, x_{(m-1)}]. \tag{4}$$

It turns out that condition (4) is appropriate for extension of Proposition 1.1 to the integral case and the more general class of n-convex functions.

DEFINITION 1.2. The n-th order divided difference of a function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0, \ldots, x_n \in I$ is defined recursively (see [6]) by

$$f[x_i] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The function f is said to be n-convex on I, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points in I, the n-th order divided difference of f satisfies

$$f[x_0, \ldots, x_n] \geqslant 0.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$. If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geqslant 0$. For $1 \leqslant k \leqslant n-2$, a function f is n-convex if and only if $f^{(k)}$ exists and is $(n-k)$-convex.

The following result is due to Popoviciu [7, 8] (see [10, 6] also).

PROPOSITION 1.3. *Let $n \geqslant 2$. Inequality (1) holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if the m-tuples $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^m p_i x_i^k = 0, \quad \textit{for all } k = 0, 1, \ldots, n-1 \tag{5}$$

$$\sum_{i=1}^m p_i (x_i - t)_+^{n-1} \geqslant 0, \quad \textit{for every } t \in [a,b]. \tag{6}$$

In fact, Popoviciu proved a stronger result – it is enough to assume that (6) holds for every $t \in [x_{(1)}, x_{(m-n+1)}]$ and then, due to (5), it is automatically satisfied for every $t \in [a,b]$. The integral analogue (see [9, 6]) is given in the next proposition.

PROPOSITION 1.4. *Let $n \geqslant 2$, $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$. Then, the inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geqslant 0 \tag{7}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if

$$\begin{aligned} \int_{\alpha}^{\beta} p(x) g(x)^k\, dx = 0, &\quad \textit{for all } k = 0,1,\ldots,n-1 \\ \int_{\alpha}^{\beta} p(x)\,(g(x)-t)_+^{n-1}\, dx \geqslant 0, &\quad \textit{for every } t \in [a,b]. \end{aligned} \tag{8}$$

In this paper we will derive inequalities of type (1) and (7) for n-convex functions by making use of the Hermite interpolation. Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$. The Hermite interpolation of a function $f \in C^n[a,b]$ is of the form

$$f(x) = P_H(x) + e_H(x)$$

where P_H is the unique polynomial of degree $n-1$, called the Hermite interpolating polynomial of f, satisfying

$$P_H^{(i)}(a_j) = f^{(i)}(a_j), \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r, \quad \sum_{j=1}^{r} k_j + r = n.$$

The associated error $e_H(x)$ can be represented in terms of the Green's function $G_{H,n}(x,s)$ for the multipoint boundary value problem

$$z^{(n)}(x) = 0, \quad z^{(i)}(a_j) = 0, \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r,$$

that is, the following result holds (see [2]):

THEOREM 1.5. *Let $f \in C^n[a,b]$, and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned} f(x) &= P_H(x) + e_H(x) \\ &= \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(x) f^{(i)}(a_j) + \int_a^b G_{H,n}(x,s) f^{(n)}(s)\, ds, \end{aligned} \tag{9}$$

where H_{ij} are the fundamental polynomials of the Hermite basis defined by

$$H_{ij}(x) = \frac{1}{i!} \frac{w(x)}{(x-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dx^k}\left(\frac{(x-a_j)^{k_j+1}}{w(x)}\right)\bigg|_{x=a_j} (x-a_j)^k, \tag{10}$$

where

$$w(x) = \prod_{j=1}^{r} (x-a_j)^{k_j+1} \tag{11}$$

and $G_{H,n}$ is the Green's function defined by

$$G_{H,n}(x,s) = \begin{cases} \sum_{j=1}^{l} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \leqslant x, \\ -\sum_{j=l+1}^{r} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \geqslant x \end{cases} \tag{12}$$

for all $a_l \leqslant s \leqslant a_{l+1}$, $l = 0,1,\dots,r$ ($a_0 = a, a_{r+1} = b$).

