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ONE PROOF OF THE GHEORGHIU INEQUALITY

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Abstract. The Gheorghiu inequality is a reverse Hölder's inequality. In this article, the Gheorghiu inequality is proven by using a property of a two – variable function. Original Gheorghiu's result is presented and compared with obtained result.

1. Introduction and preliminaries

A class of real function defined on a set Ω is noted with $\mathscr{L}$ if for any pair $f,g \in \mathscr{L}$ their linear combination $\alpha f + \beta g \in \mathscr{L}$ for all $\alpha,\beta \in \mathbb{R}$ and if $\mathscr{L}$ consists constant functions.

A linear mean $\mathbf{E}$ is a linear functional defined on $\mathscr{L}$ with property that if $f(t) \geqslant 0$ on Ω, then $\mathbf{E}(f) \geqslant 0$ and with property that $\mathbf{E}(1) = 1$. The function noted by 1 presents the basic constant function with $1(t) = 1$ for every $t \in \Omega$.

Jensen's inequality and McShane's extension are given according the [8].

THEOREM 1. (Jensen) *Let $g_1 \in \mathscr{L}$, such that $g_1(t) \in [a,A] \subset \mathbb{R}$ for all $t \in \Omega$. Let $\mathbf{E}$ be a linear mean on $\mathscr{L}$. If $\varphi : [a,A] \to \mathbb{R}$ is a continuous concave function, then $\varphi(g_1) \in \mathscr{L}$, $\mathbf{E}(\varphi(g_1)) \in [a,A]$ and*

$$\mathbf{E}(\varphi(g_1)) \leqslant \varphi(\mathbf{E}(g_1)).$$

THEOREM 2. (McShane, case on rectangular) *Let $g_1,g_2 \in \mathscr{L}$ such that $(g_1(t),g_2(t)) \in D = [a,A] \times [b,B]$ for all $t \in \Omega$. Let $\mathbf{E}$ be a linear mean on $\mathscr{L}$. If $\varphi : D \to \mathbb{R}$ is a continuous concave function, then $\varphi(g_1,g_2) \in \mathscr{L}$, $(\mathbf{E}(g_1),\mathbf{E}(g_2)) \in D$ and*

$$\mathbf{E}(\varphi(g_1,g_2) \leqslant \varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)). \tag{1}$$

In [6] author proved Theorem 3 that characterized the right hand side in the McShane inequality for a measure space.

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THEOREM 3. *Let* (Ω,Σ,μ) *be a measure space such that* $0<\mu(A)<1<\mu(B)<\infty$ *for some* $A,B\in\Sigma$ *and let bijections* $\varphi_1,\varphi_2,\psi_1,\psi_2:(0,\infty)\to(0,\infty)$ *be such that* $\frac{\psi_1\circ\varphi_1(t)}{t}\leqslant c\leqslant\frac{t}{\psi_2\circ\varphi_2(t)}$. *If*

$$\int_\Omega xy\,d\mu\leqslant\psi_1\left(\int\Omega(x)\varphi_1\circ|x|\,d\mu\right)\psi_2\left(\int\Omega(y)\varphi_2\circ|y|\,d\mu\right)$$

for all nonnegative μ*-integrable simple functions* $x,y:\Omega\to\mathbf{R}$ *(where* $\Omega(x)$ *stands for the support of x), then there exists a real* $p>1$ *such that*

$$\frac{\varphi_1(t)}{\varphi_1(1)}=t^p,\quad\frac{\psi_1(t)}{\psi_1(1)}=t^{1/p},\quad\frac{\varphi_2(t)}{\varphi_2(1)}=t^q,\quad\frac{\psi_2(t)}{\psi_2(1)}=t^{1/q},\quad t>0,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

A part of Csiszár and Móri conversion for (1) is given below. Complete conversion is given in [1]

THEOREM 4. *Let* $\varphi:D\to\mathbb{R}$ *be a concave function and suppose that* $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\geqslant0$. *If* $(B-b)\mathbf{E}(g_1)+(A-a)\mathbf{E}(g_2)\leqslant AB-ab$, *then* $\lambda\mathbf{E}(g_1)+\mu\mathbf{E}(g_2)+\nu\leqslant\mathbf{E}(\varphi(g_1,g_2))\leqslant\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2))$, *with*

$$\lambda=\frac{\varphi(A,b)-\varphi(a,b)}{A-a},\qquad\mu=\frac{\varphi(a,B)-\varphi(a,b)}{B-b},$$

and

$$\nu=\frac{AB-ab}{(A-a)(B-b)}\varphi(a,b)-\frac{b}{B-b}\varphi(a,B)-\frac{a}{A-a}\varphi(A,b).$$

