Mathematical
Inequalities
& Applications
Volume 19, Number 4 (2016), 1203–1226 doi:10.7153/mia-19-89

JENSEN, HÖLDER, MINKOWSKI, JENSEN–STEFFENSEN AND SLATER–PEČARIĆ INEQUALITIES DERIVED THROUGH N–QUASICONVEXITY

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In honour of Professors
Neven Elezović, Marko Matić and Ivan Perić
on the occasion of their 60th birthdays

(*Communicated by C. P. Niculescu*)

Abstract. Jensen, Hölder, Minkowski, Jensen-Steffensen and Slater-Pečarić type inequalities derived by the properties of γ-quasiconvex functions that we deal with here, can be seen as analog to these for superquadratic functions and refinements of these for convex functions.

1. Introduction

We deal here with inequalities satisfied by one of the many variants of convex functions. These functions are called γ-quasiconvex functions and have already been dealt with by S. Abramovich, L.-E. Persson and N. Samko. The basic facts on γ-quasiconvexity and superquadracity on which this paper is built, can be found in [4], [6], and [7].

The importance of convex functions is obvious and widely acknowledged. Numerous publications deal with convex functions, their properties and applications. In particular we refer to the classical 1964 book "Inequalities" by Hardy, Littlewood and Polya [9], the 1992 book "Convex functions, partial ordering and statistical applications" by Pecaric, Proschan and Tong [15] and to the 2006 book "Convex functions and their applications – a contemporary approach" by Niculescu and Persson [13]. Out of dealing with the classical convex functions evolved many generalizations and refinements of this notion, see in particular Capter 2 in [13].

Among the many types of refinements and generalizations of convex functions are the usual quasiconvexity, Morrey-convexity, Reitz-convexity, h-convexity, superquadracity and many others.

The subject of variants of convex functions and the comparison between them deserves at least every decade a large updated suvey which is out of the scope of this

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Jensen's inequality, Hölder's inequality, Minkowski inequality, convex function, γ-quasiconvex functions, N-quasiconvex functions, superquadratic functions.

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paper. We compare here results obtained through the use of γ-quasiconvexity and superquadracity as these notions are the subjects of this paper, see Remark 1 and Theorem 4.

The notion of γ-quasiconvexity with which we deal here is related but is not a special case of any of the cases mentioned above. Therefore it is reasonable to assume that our new and natural notion of γ-quasiconvexity will bring about new results and applications.

Currently the following is already known: The original Hardy's inequality has a "turning point" (the point where the inequality is reversed) at $p = 1$. This inequality can be proved directly by the properties of convex functions. (The proof can be found in [16] and its references.) But by using the γ-quasiconvexity we get a refined variant of the original Hardy's inequality where the turning point is any $p > 1$ (see [6] and [7]).

It is known that most of the classical inequalities can be obtained by the properties of convex functions, therefore it is reasonable to assume that using the properties of γ-quasiconvexity will bring about proofs of generalizationd and refinements of more classical inequalities.

We know that if $f := \mathbb{R}^+ \to \mathbb{R}^+$ is a concave function then $\frac{f(x)}{x}$ is not increasing (see for instance [11, page 142] and [17]). This is one of the reasons it is natural to deal with quasi-monotone functions, that is with γ-quasidecreasing functions. About the importance of this notion especially to theories related to approximation and interpolations see [11].

The notion of a γ-quasiconvex function is analog to a quasimonotone function (that is to γ-quasiincreasing functions). Therefore we hope to get in future publications analog results to those we know about quasimonotone functions, in addition to those mentioned above related to Hardy's inequalities and to those dealt with in this paper which are related in particular to Jensen, Hölder and Slater Pečarić inequalities.

γ-quasiconvex functions and superquadratic functions are closely related and therefore it is interesting to show side by side results related to these two sets.

We start with a definition of and lemmas about γ-quasiconvexity.

DEFINITION 1. Let γ be a real number. A real-valued function f defined on an interval $[0,b)$ with $0 < b \leqslant \infty$ is called γ-quasiconvex if it can be represented as the product of a comvex function and the power function x^γ.

A convex function φ on $[0,b)$, $0 < b \leqslant \infty$ is characterized by the inequality

$$\varphi(y) - \varphi(x) \geqslant C_\varphi(x)(y-x), \quad \forall x,y \in (0,b], \qquad C_\varphi \in \mathbb{R}, \tag{1.1}$$

from which we establish easily the following lemmas:

LEMMA 1. [6, Lemma1] *Let* $\psi_\gamma(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}$, *where* φ *is convex on* $[0,b)$, *that is,* ψ_γ *is a* γ*-quasiconvex function. Then*

$$\psi_\gamma(y) - \psi_\gamma(x) \geqslant \varphi(x)(y^\gamma - x^\gamma) + C_\varphi(x)\, y^\gamma (y-x), \tag{1.2}$$

holds for all $x \in [0,b)$, $y \in [0,b)$, *where* $C_\varphi(x)$ *is defined by (1.1).*

The following is derived by some computation on the right handside of (1.2), see also [7, Lemma 2]:

LEMMA 2. [7] *Let* φ *be convex differentiable function and let* $\psi_k(x) = x^k\varphi(x)$, $k = 0,1,\ldots,N$, *then the function* $\psi_N(x) = x^N\varphi(x)$, *satisfies for* $x,y \in [a,b)$, $a \geqslant 0$

$$\begin{aligned} &\psi_N(y) - \psi_N(x) \\ &\geqslant (\psi_N(x))'(y-x) + (y-x)^2 \sum_{k=1}^{N} y^{k-1}(\psi_{N-k}(x))' \\ &= (\psi_N(x))'(y-x) + (y-x)^2 \frac{\partial}{\partial x}\left(\frac{x^N - y^N}{x-y}\varphi(x)\right). \end{aligned} \tag{1.3}$$

Now we quote a definition and some basic properties of superquadratic functions.