The following are some special cases of the Hermite interpolation of functions:

(i) $(m,n-m)$ conditions: $r = 2$, $a_1 = a$, $a_2 = b$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$ and $k_2 = n-m-1$. In this case

$$f(x) = \sum_{i=0}^{m-1} \tau_i(x) f^{(i)}(a) + \sum_{i=0}^{n-m-1} \eta_i(x) f^{(i)}(b) + \int_a^b G_{m,n}(x,s) f^{(n)}(s) ds,$$

where

$$\tau_i(x) = \frac{1}{i!}(x-a)^i \left(\frac{x-b}{a-b}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-a}{b-a}\right)^k, \tag{13}$$

$$\eta_i(x) = \frac{1}{i!}(x-b)^i \left(\frac{x-a}{b-a}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-b}{a-b}\right)^k \tag{14}$$

and the Green's function $G_{m,n}$ is of the form

$$G_{m,n}(x,s) = \begin{cases} \sum_{j=0}^{m-1} \left[\sum_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{x-a}{b-a}\right)^p \right] \frac{(x-a)^j (a-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{b-x}{b-a}\right)^{n-m}, & s \leqslant x, \\ -\sum_{i=0}^{n-m-1} \left[\sum_{q=0}^{n-m-1-i} \binom{m+q-1}{q} \left(\frac{b-x}{b-a}\right)^q \right] \frac{(x-b)^i (b-s)^{n-i-1}}{i!(n-i-1)!} \left(\frac{x-a}{b-a}\right)^{m}, & s \geqslant x \end{cases} \tag{15}$$

(ii) Taylor's two-point condition: $m \in \mathbb{N}$, $n = 2m$, $r = 2$, $a_1 = a$, $a_2 = b$ and $k_1 = k_2 = m-1$. In this case

$$\begin{aligned} f(x) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \binom{m+k-1}{k} \Big[& \frac{(x-a)^i}{i!} \left(\frac{x-b}{a-b}\right)^m \left(\frac{x-a}{b-a}\right)^k f^{(i)}(a) \\ & + \frac{(x-b)^i}{i!} \left(\frac{x-a}{b-a}\right)^m \left(\frac{x-b}{a-b}\right)^k f^{(i)}(b) \Big] + \int_a^b G_{2T,m}(x,s) f^{(2m)}(s) ds, \end{aligned}$$

where the Green's function $G_{2T,m}$ is of the form

$$G_{2T,m}(x,s) = \frac{(-1)^m}{(2m-1)!} \begin{cases} p^m(x,s) \sum_{k=0}^{m-1} \binom{m+k-1}{k} (x-s)^{m-1-k} q^k(x,s), & s \leqslant x, \\ q^m(x,s) \sum_{k=0}^{m-1} \binom{m+k-1}{k} (s-x)^{m-1-k} p^k(x,s), & x \leqslant s, \end{cases}$$

where $p(x,s) = \frac{(s-a)(b-x)}{(b-a)}$ and $q(x,s) = p(s,x)$.

The following lemma yields the sign of the Green's function (12) on certain intervals (see Lemma 2.3.3, page 75, in [2]).

LEMMA 1.6. *The Green's function $G_{H,n}$ given by (12) and w given by (11) satisfy*

$$\frac{G_{H,n}(x,s)}{w(x)} > 0, \quad \text{for } a_1 \leqslant x \leqslant a_r, \quad a_1 < s < a_r.$$

Integration by parts easily yields that for any function $f \in C^2[a,b]$ the following holds

$$f(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) + \int_a^b G(x,s)f''(s)ds, \tag{16}$$

where the function $G : [a,b] \times [a,b] \to \mathbb{R}$ is the Green's function of the boundary value problem

$$z''(x) = 0, \quad z(a) = z(b) = 0$$

and is given by

$$G(x,s) = \begin{cases} \frac{(x-b)(s-a)}{b-a}, & \text{for } a \leqslant s \leqslant x, \\ \frac{(s-b)(x-a)}{b-a}, & \text{for } x \leqslant s \leqslant b. \end{cases} \tag{17}$$

The function G is continuous, symmetric and convex with respect to both variables x and s.

2. Main results

We will start this section with several identities.