More general conversion and refinement in Theorem 5 is proven in [2] by considering the functions

$$M_{ij}(t,s)=\frac{(\lambda_i+\lambda_j)t+(\mu_i+\mu_j)s+\nu_i+\nu_j}{2}+\frac{|(\lambda_i-\lambda_j)t+(\mu_i-\mu_j)s+\nu_i-\nu_j|}{2}$$

and

$$m_{ij}(t,s)=\frac{(\lambda_i+\lambda_j)t+(\mu_i+\mu_j)s+\nu_i+\nu_j}{2}-\frac{|(\lambda_i-\lambda_j)t+(\mu_i-\mu_j)s+\nu_i-\nu_j|}{2}.$$

Given coefficients are $\lambda_1=\lambda_4=\frac{\varphi(A,b)-\varphi(a,b)}{A-a}$, $\mu_1=\mu_3=\frac{\varphi(a,B)-\varphi(a,b)}{B-b}$; $\lambda_2=\lambda_3=\frac{\varphi(A,B)-\varphi(a,B)}{A-a}$; $\mu_2=\mu_4=\frac{\varphi(A,B)-\varphi(A,b)}{B-b}$; $\nu_1=\varphi(a,b)-\lambda_1a-\mu_1b$; $\nu_2=\varphi(A,B)-\lambda_2A-\mu_2B$; $\nu_3=\varphi(a,B)-\lambda_3a-\mu_3B$ and $\nu_4=\varphi(A,b)-\lambda_4A-\mu_4b$.

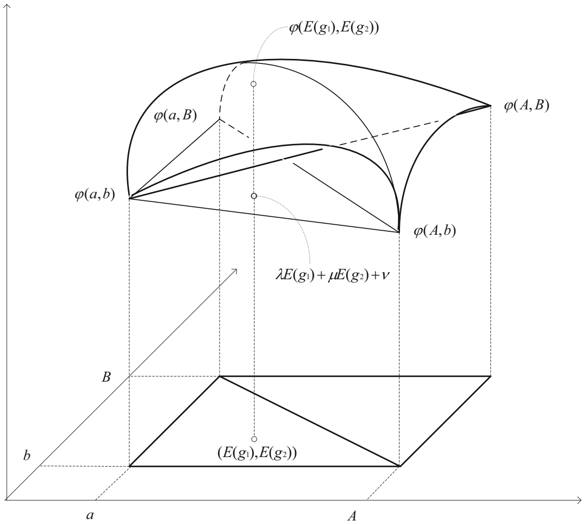

Figure 1: *Conversion by Csiszár and Móri*

THEOREM 5. *Suppose $\varphi : D \to \mathbb{R}$ is a continuous and concave function. If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B) \geqslant 0$, then*

$$M_{12}(\mathbf{E}(g_1),\mathbf{E}(g_2)) \leqslant \mathbf{E}(m_{34}(g_1,g_2)) \leqslant \mathbf{E}(\varphi(g_1,g_2)) \leqslant \varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)). \qquad (2)$$

If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B) \leqslant 0$, then

$$M_{34}(\mathbf{E}(g_1),\mathbf{E}(g_2)) \leqslant \mathbf{E}(m_{12}(g_1,g_2)) \leqslant \mathbf{E}(\varphi(g_1,g_2)) \leqslant \varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)).$$

2. Main result and applications

Conversion of (1) by a two-variable function is given. Under the special conditions the Georhiu-type inequality is proven. The following general result is given in [4].