DEFINITION 2. [4, Definition 2.1] A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geqslant C(x)(y-x) \tag{1.4}$$

for all $y \geqslant 0$.

From this definition we get that when φ is a superquadratic function, if $\varphi \geqslant 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$, see [4].

When $\varphi : [0,b) \to \mathbb{R}$ is differentiable non-negative, incresing convex and $\varphi(0) = 0$ the function $\psi_N(x) = x^N\varphi(x)$ is not only N-quasiconvex where N is a non-negative integer but also superquadratic. In particular the power functions $f(x) = x^p$, $p \geqslant 2$, $x \geqslant 0$ are superquadratic functions as well as 1-quasiconvex functions. The power functions $f(x) = x^p$, $p \geqslant N+1$, $N \geqslant 1$, $x \geqslant 0$, are also N-quasiconvex functions.

In Section 2 we deal with Jensen's type and Slater-Pečarić type inequalities when the coefficients $\alpha_i \geqslant 0$, $i = 1,\ldots,n$. In Section 5 we deal with inequalities for which the coefficients are not always non-negative. We call these coefficients *Steffensen's coefficients*. For such coefficients and for a function φ we get:

LEMMA 3. *Let* $\varphi : [0,\infty) \to \mathbb{R}$ *be a given function, let* $\boldsymbol{x}$ *be a nonnegative monotonic* n*-tuple in* $\mathbb{R}^n$, *and* $\boldsymbol{\rho}$ *a real* n*-tuple satisfying Steffensen's coefficients, that is*

$$\begin{gathered} 0 \leqslant P_j \leqslant P_n, \qquad j = 1,\ldots,n, \quad P_n > 0, \\ P_j = \sum_{i=1}^{j} \rho_i, \quad \overline{P}_j = \sum_{i=j}^{n} \rho_i, \qquad j = 1,\ldots,n \end{gathered} \tag{1.5}$$

Then

$$\begin{aligned} \sum_{i=1}^{n} \rho_i\varphi(x_i) = &\sum_{j=1}^{k-1} P_j\left(\varphi(x_j) - \varphi(x_{j+1})\right) + P_k\varphi(x_k) \\ &+ \overline{P}_{k+1}\varphi(x_{k+1}) + \sum_{j=k+2}^{n} \overline{P}_j\left(\varphi(x_j) - \varphi(x_{j-1})\right). \end{aligned} \tag{1.6}$$

Identity (1.6) is used in the proofs in Section 5 related to N-quasiconvex functions in a similar way as they are used in [15] and [1] for convex functions, in [2] for superquadratic functions and in [7] for 1-quasiconvex functions.

By using the results stated in Section 2 we get in Section 3 Hölder's type inequalities which are of the type

$$\int fg d\nu \lesseqgtr \left(\int g^q d\nu\right)^{1/q} \left(\int f^p d\nu\right)^{1/p} H(f,g)$$

that lately are widely discussed (see for instance [10], [12], [14], [19] and their references).

In Section 4 we prove Minkowski type inequalities by using again the results stated in Section 2.

In Section 5 we get more inequalities which are derived from the results from Section 2.

In Section 6 we get inequalities related to differences of "Jensen's gap" motivated by the work of Dragomir in [8]. The results in this section are analog to the results in [3].

2. Jensen and Slater-Pečarić type inequalities for N-quasiconvex functions

We quote first extensions of Jensen and of Slater-Pečarić inequalities for superquadratic functions which are proved in [4] and stated in Lemma A and in Theorem B.

LEMMA A. [4, Lemma 2.3] *Supppose that ψ is superquadratic on $[0,b)$ then*

$$\int_\Omega \psi(f(s))\,d\mu(s) - \psi\left(\int_\Omega f(s)\,d\mu(s)\right) \geqslant \int_\Omega \psi\left(\left|f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right|\right) d\mu(s), \tag{2.1}$$

where f is any non-negative μ-integrable function on a probability measure space (Ω,μ) and $\int_\Omega f(s)\,d\mu(s) > 0$.