THEOREM 2.1. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n$, $f \in C^n[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let H_{ij} and $G_{H,n}$ be given by (10) and (12). Then*

$$\sum_{k=1}^{m} p_k f(x_k) = \sum_{j=1}^{r}\sum_{i=0}^{k_j}\sum_{k=1}^{m} p_k H_{ij}(x_k) f^{(i)}(a_j) + \int_a^b \sum_{k=1}^{m} p_k G_{H,n}(x_k,s) f^{(n)}(s)\,ds. \tag{18}$$

Proof. By applying identity (9) at x_k, multiplying it by p_k and summing up we obtained the required identity. □

The integral version of the previous theorem is the following:

THEOREM 2.2. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n$, $f \in C^n[a,b]$, $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ and let H_{ij} and $G_{H,n}$ be given by (10) and (12). Then*

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))\,dx = &\sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)\int_\alpha^\beta p(x) H_{ij}(x)\,dx \\ &+ \int_a^b \left(\int_\alpha^\beta p(x) G_{H,n}(g(x),s)\,dx\right) f^{(n)}(s)\,ds.\end{aligned}$$

THEOREM 2.3. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $f \in C^n[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let H_{ij} and $G_{H,n-2}$ be given by (10) and (12). Then*

$$\sum_{k=1}^{m} p_k f(x_k) = \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \sum_{k=1}^{m} p_k G(x_k,s)H_{ij}(s)\,ds + \int_a^b\int_a^b \sum_{k=1}^{m} p_k G(x_k,s)G_{H,n-2}(s,t)f^{(n)}(t)\,dt\,ds. \tag{19}$$

Proof. Applying identity (16) at x_k, multiplying it by p_k and summing up we obtain

$$\sum_{k=1}^{m} p_k f(x_k) = \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k + \int_a^b \sum_{k=1}^{m} p_k G(x_k,s)f''(s)\,ds. \tag{20}$$

By Theorem 1.5, $f''(s)$ can be expressed as

$$f''(s) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(s)f^{(i+2)}(a_j) + \int_a^b G_{H,n-2}(s,t)f^{(n)}(t)\,dt. \tag{21}$$

Inserting (21) in (20) we get (19). □

We also state the integral version of the previous theorem.

THEOREM 2.4. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $f \in C^n[a,b]$, $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ and let H_{ij} and $G_{H,n-2}$ be given by (10) and (12). Then*

$$\int_\alpha^\beta p(x)f(g(x))\,dx = \frac{f(b)-f(a)}{b-a}\int_\alpha^\beta p(x)g(x)\,dx + \frac{bf(a)-af(b)}{b-a}\int_\alpha^\beta p(x)\,dx + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \left(\int_\alpha^\beta p(x)G(g(x),s)\,dx\right) H_{ij}(s)\,ds + \int_a^b\int_a^b \left(\int_\alpha^\beta p(x)G(g(x),s)\,dx\right) G_{H,n-2}(s,t)f^{(n)}(t)\,dt\,ds.$$

Next we will use the identities proven above to derive inequalities.

THEOREM 2.5. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let H_{ij} and $G_{H,n}$ be given by (10) and (12). If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\sum_{k=1}^{m} p_k G_{H,n}(x_k,s) \geqslant 0 \quad \text{for all } s \in [a,b], \tag{22}$$

then

$$\sum_{k=1}^{m} p_k f(x_k) \geqslant \sum_{j=1}^{r}\sum_{i=0}^{k_j}\sum_{k=1}^{m} p_k H_{ij}(x_k) f^{(i)}(a_j). \tag{23}$$

If the inequality in (22) is reversed, then the inequality in (23) is reversed also.

Proof. If (22) holds, then the second term on the right hand side (18) is nonnegative. □

THEOREM 2.6. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n$, $\mathbf{x} \in [a,b]^m$, $p : [\alpha,\beta] \to \mathbb{R}$ and let H_{ij} and $G_{H,n}$ be given by (10) and (12). If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\int_{\alpha}^{\beta} p(x) G_{H,n}(g(x),s)\,dx \geqslant 0 \quad \textit{for all } s \in [a,b], \tag{24}$$

then

$$\int_{\alpha}^{\beta} p(x) f(g(x))\,dx \geqslant \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j) \int_{\alpha}^{\beta} p(x) H_{ij}(x)\,dx. \tag{25}$$

If the inequality in (24) is reversed, then the inequality in (25) is reversed also.