THEOREM 6. (General result) *Let $\varphi, \psi : D \to \mathbb{R}$ be continuous, let φ be concave and for $g_1, g_2 \in \mathscr{L}$ let us assume that $(g_1(t), g_2(t)) \in D$ for all $t \in \Omega$. Let E be a linear mean on $\mathscr{L}$. Suppose that $\varphi(D) \subseteq U$ and $\psi(D) \subseteq V$ and suppose that $\mathscr{F} : U \times V \subseteq \mathbb{R}^2 \to \mathbb{R}$ is increasing in the first variable.*

If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B) \geqslant 0$, then

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{12}(t,s), \psi(t,s)\right) \leqslant \mathscr{F}\left(\mathbf{E}(\varphi(g_1,g_2)), \psi(\mathbf{E}(g_1),\mathbf{E}(g_2))\right).$$

If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B) \leqslant 0$, then

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{34}(t,s), \psi(t,s)\right) \leqslant \min_{(t,s)\in D} \mathscr{F}\left(\mathbf{E}(\varphi(g_1,g_2)), \psi(\mathbf{E}(g_1),\mathbf{E}(g_2))\right).$$

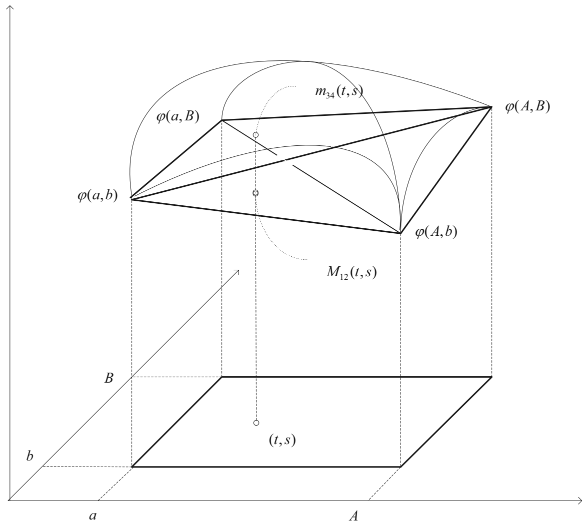

Figure 2: *Conversions in Theorem 5*

Proof. If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\geqslant 0$, then by Theorem 5, inequality (2) holds. Since $(\mathbf{E}(g_1),\mathbf{E}(g_2))\in D$, then

$$\min_{(t,s)\in D}\mathscr{F}\left(M_{12}(t,s),\psi(t,s)\right)\leqslant\mathscr{F}\left((M_{12}(E(g_1),E(g_2)),\psi(\mathbf{E}(g_1),\mathbf{E}(g_2))\right).$$

By Theorem 5, we have that $M_{12}(E(g_1),E(g_2))\leqslant\mathbf{E}(\varphi(g_1,g_2))$. Since $\mathscr{F}$ is increasing in the first variable, we get

$$\min_{(t,s)\in D}\mathscr{F}\left(M_{12}(t,s),\psi(t,s)\right)\leqslant\mathscr{F}\left(\mathbf{E}(\varphi(g_1,g_2)),\psi(\mathbf{E}(g_1),\mathbf{E}(g_2))\right)$$

and obtain the desired inequality. □

A multiplicative conversion is made by taking $\mathscr{F}(x,y)=\dfrac{x}{y}$.

COROLLARY 1. *Suppose that assumptions of Theorem 6 hold with $\varphi(D)>0$ additionally.*

If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\geqslant 0$, then

$$\min_{(t,s)\in D}\frac{M_{12}(t,s)}{\varphi(t,s)}\cdot\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2))\leqslant\mathbf{E}(\varphi(g_1,g_2))\leqslant\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)).$$

In opposite, if $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\leqslant 0$, then

$$\min_{(t,s)\in D}\frac{M_{34}(t,s)}{\varphi(t,s)}\cdot\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2))\leqslant\mathbf{E}(\varphi(g_1,g_2))\leqslant\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)).$$

A conversion by medium value is given by the next lemma.

LEMMA 1. *Assume that* φ, g_1, g_2 *and* $\mathbf{E}$ *are as in Theorem* 6. *Let* $\alpha, \beta \geqslant 0$ *such that* $\alpha + \beta = 1$.