The discrete version of (2.1) is:

Suppose that ψ is superquadratic on $[0,b)$. Let $0 \leqslant x_i < b$, $i = 1,\dots,n$ and let $\overline{x} = \sum_{i=1}^n \alpha_i x_i$ where $\alpha_i \geqslant 0$ and $\sum_{i=1}^n \alpha_i = 1$. Then

$$\sum_{i=1}^n \alpha_i \psi(x_i) - \psi(\overline{x}) \geqslant \sum_{i=1}^n \alpha_i \psi(|x_i - \overline{x}|)\ . \tag{2.2}$$

LEMMA B. [4, Theorem 2.4] *Suppose that ψ is superquadratic and that $C(f(s))$ is given as in Definition 2. If μ is a probability measure, f is any non-negative μ-measurable function, $\int C(f(s))\,d\mu(s) \neq 0$, and m and M as defined by*

$$m = \int f(s)\,d\mu(s) \quad \textit{and} \quad M = \frac{\int f(s)C(f(s))\,d\mu(s)}{\int C(f(s))\,d\mu(s)}.$$

then

$$\begin{aligned} &\psi(m)+\int \psi(|f(s)-m|)\,d\mu(s) \\ &\leqslant \int \psi(f(s))\,d\mu(s) \\ &\leqslant \psi(M)-\int \psi(|f(s)-M|)\,d\mu(s). \end{aligned}$$

The discrete version is: Suppose that ψ is superquadratic and C is as in Definition 2. Let $x_i \geqslant 0,\ i=1,\dots,n$ and let $\alpha_i \geqslant 0,\ \sum_{i=1}^{n}\alpha_i = 1$. If $\sum_{i=1}^{n}\alpha_i C(x_i) \neq 0$ we define $M = \frac{\sum_{i=1}^{n}\alpha_i x_i C(x_i)}{\sum_{i=1}^{n}\alpha_i C(x_i)}$. Then

$$\sum_{i=1}^{n}\alpha_i\psi(x_i) \leqslant \psi(M)-\sum_{i=1}^{n}\alpha_i\psi(|x_i-M|)\ ,$$

The following Theorem 1 may be considered an analog of lemmas A and B. In it we get refinements of Jensen's inequality and Slater-Pečarić inequality (see [1] and [15]). The refinements are obtained just by using (1.3) in Lemma 2 for each i and then summing up for $i=1,\dots,n$.

THEOREM 1. *Let $\varphi:[a,b)\to\mathbb{R},\ a\geqslant 0$ be convex differentiable function, and let $\psi_k(x)$ be $\psi_k(x)=x^k\varphi(x),\ k=0,1,\dots,N$, where $\psi_0=\varphi$. Let $\alpha_i\geqslant 0,\ x_i\in[a,b),\ i=1,\dots,n,\ \sum_{i=1}^{n}\alpha_i=1$. Then:*

1) A Jensen's type inequality holds where $\bar{x}=\sum_{i=1}^{n}\alpha_i x_i$:

$$\begin{aligned} &\sum_{i=1}^{n}\alpha_i\psi_N(x_i)-\psi_N(\bar{x}) \\ &\geqslant \sum_{i=1}^{n}\alpha_i\varphi(\bar{x})\left(x_i^N-\bar{x}^N\right)+\sum_{i=1}^{n}\alpha_i\varphi'(\bar{x})\,x_i^N(x_i-\bar{x}) \\ &=\sum_{i=1}^{n}\sum_{k=1}^{N}\alpha_i(x_i-\bar{x})^2 x_i^{k-1}\left(\psi_{N-k}(\bar{x})\right)' \\ &=\sum_{i=1}^{n}\alpha_i(x_i-\bar{x})^2\frac{\partial}{\partial\bar{x}}\left(\frac{\bar{x}^N-x_i^N}{\bar{x}-x_i}\varphi(\bar{x})\right). \end{aligned} \tag{2.3}$$

If φ is also non-negative and increasing then for $N=2,\dots,$ the above inequality refines Jensen's inequality. For $N=1$ we get for $\psi_1(x)=x\varphi(x)$

$$\sum_{i=1}^{n}\alpha_i\psi_1(x_i)-\psi_1(\bar{x})\geqslant\sum_{i=1}^{n}\alpha_i\varphi'(\bar{x})\,x_i(x_i-\bar{x})=\sum_{i=1}^{n}\alpha_i\varphi'(\bar{x})(x_i-\bar{x})^2. \tag{2.4}$$

If φ is increasing and convex (and not necessarily non-negative) then again (2.4) is a refinement of Jensen's inequality.

2) For a fixed $C \in [a,b)$ we get when $\alpha_i \geqslant 0,\ i=1,\dots,n,\ \sum_{i=1}^{n}\alpha_i = 1$ that

$$\begin{aligned}
&C^N\varphi(C) - \sum_{i=1}^{n}\alpha_i x_i^N \varphi(x_i)\\
&= \psi_N(C) - \sum_{i=1}^{n}\alpha_i \psi_N(x_i)\\
&\geqslant \sum_{i=1}^{n}\alpha_i \left(x_i^N\varphi(x_i)\right)'(C-x_i) + \sum_{i=1}^{n}\alpha_i (C-x_i)^2 \sum_{k=1}^{N} C^{k-1}\left(\psi_{N-k}(x_i)\right)'\\
&= \sum_{i=1}^{n}\alpha_i \left(x_i^N\varphi(x_i)\right)'(C-x_i) + \sum_{i=1}^{n}\alpha_i (C-x_i)^2 \frac{\partial}{\partial x_i}\left(\frac{x_i^N - C^N}{x_i - C}\varphi(x_i)\right).
\end{aligned}$$