THEOREM 2.7. *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let H_{ij} and $G_{H,n-2}$ be given by (10) and (12). Let $f : [a,b] \to \mathbb{R}$ be n-convex and*

$$\sum_{k=1}^{m} p_k G(x_k,s) \geqslant 0 \quad \textit{for all } s \in [a,b], \tag{26}$$

and consider the inequality

$$\begin{aligned}\sum_{k=1}^{m} p_k f(x_k) \geqslant{}& \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k \\ &+ \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \sum_{k=1}^{m} p_k G(x_k,s) H_{ij}(s)\,ds. \end{aligned}\tag{27}$$

(i) *If k_j for $j = 2,\ldots,r$ are odd, then (27) holds.*

(ii) *If k_j for $j = 2,\ldots,r-1$ are odd and k_r is even, then the reverse of (27) holds.*

Proof. (i) Assume first that $f \in C^n[a,b]$. Due to the assumptions w given by (11) satisfies $w(x) \geqslant 0$ for all x and, hence, by Lemma 1.6, $G_{H,n-2}(s,t) \geqslant 0$ for all $s,t \in [a,b]$. Therefore, the last term on the right hand side of (19) is nonnegative, so inequality (27) holds. The inequality for general f follows since every n-convex function can be obtained, by making use of the Bernstein polynomials, as a uniform limit of n-convex functions with a continuous n-th derivative (see [6]).

(ii) Under these assumptions $w(x) \leqslant 0$, so $G_{H,n-2}(s,t) \leqslant 0$. The rest of the proof is the same as in (i). □

THEOREM 2.8. *Let* $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $g : [\alpha,\beta] \to \mathbb{R}$, $p : [\alpha,\beta] \to \mathbb{R}$ *and let* H_{ij} *be given by (10). Let* $f : [a,b] \to \mathbb{R}$ *be* n*-convex and*

$$\int_{\alpha}^{\beta} p(x)G(g(x),s)\,dx \geqslant 0 \quad \text{for all } s \in [a,b],$$

and consider the inequality

$$\int_{\alpha}^{\beta} p(x)f(g(x))\,dx \geqslant \frac{f(b)-f(a)}{b-a}\int_{\alpha}^{\beta} p(x)g(x)\,dx + \frac{bf(a)-af(b)}{b-a}\int_{\alpha}^{\beta} p(x)\,dx + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \left(\int_{\alpha}^{\beta} p(x)G(g(x),s)\,dx\right) H_{ij}(s)\,ds. \tag{28}$$

(*i*) *If* k_j *for* $j = 2,\ldots,r$ *are odd, then (28) holds.*

(*ii*) *If* k_j *for* $j = 2,\ldots,r-1$ *are odd and* k_r *is even, then the reverse of (28) holds.*

In the case of the $(m,n-m)$ conditions we have the following corollary.

COROLLARY 2.9. *Let* τ_i *and* η_i *be given by (13) and (14) and let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *be such that (26) holds. Let* $f : [a,b] \to \mathbb{R}$ *be* n*-convex and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geqslant \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k + \int_a^b \left(\sum_{k=1}^{m} p_k G(x_k,s)\right)\left(\sum_{i=0}^{l-1}\tau_i(s)f^{(i+2)}(a) + \sum_{i=0}^{n-l-1}\eta_i(s)f^{(i+2)}(b)\right) ds. \tag{29}$$

(*i*) *If* $n-l$ *is even, then (29) holds.*

(*ii*) *If* $n-l$ *is odd, then the reverse of (29) holds.*

In the case of Taylor's two point conditions we have the following corollary.