If $\varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B) \geqslant 0$ *then*

$$(\alpha\lambda_1 + \beta\lambda_2)\mathbf{E}(g_1) + (\alpha\mu_1 + \beta\mu_2)\mathbf{E}(g_2) + \alpha\nu_1 + \beta\nu_2 \leqslant \mathbf{E}(\varphi(g_1, g_2)).$$

If $\varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B) \leqslant 0$, *then*

$$(\alpha\lambda_3 + \beta\lambda_4)\mathbf{E}(g_1) + (\alpha\mu_3 + \beta\mu_4)\mathbf{E}(g_2) + \alpha\nu_3 + \beta\nu_4 \leqslant \mathbf{E}(\varphi(g_1, g_2)).$$

Proof. Considering that

$$M_{12}(\mathbf{E}(g_1), \mathbf{E}(g_2) = \max\{\lambda_1\mathbf{E}(g_1) + \mu_1\mathbf{E}(g_1) + \nu_1, \lambda_2\mathbf{E}(g_1) + \mu_2\mathbf{E}(g_2) + \nu_2\},$$

we obtain the first inequality if $\varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B) \geqslant 0$. □

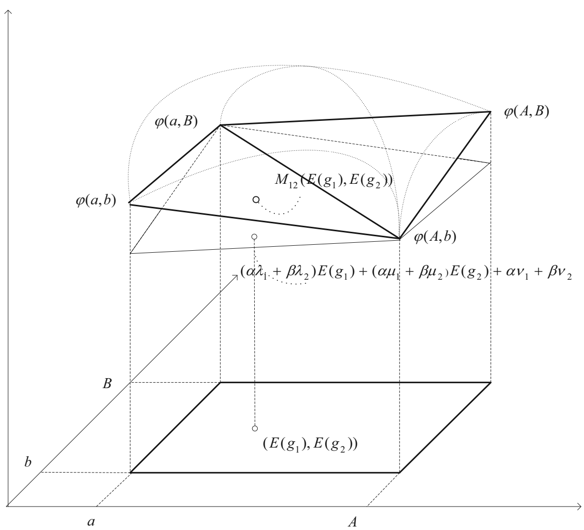

Figure 3: *Conversion by a medium value*

A conversion with very special condition is given bellow.

PROPOSITION 1. Suppose that assumptions of Lemma 1 hold.

(i) If $\varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B) \geqslant 0$ and $\nu_1 \cdot \nu_2 < 0$, then

$$U_{12}\mathbf{E}(g_1) + V_{12}\mathbf{E}(g_2) \leqslant \mathbf{E}(\varphi(g_1, g_2)) \leqslant \varphi(\mathbf{E}(g_1), \mathbf{E}(g_2)),$$

where:

$$U_{12} = \frac{\nu_2\lambda_1 - \nu_1\lambda_2}{\nu_2 - \nu_1}, \; V_{12} = \frac{\nu_2\mu_1 - \nu_1\mu_2}{\nu_2 - \nu_1}.$$

(ii) If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\leqslant 0$ and $\nu_3\cdot\nu_4<0$, then

$$U_{34}=\frac{\nu_4\lambda_3-\nu_3\lambda_4}{\nu_4-\nu_3},\ V_{34}=\frac{\nu_4\mu_3-\nu_3\mu_4}{\nu_4-\nu_3}.$$

Proof. Solving the system $\begin{cases}\alpha+\beta=1\\ \alpha\nu_1+\beta\nu_2=0\end{cases}$ by α,β we obtain that $\alpha\lambda_1+\beta\lambda_2=U_{1,2}$ and $\alpha\mu_1+\beta\mu_2=V_{1,2}$. □

Considering the Corollary 1, the next Proposition is given.

PROPOSITION 2. Let $\varphi:D\to\mathbb{R}$ be a continuous concave positive function, $g_1,g_2\in\mathscr{L}$ and a linear mean $\mathbf{E}$ on $\mathscr{L}$.

(i) If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\geqslant 0$ and $\nu_1\cdot\nu_2<0$, then:

$$\min_{(t,s)\in D}\frac{U_{12}t+V_{12}s}{\varphi(t,s)}\cdot\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2))\leqslant\mathbf{E}(\varphi(g_1,g_2))\leqslant\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)).\tag{3}$$

(ii) If $\varphi(a,b)+\varphi(A,B)-\varphi(A,b)-\varphi(a,B)\leqslant 0$ and $\nu_3\cdot\nu_4<0$, then:

$$\min_{(t,s)\in D}\frac{U_{34}t+V_{34}s}{\varphi(t,s)}\cdot\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2))\leqslant\mathbf{E}(\varphi(g_1,g_2))\leqslant\varphi(\mathbf{E}(g_1),\mathbf{E}(g_2)),$$

where values U_{12} V_{12}, U_{34} and V_{34} are given in Proposition 2.