3) Especially if $\sum_{i=1}^{n}\alpha_i\psi_N'(x_i) > 0$, and if $C = M_{\psi_N} = \frac{\sum_{i=1}^{n}\alpha_i x_i \psi_N'(x_i)}{\sum_{i=1}^{n}\alpha_i\psi_N'(x_i)} \in [a,b)$, then by using $\sum_{i=1}^{n}\alpha_i\psi_N'(x_i)\left(M_{\psi_N} - x_i\right) = 0$ we get a Slater-Pečarić type inequality

$$\begin{aligned}
&\psi_N\left(M_{\psi_N}\right) - \sum_{i=1}^{n}\alpha_i\psi_N(x_i)\\
&\geqslant \sum_{i=1}^{n}\sum_{k=1}^{N}\alpha_i\left(M_{\psi_N} - x_i\right)^2 M_{\psi_N}^{k-1}\left(\psi_{N-k}(x_i)\right)'\\
&= \sum_{i=1}^{n}\alpha_i\left(M_{\psi_N} - x_i\right)^2 \frac{\partial}{\partial x_i}\left(\frac{M_{\psi_N}^N - x_i^N}{M_{\psi_N} - x_i}\varphi(x_i)\right).
\end{aligned}$$

If φ is also non-negative and increasing then for $N = 1,\dots$ the above inequality is a refinement of Slater Pečarić inequality.

Theorem 1 Case 1 appears in [5, Corollary 1].

We get in [7, Theorem 1] the integral form of Jensen's type inequality for γ-quasiconvex functions and the special case when $\gamma = 1$ is:

LEMMA C. [7] *Let f be a non-negative function. Let f and $\varphi \circ f$ be μ-integrable functions on the probability measure space (Ω,μ) and $\int_\Omega f(s)\,d\mu(s) > 0$. Let also $\psi(x) = x\varphi(x)$. If φ is a differentiable convex on $[0,b)$, $0 < b \leqslant \infty$*

$$\begin{aligned}
&\int_\Omega \psi(f(s))\,d\mu(s) - \psi\left(\int_\Omega f(s)\,d\mu(s)\right)\\
&\geqslant \int_\Omega \varphi'\left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)\left(f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right)^2 d\mu(s).
\end{aligned}$$

hold. If φ is also increasing we get a refinement of Jensen's inequality.

EXAMPLE 1. Let $\varphi(x) = e^{x^3}$, $\psi(x) = xe^{x^3}$ then from the convexity of ψ we get that $\int_0^1 \psi(x)\,dx \geqslant \frac{e^{\frac{1}{8}}}{2}$ and from the 1-quasiconvexity we get the better result $\int_0^1 \psi(x)\,dx \geqslant \frac{5e^{\frac{1}{8}}}{8}$.

REMARK 1. In [7, Proposition 5] it is proved that: Let f be a non-negative function. Let f and $\varphi \circ f$ be μ-integrable functions on the probability measure space (Ω, μ) and $\int_\Omega f(s)\,d\mu(s) > 0$. Let also $\psi(x) = x\varphi(x)$. If φ is a differentiable non-negative convex increasing on $[0,b)$, $0 < b \leqslant \infty$ and $\varphi(0) = \lim_{z \to 0^+} z\varphi'(z) = 0$ then ψ is also superquadratic and the inequalities

$$\begin{aligned}
&\int_\Omega \psi(f(s))\,d\mu(s) - \psi\left(\int_\Omega f(s)\,d\mu(s)\right) \\
\geqslant &\int_\Omega \varphi'\left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)\left(f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right)^2 d\mu(s) \\
\geqslant &\int_\Omega \psi\left(\left|f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right|\right) d\mu(s),
\end{aligned}$$

hold when $0 < f(s) \leqslant 2\int_\Omega f(\sigma)\,d\mu(\sigma)$ for every $s \in \Omega$, in particular when $0 < a \leqslant f(s) \leqslant 2a$, $s \in \Omega$.

The discrete form says there that: when $0 < x_i \leqslant 2\overline{x}$, $i = 1, \ldots, n$ and $\psi(x) = x\varphi(x)$ where $\varphi(x)$ is non-negative increasing differentiable and convex then Inequality (2.4) is better than (2.2) when $\varphi(0) = \lim_{z \to 0^+} z\varphi'(z) = 0$.

3. Hölder type inequalities derived from γ-quasiconvexity and superquadracity

In this section we use Jensen's type inequalities to prove new Hölder type inequalities and reversed Hölder type inequalities. We use in particular Lemma A, Lemma C and the following lemmas D and E to get refinements for $p \geqslant 2$ of Hölder inequality, lower bounds for $1 < p \leqslant 2$ and upper bounds when $0 < p < 1$.