COROLLARY 2.10. *Let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *be such that (26) holds. Let* $f : [a,b] \to \mathbb{R}$ *be* n*-convex and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geqslant \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k + \int_a^b \left(\sum_{k=1}^{m} p_k G(x_k,s)\right) \times \left(\sum_{i=0}^{l-1}\sum_{k=0}^{l-i-1}\binom{l+k-1}{k}\left[\frac{(s-a)^i}{i!}\left(\frac{s-b}{a-b}\right)^l\left(\frac{s-a}{b-a}\right)^k f^{(i+2)}(a) + \frac{(s-b)^i}{i!}\left(\frac{s-a}{b-a}\right)^l\left(\frac{s-b}{a-b}\right)^k f^{(i+2)}(b)\right]\right) ds. \tag{30}$$

(i) *If l is even, then (30) holds.*

(ii) *If l is odd, then the reverse of (30) holds.*

THEOREM 2.11. *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy (2), and let H_{ij} and $G_{H,n-2}$ be given by (10) and (12). Let $f : [a,b] \to \mathbb{R}$ be n-convex and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geqslant \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b \sum_{k=1}^{m} p_k G(x_k, s) H_{ij}(s)\, ds \tag{31}$$

and the function

$$F(x) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b G(x,s) H_{ij}(s)\, ds. \tag{32}$$

(i) *If k_j for $j = 2, \ldots, r$ are odd, then (31) holds. Furthermore, if the function F is convex, then inequality (1) holds.*

(ii) *If k_j for $j = 2, \ldots, r-1$ are odd and k_r is even, then the reverse of (31) holds. Furthermore, if the function F is concave, then the reverse of inequality (1) holds.*

Proof. The function $G(x,s)$ is convex in the first variable, so assumption (26) is satisfied by Proposition 1.1. Now, the claims of the theorem follow from Theorem 2.7 and Proposition 1.1. $\square$

THEOREM 2.12. *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geqslant 2$, $\sum_{j=1}^{r} k_j + r = n-2$, let $g : [\alpha, \beta] \to \mathbb{R}$ and $p : [\alpha, \beta] \to \mathbb{R}$ satisfy (8), and let H_{ij} and $G_{H,n-2}$ be given by (10) and (12). Let $f : [a,b] \to \mathbb{R}$ be n-convex and consider the inequality*

$$\int_\alpha^\beta p(x) f(x)\, dx \geqslant \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b \left(\int_\alpha^\beta p(x) G(g(x), s)\, dx \right) H_{ij}(s)\, ds \tag{33}$$

and the function F given by (32).

(i) *If k_j for $j = 2, \ldots, r$ are odd, then (33) holds. Furthermore, if the function F is convex, then inequality (7) holds.*

(ii) *If k_j for $j = 2, \ldots, r-1$ are odd and k_r is even, then the reverse of (33) holds. Furthermore, if the function F is concave, then the reverse of inequality (7) holds.*

3. Bounds for identities related to the Popoviciu-type inequalities

Let $f,h : [a,b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(f,h) = \frac{1}{b-a}\int_a^b f(x)h(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right).$$

The following results can be found in [4].

PROPOSITION 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L[a,b]$. Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(f,f)|\int_a^b (x-a)(b-x)[h'(x)]^2\,dx\right)^{\frac{1}{2}}. \tag{34}$$

The constant $\frac{1}{\sqrt{2}}$ in (34) is the best possible.

PROPOSITION 3.2. *Let $h : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function and let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,h)| \leqslant \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{35}$$

The constant $\frac{1}{2}$ in (35) is the best possible.

For m-tuples $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$, $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$ and the functions G and $G_{H,n}$ given by (17) and (12) denote

$$\delta_1(t) = \sum_{k=1}^m p_k G_{H,n}(x_k,t), \qquad \text{for } t \in [a,b]. \tag{36}$$

$$\delta_2(t) = \int_a^b \sum_{k=1}^m p_k G(x_k,s)G_{H,n-2}(s,t)\,ds, \qquad \text{for } t \in [a,b]. \tag{37}$$

Now, we are ready to state the main results of this section.