Gheorghiu's type inequality is a converse of Hölder's type inequality. The original Gheorghiu inequality from [9] will be presented in the next section. Here the proof for a refinement is presented.

THEOREM 7. *Let $\mathbf{E}$ be a linear mean on $\mathscr{L}$ and for $g_1,g_2\in\mathscr{L}$ let us assume that $g_1(\Omega)\subset[a,A]$ and $g_2(\Omega)\subset[b,B]$ for positive real numbers a,b. Let p,q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ holds. Then*

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}(abAB)^{\frac{1}{pq}}\left((AB)^{\frac{1}{p}}-(ab)^{\frac{1}{p}}\right)^{\frac{1}{p}}\left((AB)^{\frac{1}{q}}-(ab)^{\frac{1}{q}}\right)^{\frac{1}{q}}}{AB-ab}\cdot(\mathbf{E}(g_1))^{\frac{1}{p}}(\mathbf{E}(g_2))^{\frac{1}{q}}$$
$$\leqslant\mathbf{E}\left(g_1^{\frac{1}{p}}\cdot g_2^{\frac{1}{q}}\right)\leqslant(\mathbf{E}(g_1))^{\frac{1}{p}}(\mathbf{E}(g_2))^{\frac{1}{q}}.\tag{4}$$

Proof. The function $\varphi(x,y)=x^{\frac{1}{p}}y^{\frac{1}{q}}$ is continuous, concave and positive for all $(x,y)\in[a,A]\times[b,B]$. Because $\left(A^{\frac{1}{p}}-a^{\frac{1}{p}}\right)\left(B^{\frac{1}{q}}-b^{\frac{1}{q}}\right)>0$, for application of Proposition 2 it is enough to prove that

$$\nu_1=a^{\frac{1}{p}}b^{\frac{1}{q}}-ab^{\frac{1}{q}}\frac{A^{\frac{1}{p}}-a^{\frac{1}{p}}}{A-a}-a^{\frac{1}{p}}b\frac{B^{\frac{1}{q}}-b^{\frac{1}{q}}}{B-b}\geqslant 0$$

and

$$v_2 = A^{\frac{1}{p}}B^{\frac{1}{q}} - AB^{\frac{1}{q}}\frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - A^{\frac{1}{p}}B\frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \leqslant 0.$$

Since the function $f(x) = x^{\frac{1}{p}}$ is concave for $p > 1$, $f'(x)$ is continuous and decreasing. So there exists $c \in [a,A]$ such that $f'(c) = \dfrac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a}$ and $f'(a) \geqslant f'(c) \geqslant f'(A)$ which gives

$$\frac{1}{p}a^{\frac{1}{p}-1} \geqslant \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \geqslant \frac{1}{p}A^{\frac{1}{p}-1}$$

Multiplying with $ab^{\frac{1}{q}}$ and $AB^{\frac{1}{q}}$ we get

$$\frac{a^{\frac{1}{p}}b^{\frac{1}{q}}}{p} \geqslant ab^{\frac{1}{q}}\frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \quad \text{and} \quad AB^{\frac{1}{q}}\frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \geqslant \frac{A^{\frac{1}{p}}B^{\frac{1}{q}}}{p}.$$

Similar consideration on $f(x) = x^{\frac{1}{q}}$ gives

$$\frac{1}{q}b^{\frac{1}{q}-1} \geqslant \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \geqslant \frac{1}{p}A^{\frac{1}{p}-1}.$$

Multiplying with $a^{\frac{1}{p}}b$ and $A^{\frac{1}{p}}B$ we get

$$\frac{a^{\frac{1}{p}}b^{\frac{1}{q}}}{q} \geqslant a^{\frac{1}{p}}b\frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \quad \text{and} \quad A^{\frac{1}{p}}B\frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \geqslant \frac{A^{\frac{1}{p}}B^{\frac{1}{q}}}{q}.$$