LEMMA D. [7, Corollary 1] *Let $0 < p \leqslant 1$, and let f be a μ-measurable and positive function on the probability measure space (μ, Ω) and $x = \int_\Omega f(s)\,d\mu(s) > 0$. Then*

$$-I_1 + \left(\int_\Omega f(s)\,d\mu(s)\right)^p \leqslant \int_\Omega (f(s))^p\,d\mu(s) \leqslant \left(\int_\Omega f(s)\,d\mu(s)\right)^p,$$

where

$$I_1 = p\left(\int_\Omega f(s)\,d\mu(s)\right)^p \left(1 - \int_\Omega f(s)\,d\mu(s)\int_\Omega (f(s))^{-1}\,d\mu(s)\right) > 0.$$

LEMMA E. [7, Corollary 2] *Let $0 < p \leqslant 1$, let f be a non-negative μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\,d\mu(s) > 0$. Then*

$$-I_2 + \left(\int_\Omega f(s)\,d\mu(s)\right)^p \leqslant \int_\Omega (f(s))^p\,d\mu(s) \leqslant \left(\int_\Omega f(s)\,d\mu(s)\right)^p, \tag{3.1}$$

where

$$I_2 = p\left(\int_\Omega f(s)\,d\mu(s)\right)^{p-1} \int_\Omega \frac{(f(s) - x)^2}{f(s)}\,d\mu(s). \tag{3.2}$$

As the power functions $\varphi(x)=x^p,\ x\geqslant 0$ are superquadratic when $p\geqslant 2$ and subquadratic when $1\leqslant p\leqslant 2$, we get from Lemma A that for $p\geqslant 2$

$$\int_\Omega (f(s))^p d\mu(s)-\left(\int_\Omega f(s)\,d\mu(s)\right)^p \geqslant \int_\Omega \left(\left|f(s)-\int_\Omega f(\sigma)\,d\mu(\sigma)\right|\right)^p d\mu(s), \tag{3.3}$$

holds, where f is any non-negative μ-integrable function on a probability measure space (Ω,μ) and $\int_\Omega f(s)\,d\mu(s)>0$.

In [18, Theorem 1.4] a refinement of Hölder's inequality is proved:

THEOREM 2. *For $p\geqslant 2$ and for any two non-negative ν-measurable functions f and g and for $\frac{1}{p}+\frac{1}{q}=1$ we get a refinement of Hölder inequality*

$$\begin{aligned}\int_\Omega fg\,d\nu &\leqslant \left(\int_\Omega f^p d\nu-\int_\Omega\left|f-g^{q-1}\frac{\int_\Omega fg\,d\nu}{\int_\Omega g^q d\nu}\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}\\ &=\left(\int_\Omega f^p d\nu-\int_\Omega\left|fg^{1-q}-\frac{\int_\Omega fg\,d\nu}{\int_\Omega g^q d\nu}\right|^p g^q d\nu\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}.\end{aligned}$$

In the case $1<p\leqslant 2$ we get for any two non-negative ν-measurable functions f and g when $\int_\Omega f^p d\nu\geqslant\int_\Omega\left|f-g^{q-1}\frac{\int_\Omega fg\,d\nu}{\int_\Omega g^q d\nu}\right|^p d\nu$, that

$$\begin{aligned}&\left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}\\ &\geqslant\int_\Omega fg\,d\nu\geqslant\left(\int_\Omega f^p d\nu-\int_\Omega\left|f-g^{q-1}\frac{\int_\Omega fg\,d\nu}{\int_\Omega g^q d\nu}\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}.\end{aligned}$$

From Lemma C we get that for the 1-quasiconvex functions $\varphi(x)=x^p,\ x\geqslant 0$, $p\geqslant 2$ the inequality

$$\begin{aligned}&\int_\Omega (f(s))^p d\mu(s)-\left(\int_\Omega f(s)\,d\mu(s)\right)^p\\ &\geqslant (p-1)\left(\int_\Omega f(s)\,d\mu(s)\right)^{p-2}\int_\Omega\left(f(s)-\int_\Omega f(s)\,d\mu(s)\right)^2 d\mu(s)\end{aligned}\tag{3.4}$$

holds.

Theorem 3, which is another refinement of Hölder inequality, follows in the same way that Hölder's inequality follows from Jensen's inequality by fixing a non-negative ν-measurable functions f and g and applying (3.4) with fg^{1-q} in place of f and $\frac{g^q d\nu}{\int_\Omega g^q d\nu}$ in place of $d\mu$ where $\frac{1}{p}+\frac{1}{q}=1$:

THEOREM 3. *Let $p \geqslant 2$ and define q by $\frac{1}{p}+\frac{1}{q}=1$. Then for any two nonnegative ν-measurable functions f and g*

$$\begin{aligned}&\int_\Omega fgd\nu \\ &\leqslant \left(\int_\Omega f^p d\nu-(p-1)\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^{p-2}\int_\Omega\left(fg^{(1-q)}-\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^2 g^q d\nu\right)^{\frac{1}{p}} \\ &\quad \times\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}.\end{aligned} \tag{3.5}$$

If $1<p\leqslant 2$ we get when $\int_\Omega f^p d\nu \geqslant (p-1)\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^{p-2}\int_\Omega\left(fg^{(1-q)}-\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^2 g^q d\nu$, that

$$\begin{aligned}&\left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \\ &\geqslant \int_\Omega fgd\nu \\ &\geqslant \left(\int_\Omega f^p d\nu-(p-1)\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^{p-2}\int_\Omega\left(fg^{(1-q)}-\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^2 g^q d\nu\right)^{\frac{1}{p}} \\ &\quad \times\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}.\end{aligned} \tag{3.6}$$

The last inequalities emphasize that through the 1-quasiconvexity and 1-quasiconcavity notions we get refined Hölder inequality for $p \geqslant 2$ in (3.5) and a lower bound in (3.6) for $1<p\leqslant 2$.