THEOREM 3.3. *Let $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let H_{ij}, δ_1 and δ_2 be given by (10), (36) and (37).*

(i) If $\sum_{j=1}^r k_j + r = n$, then

$$\sum_{k=1}^m p_k f(x_k) = \sum_{j=1}^r \sum_{i=0}^{k_j} \sum_{k=1}^m p_k H_{ij}(x_k) f^{(i)}(a_j) + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b \delta_1(s)\,ds + R_n^1(f;a,b), \tag{38}$$

where the remainder $R_n^1(f;a,b)$ *satisfies the estimation*

$$|R_n^1(f;a,b)| \leqslant \left(\frac{b-a}{2}|T(\delta_1,\delta_1)|\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}. \tag{39}$$

(ii) *If* $\sum_{j=1}^r k_j + r = n-2$, *then*

$$\begin{aligned}\sum_{k=1}^m p_k f(x_k) = {} & \frac{f(b)-f(a)}{b-a}\sum_{k=1}^m p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^m p_k \\ & + \sum_{j=1}^r \sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \sum_{k=1}^m p_k G(x_k,s)H_{ij}(s)\,ds \\ & + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \delta_2(s)\,ds + R_n^2(f;a,b),\end{aligned} \tag{40}$$

where the remainder $R_n^2(f;a,b)$ *satisfies the estimation*

$$|R_n^2(f;a,b)| \leqslant \left(\frac{b-a}{2}|T(\delta_2,\delta_2)|\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.$$

Proof. (i) Applying Proposition 3.1 with $f \to \delta_1$ and $h \to f^{(n)}$ we get

$$\begin{aligned}&\left|\int_a^b \delta_1(s)f^{(n)}(s)\,ds - \frac{1}{b-a}\int_a^b \delta_1(s)\,ds\int_a^b f^{(n)}(s)\,ds\right| \\ &\qquad \leqslant \left(\frac{b-a}{2}|T(\delta_1,\delta_1)|\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.\end{aligned} \tag{41}$$

From identities (18) and (38) we obtain

$$\int_a^b \delta_1(s)f^{(n)}(s)\,ds = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \delta_1(s)\,ds + R_n^1(f;a,b),$$

where the estimate (39) follows from (41).

(ii) Analogous as in (i). $\square$

By using Proposition 3.2 we obtain the following Grüss type inequality.

THEOREM 3.4. *Let* $-\infty < a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b < \infty$, $r \geqslant 2$, *let* $\mathbf{x}$, $\mathbf{p}$, H_{ij}, δ_1, δ_2 *and* n *be as in Theorem 3.3 and let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is an absolutely continuous function with* $f^{(n+1)} \geqslant 0$. *Then representations (38) and (40) hold with the remainders* $R_n^i(f;a,b)$, $i=1,2$, *satisfying the bounds*

$$|R_n^i(f;a,b)| \leqslant \|\delta_i'\|_\infty \left[\frac{b-a}{2}\left(f^{(n-1)}(b)+f^{(n-1)}(a)\right) - f^{(2n-2)}(b) + f^{(2n-2)}(a)\right]. \tag{42}$$

Proof. If we apply Proposition 3.2 with $f \to \delta_i$ and $h \to f^{(n)}$ we obtain

$$\left|\int_a^b \delta_i(s) f^{(n)}(s)ds - \frac{1}{b-a}\int_a^b \delta_i(s)ds \int_a^b f^{(n)}(s)ds\right| \leqslant \frac{1}{2}\|\delta_i'\|_\infty \int_a^b (s-a)(b-s)f^{(n+1)}(s)ds.$$

Since

$$\begin{aligned}\int_a^b (s-a)(b-s)f^{(n+1)}(s)\,ds &= \int_a^b (2s-a-b)f^{(n)}(s)\,ds \\ &= (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right], \quad (43)\end{aligned}$$

using (43) and identities (18) or (19) we deduce (42). □

REMARK 3.5. We can construct linear functionals by taking differences of the left and right hand sides of the inequalities from Theorems 2.5, 2.6, 2.7 and 2.8. By using similar methods as in [1, 3] we can prove mean value results for these functionals, as well as construct new families of exponentially convex functions and Cauchy-type means. Then, by using some known properties of exponentially convex functions, we can derive new inequalities and prove monotonicity of the obtained Cauchy-type means analogously as in [1, 3].

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(Received January 25, 2016)

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