Now we have

$$v_1 = a^{\frac{1}{p}}b^{\frac{1}{q}} - ab^{\frac{1}{q}}\frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - a^{\frac{1}{p}}b\frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b}$$

$$v_1 \geqslant a^{\frac{1}{p}}b^{\frac{1}{q}} - \frac{a^{\frac{1}{p}}b^{\frac{1}{q}}}{p} - \frac{a^{\frac{1}{p}}b^{\frac{1}{q}}}{q} = a^{\frac{1}{p}}b^{\frac{1}{q}}\left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

$$v_2 = A^{\frac{1}{p}}B^{\frac{1}{q}} - AB^{\frac{1}{q}}\frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - A^{\frac{1}{p}}B\frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b}$$

$$v_2 \leqslant A^{\frac{1}{p}}B^{\frac{1}{q}} - \frac{A^{\frac{1}{p}}B^{\frac{1}{q}}}{p} - \frac{A^{\frac{1}{p}}B^{\frac{1}{q}}}{q} = A^{\frac{1}{p}}B^{\frac{1}{q}}\left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

Note that (4) is equal to (3) by coefficients

$$U_{12} = \frac{B^{\frac{1}{q}}b^{\frac{1}{q}}\left((AB)^{\frac{1}{p}} - (ab)^{\frac{1}{p}}\right)}{AB-ab} \quad \text{and} \quad V_{12} = \frac{A^{\frac{1}{p}}a^{\frac{1}{p}}\left((AB)^{\frac{1}{q}} - (ab)^{\frac{1}{q}}\right)}{AB-ab}.$$

It is necessary to minimize $\min_{(t,s)\in D}\dfrac{U_{12}t+V_{12}s}{t^{\frac{1}{p}}s^{\frac{1}{q}}}=\min_{(t,s)\in D}\left(U_{12}\cdot\left(\dfrac{t}{s}\right)^{\frac{1}{q}}+V_{12}\cdot\left(\dfrac{s}{t}\right)^{\frac{1}{p}}\right)$. Supstitution $z=\dfrac{t}{s}$ and easy calculus gives $\min_{(t,s)\in D}\dfrac{U_{12}t+V_{12}s}{\varphi(t,s)}=U_{12}^{\frac{1}{p}}V_{12}^{\frac{1}{q}}p^{\frac{1}{p}}q^{\frac{1}{q}}$. Substituting the all above in (3) we get (4) and the proof is done. □

REMARK 1. Using substitutions $g_1 \rightarrowtail g_1^p$ and $g_1 \rightarrowtail g_1^p$ in the previous Theoram we get the following

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(A^paB-a^pbA\right)^{\frac{1}{q}}}{A^pB^q-a^pb^q}\cdot(\mathbf{E}(g_1^p))^{\frac{1}{p}}(\mathbf{E}(g_2^q))^{\frac{1}{q}}$$
$$\leqslant \mathbf{E}(g_1\cdot g_2)\leqslant(\mathbf{E}(g_1^p))^{\frac{1}{p}}(\mathbf{E}(g_2^q))^{\frac{1}{q}}. \tag{5}$$

Normalized Gheorghiu inequality in the case that $(\Omega,p,\mathscr{F})$ is a probability space is given in [5]. Functions $g_1=X$ and $g_2=Y$ are random variables and $\mathbf{E}(g_1)=E[X]$ is the mathematical expectation of random variable X.

COROLLARY 2. *Suppose that random variables X and Y capture their values $0<\alpha\leqslant X\leqslant 1$ and $0<\beta\leqslant Y\leqslant 1$. Equality $\frac{1}{p}+\frac{1}{q}=1$ implies*

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}(\beta-\alpha\beta^q)^{\frac{1}{p}}(\alpha-\alpha^p\beta)^{\frac{1}{q}}}{1-\alpha^p\beta^q}(E[X^p])^{\frac{1}{p}}(E[Y^q])^{\frac{1}{q}}\leqslant E[XY]\leqslant(E[X^p])^{\frac{1}{p}}(E[Y^q])^{\frac{1}{q}}$$

in the case of positive p and q.

3. Original Gheorghiu's inequality

In the article [9], the converse of Hölder's inequality was obtained. Here it is presented in the next theorem.