From Remark 1 it follows that:

THEOREM 4. *Under the same conditions as in Theorems 2 and 3 we get that the refinement of Hölder inequality derived from the 1-quasiconvexity of x^p, $x \geqslant 0$, $p \geqslant 2$ is better that the refinement derived from its superquadracity when $0 \leqslant fg^{(1-q)} \leqslant 2\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}$ that is we get that*

$$\begin{aligned}\int_\Omega fgd\nu &\leqslant \left(\int_\Omega f^p d\nu-\Delta_1\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \\ &\leqslant \left(\int_\Omega f^p d\nu-\Delta_2\right)^{\frac{1}{p}}\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\Delta_1=(p-1)\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^{p-2}\int_\Omega\left(fg^{(1-q)}-\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^2 g^q d\nu$$

and

$$\Delta_2 = \int_\Omega \left| fg^{(1-q)} - \frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu} \right|^p g^q d\nu.$$

From Lemma E we get a two sided Hölder type inequality:

THEOREM 5. *Let* $0 < p \leqslant 1$, f *and* g *be non-negative* μ*-measurable functions on the probability measure space* (Ω, ν) *then*

$$\begin{aligned} &\left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \\ &\leqslant \int_\Omega fgd\nu \\ &\leqslant \left(\int_\Omega f^p d\nu + p\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^{p-1} \int_\Omega \left(fg^{(1-q)} - \frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^2 \frac{g^{2q-1}}{f} d\nu\right)^{\frac{1}{p}} \\ &\times \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

Proof. To get a refinement of Hölder inequality, from Lemma E we fix as before a non-negative ν-measurable functions f and g and apply (3.1) and (3.2) with fg^{1-q} in place of f and $\frac{g^q d\nu}{\int_\Omega g^q d\nu}$ in place of $d\mu$ where $\frac{1}{p} + \frac{1}{q} = 1$ and get the right side of (3.7) by a simple computation, and together with Hölder inequality for $0 < p \leqslant 1$ which says that

$$\left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \leqslant \int_\Omega fgd\nu$$

(3.7) is obtained. □

Similarly we get from Lemma D that

THEOREM 6. *Let* $0 < p \leqslant 1$, f *and* g *be non-negative* μ*-measurable functions on the probability measure space* (Ω, ν), *then*

$$\begin{aligned} &\int_\Omega fgd\nu \\ &\leqslant \left(\int_\Omega f^p d\nu + p\left(\frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu}\right)^p \left(\int_\Omega g^q d\nu - \frac{\int_\Omega fgd\nu}{\int_\Omega g^q d\nu} \int_\Omega \frac{g^{2q-1}}{f} d\nu\right)\right)^{\frac{1}{p}} \\ &\times \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}}. \end{aligned}$$

Hölder type inequality for $0 < p \leqslant \frac{1}{2}$ and for $\frac{1}{2} \leqslant p < 1$ which we get now are derived again from the theorems related to 1-quasiconvex functions but are obtained by different substitutions that those employed up to now.

THEOREM 7. *Let* $0 < p \leqslant \frac{1}{2}$ *and define* $\frac{1}{p} + \frac{1}{q} = 1$. *then for any positive* ν*-measurable function* f *and* g

$$\int_\Omega fg d\nu \geqslant \left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \times \left[1 + \left(\frac{1}{p} - 1\right) \int_\Omega \left(\frac{f^p \int_\Omega g^q d\nu - g^q \int_\Omega f^p d\nu}{\int_\Omega f^p d\nu}\right)^2 \frac{g^{-q}}{\int_\Omega g^q d\nu} d\nu\right] \tag{3.8}$$

is derived, which is a refinement of Hölder inequality.

For $\frac{1}{2} \leqslant p < 1$, *we get the reverse of inequality (3.8) and together with Hölder inequality for* $0 < p < 1$

$$\begin{aligned} &\left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}} \\ &\leqslant \int_\Omega fg d\nu \\ &\leqslant \left(\int_\Omega g^q d\nu\right)^{\frac{1}{q}} \left(\int_\Omega f^p d\nu\right)^{\frac{1}{p}} \\ &\quad \times \left[1 + \left(\frac{1}{p} - 1\right) \int_\Omega \left(\frac{f^p \int_\Omega g^q d\nu - g^q \int_\Omega f^p d\nu}{\int_\Omega f^p d\nu}\right)^2 \frac{g^{-q}}{\int_\Omega g^q d\nu} d\nu\right] \end{aligned} \tag{3.9}$$

is derived.

Proof. For simplicity we denote $\int_\Omega$ as $\int$. For $\frac{1}{p} \geqslant 2$ we use the inequality for the 1-quasiconvex function $x^{\frac{1}{p}}$

$$\int f^{\frac{1}{p}} d\mu \geqslant \left(\int f d\mu\right)^{\frac{1}{p}} \left(1 + \left(\frac{1}{p} - 1\right) \int \left(\frac{f - \int f d\mu}{\int f d\mu}\right)^2 d\mu\right). \tag{3.10}$$

We fix now non-negative ν measurable functions f and g and apply (3.10) with $f^p g^{-q}$ in place of f and $d\mu = \frac{g^q d\nu}{\int g^q d\nu}$. Therefore $\int f d\mu$ is replaced by $\frac{\int f^p d\nu}{\int g^q d\nu}$, $\int f^{\frac{1}{p}} d\mu$ is replaced by $\frac{\int fg d\nu}{\int g^q d\nu}$, and apply (3.10) and get

$$\frac{\int fg d\nu}{\int g^q d\nu} \geqslant \left(\frac{\int f^p d\nu}{\int g^q d\nu}\right)^{\frac{1}{p}} \left[1 + \left(\frac{1}{p} - 1\right) \int \left(\frac{f^p g^{-q} - \frac{\int f^p d\nu}{\int g^q d\nu}}{\frac{\int f^p d\nu}{\int g^q d\nu}}\right)^2 \frac{g^q}{\int g^q d\nu} d\nu\right]. \tag{3.11}$$

from which (3.8) is obtained.

The proof of (3.9) is similar using the 1-quasiconcavity of $x^{\frac{1}{p}}$, $\frac{1}{2} \leqslant p < 1$. □

Similarly, using the superquadracity of $x^{\frac{1}{p}}$, $x \geqslant 0$, $\frac{1}{p} \geqslant 2$ and the subquadracity of $x^{\frac{1}{p}}$, $1 \leqslant \frac{1}{p} \leqslant 2$ and under the same condition as in 7 we get in a similar way when

$0 < p \leqslant \frac{1}{2}$ the inequality

$$\int fgd\nu \geqslant \left(\int f^p d\nu\right)^{\frac{1}{p}} \left(\int g^p d\nu\right)^{\frac{1}{q}} \times \left[1 + \int \left|\frac{f^p \int g^q d\nu - g^q \int f^p d\nu}{\int f^p d\nu}\right|^{\frac{1}{p}} \frac{gd\nu}{\int g^p d\nu}\right]$$

and the reverse inequality holds when $\frac{1}{2} \leqslant p < 1$, and together with Hölder inequality for $0 < p < 1$ we get

$$\left(\int f^p d\nu\right)^{\frac{1}{p}} \left(\int g^p d\nu\right)^{\frac{1}{q}} \leqslant \int fgd\nu \leqslant \left(\int f^p d\nu\right)^{\frac{1}{p}} \left(\int g^p d\nu\right)^{\frac{1}{q}} \times \left[1 + \int \left|\frac{f^p \int g^q d\nu - g^q \int f^p d\nu}{\int f^p d\nu}\right|^{\frac{1}{p}} \frac{gd\nu}{\int g^p d\nu}\right]$$

4. Minkowski type inequalities using 1-quasiconvexity

By using Theorem 3 we get Minkowski type inequalities:

THEOREM 8. *Let $p \geqslant 2$ and let $\frac{1}{q} = 1 - \frac{1}{p}$. Then for any two non-negative ν-measurable functions f and g*

$$\left(\int (f+g)^p d\nu\right)^{\frac{1}{p}} \leqslant \left(\int f^p d\nu - D\left(\int f(f+g)^{p-1} d\nu\right)^{p-2}\right)^{\frac{1}{p}} + \left(\int g^p d\nu - D\left(\int g(f+g)^{p-1} d\nu\right)^{p-2}\right)^{\frac{1}{p}} \tag{4.1}$$

where

$$D = (p-1)\left(\int \left(\frac{\left(g\int f(f+g)^{p-1} d\nu - f\int g(f+g)^{p-1} d\nu\right)^2 (f+g)^{p-2}}{\left(\int (f+g)^p d\nu\right)^p}\right) d\nu\right). \tag{4.2}$$

Proof. Inequality (4.1) follows from inequality (3.5) in the same way that Minkowski's inequality follows from Hölder's and as Minkowski's inequality for superquadratic functions x^p, $p \geqslant 2$, $x \geqslant 0$ follows from Hölder's inequality for superquadratic functions (see [18]).

Let $p \geqslant 2$ and apply (3.5) with g replaced by $(f+g)^{p-1}$ and we get

$$\int f(f+g)^{p-1}d\nu \leqslant \left(\int (f+g)^p d\nu\right)^{\frac{1}{q}} \times \left[\int f^p d\nu - (p-1)\left(\frac{\int f(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu}\right)^{p-2} \times \int \left(\frac{f}{f+g} - \frac{\int f(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu}\right)^2 (f+g)^p d\nu\right]^{\frac{1}{p}}.$$

Interchanging the roles of f and g yields

$$\int g(f+g)^{p-1}d\nu \leqslant \left(\int (f+g)^p d\nu\right)^{\frac{1}{q}} \left[\int g^p d\nu - (p-1)\left(\frac{\int g(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu}\right)^{p-2} \times \int \left(\frac{g}{f+g} - \frac{\int g(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu}\right)^2 (f+g)^p d\nu\right]^{\frac{1}{p}}.$$

Adding the last two inequalities gives after simple computation Inequality (4.1). □

The following Theorem 9 follows from inequality (3.6) by a similar argument as Theorem 8 follows from inequality (3.5).