THEOREM 8. *Suppose that $a_1,a_2,\ldots a_n,b_1,b_2,\ldots b_n$ are given $2n$ positive real numbers. Let pair (a,A) represents the minimal and maximal number among $a_1,a_2,\ldots,$ a_n and in the same manner let (b,B) be the pair of those among the $b_1,b_2,\ldots,b_n$. Assume that p is a real number greater than 1. Then we have*

$$1\leqslant\frac{\left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^{\frac{p}{p-1}}\right)^{p-1}}{\left(\sum_{k=1}^n a_kb_k\right)^p}\leqslant\mu, \tag{6}$$

where

$$\mu=\frac{(p-1)^{p-1}}{p^p}\cdot\frac{A^{p-1}}{a^{p-1}}\cdot\frac{B}{b}\cdot\frac{\left(1-\frac{a^pb^{\frac{p}{p-1}}}{A^pB^{\frac{p}{p-1}}}\right)^p}{\left(1-\frac{ab^{\frac{1}{p-1}}}{AB^{\frac{1}{p-1}}}\right)\left(1-\frac{a^{p-1}b}{A^{p-1}B}\right)^{p-1}}. \tag{7}$$

The left inequality has been demonstrated by Hölder and Jensen. Theorem 8 could be modulated in the terms that are given in the introduction of this paper.

THEOREM 7'. *Suppose that* $\Omega=\{1,2,3,\ldots,n\}$ *and* $g_1,g_2:\Omega\to\mathbb{R}$ *are given real functions. Let* $a=\min\{g_1(k),k\in\Omega\}$, $A=\max\{g_2(k),k\in\Omega\}$, $b=\min\{g_2(k),k\in\Omega\}$ *and* $B=\max\{g_2(k),k\in\Omega\}$. *Let* $\mathbf{E}(g)=\frac{1}{n}\sum_{k=1}^{n}g(k)$. *If* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$1\leqslant\frac{\mathbf{E}(g_1^p)^{\frac{1}{p}}\cdot\mathbf{E}(g_2^q)^{\frac{1}{q}}}{\mathbf{E}(g_1\cdot g_2)}\leqslant\mu^{\frac{1}{p}},\tag{8}$$

where μ *is given by (7).*

Inequality (8) can be expressed as the chain of inequalities alike the (5):

$$\mu^{-\frac{1}{p}}\cdot\mathbf{E}(g_1^p)^{\frac{1}{p}}\cdot\mathbf{E}(g_2^q)^{\frac{1}{q}}\leqslant\mathbf{E}(g_1\cdot g_2)\leqslant\mathbf{E}(g_1^p)^{\frac{1}{p}}\cdot\mathbf{E}(g_2^q)^{\frac{1}{q}}\tag{9}$$

The next proposition shows that left inequality in (5) is better than the left inequality in (9).

PROPOSITION 3. Under the assumptions of Theorem 7', the next is valid:

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(A^paB-a^pbA\right)^{\frac{1}{q}}}{A^pB^q-a^pb^q}=p\cdot\mu^{-\frac{1}{p}}.\tag{10}$$

Proof. Using elementary algebra we get

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}a^{\frac{1}{q}}b^{\frac{1}{p}}\left(AB^{\frac{q}{p}}-ab^{\frac{q}{p}}\right)^{\frac{1}{p}}\left(A^{\frac{p}{q}}B-a^{\frac{p}{q}}b\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}A^{\frac{p}{q^2}+1-p}B^{\frac{q}{p^2}+1-q}\left(A^pB^q-a^pb^q\right)}.$$

Separately, using relation $p-1=\frac{p}{q}$, we have $\frac{p}{q^2}+1-p=-\frac{1}{q}$ and $\frac{q}{p^2}+1-q=-\frac{1}{p}$. The proof is prolonging with

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}A^{\frac{1}{q}}a^{\frac{1}{q}}B^{\frac{1}{p}}b^{\frac{1}{p}}\left(AB^{\frac{q}{p}}-ab^{\frac{q}{p}}\right)^{\frac{1}{p}}\left(A^{\frac{p}{q}}B-a^{\frac{p}{q}}b\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}\left(A^pB^q-a^pb^q\right)}.$$

By selective multiplying factors and brackets with the same exponent we have

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}\left(AbB^{\frac{q}{p}+1}-ab^{\frac{q}{p}+1}B\right)^{\frac{1}{p}}\left(aA^{\frac{p}{q}+1}B-a^{\frac{p}{q}+1}bA\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}\left(A^pB^q-a^pb^q\right)}.$$

Considering that $\frac{q}{p}+1=q$ and $\frac{p}{q}+1=q$ we finally obtain that

$$\mu^{-\frac{1}{p}}=\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(aA^pB-a^pbA\right)^{\frac{1}{q}}}{p\left(A^pB^q-a^pb^q\right)}.$$

The last equation is the same as (10) and the proof is finished. □

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