THEOREM 9. *Let $1 < p \leqslant 2$ and let $\frac{1}{q} = 1 - \frac{1}{p}$. Then for any two non-negative ν-measurable functions f and g*

$$\left(\int f^p d\nu\right)^{\frac{1}{p}} + \left(\int g^p d\nu\right)^{\frac{1}{p}} \geqslant \left(\int (f+g)^p d\nu\right)^{\frac{1}{p}} \geqslant \left(\int f^p d\nu - D\left(\int f(f+g)^{p-1}d\nu\right)^{p-2}\right)^{\frac{1}{p}} + \left(\int g^p d\nu - D\left(\int g(f+g)^{p-1}d\nu\right)^{p-2}\right)^{\frac{1}{p}} \tag{4.3}$$

where

$$D = (p-1)\left(\int\left(\frac{\left(g\int f(f+g)^{p-1}d\nu - f\int g(f+g)^{p-1}d\nu\right)^2 (f+g)^{p-2}}{\left(\int (f+g)^p d\nu\right)^p}\right)d\nu\right). \tag{4.4}$$

and $\int f^p d\nu \geqslant D\left(\int f(f+g)^{p-1}d\nu\right)^{p-2}$, $\int g^p d\nu \geqslant D\left(\int g(f+g)^{p-1}d\nu\right)^{p-2}$.

Now we get Minkowski's type inequalities when $0<p\leqslant\frac{1}{2}$ and when $\frac{1}{2}\leqslant p<1$.

THEOREM 10. *Let $0<p\leqslant\frac{1}{2}$ and define $\frac{1}{p}+\frac{1}{q}=1$. Then for any two non-negative ν-measurable functions f and g*

$$\begin{aligned}&\left(\int (f+g)^p\,d\nu\right)^{\frac{1}{p}} \\ \geqslant&\left(\int f^p d\nu\right)^{\frac{1}{p}} \\ &\times\left[1+\left(\frac{1}{p}-1\right)\int\left(\frac{(f+g)^p\int f^p d\nu-f^p\int (f+g)^p\,d\nu}{\int f^p d\nu}\right)^2\frac{(f+g)^{-p}}{\int (f+g)^p\,d\nu}d\nu\right] \\ &+\left(\int g^p d\nu\right)^{\frac{1}{p}} \\ &\times\left[1+\left(\frac{1}{p}-1\right)\int\left(\frac{(f+g)^p\int g^p d\nu-g^p\int (f+g)^p\,d\nu}{\int g^p d\nu}\right)^2\frac{(f+g)^{-p}}{\int (f+g)^p\,d\nu}d\nu\right].\end{aligned}\tag{4.5}$$

When $\frac{1}{2}\leqslant p<1$ we get

$$\begin{aligned}&\left(\int f^p d\nu\right)^{\frac{1}{p}}+\left(\int g^p d\nu\right)^{\frac{1}{p}} \\ \leqslant&\left(\int (f+g)^p\,d\nu\right)^{\frac{1}{p}} \\ \leqslant&\left(\int f^p d\nu\right)^{\frac{1}{p}} \\ &\times\left[1+\left(\frac{1}{p}-1\right)\int\left(\frac{(f+g)^p\int f^p d\nu-f^p\int (f+g)^p\,d\nu}{\int f^p d\nu}\right)^2\frac{(f+g)^{-p}}{\int (f+g)^p\,d\nu}d\nu\right] \\ &+\left(\int g^p d\nu\right)^{\frac{1}{p}} \\ &\times\left[1+\left(\frac{1}{p}-1\right)\int\left(\frac{(f+g)^p\int g^p d\nu-g^p\int (f+g)^p\,d\nu}{\int g^p d\nu}\right)^2\frac{(f+g)^{-p}}{\int (f+g)^p\,d\nu}d\nu\right].\end{aligned}\tag{4.6}$$

Proof. We use inequality (3.10) for $\frac{1}{p}>2$ to get (4.5). We fix non-negative ν-measurable functions f and g and apply (3.10) with $\left(\frac{f}{f+g}\right)^p$ in place of f and $d\mu=\frac{(f+g)^p d\nu}{\int (f+g)^p d\nu}$. Therefore $\frac{\int f^p d\nu}{\int (f+g)^p d\nu}$ is in place of $\int f\,d\mu$, $\frac{f}{f+g}$ is in place of $f^{\frac{1}{p}}$,

$\frac{\int f(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu}$ is in place of $\int f^{\frac{1}{p}}d\mu$ and get

$$\begin{aligned} &\frac{\int f(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu} \\ &\geqslant \left(\frac{\int f^p d\nu}{\int (f+g)^p d\nu}\right)^{\frac{1}{p}} \\ &\times \left[1+\left(\frac{1}{p}-1\right)\int \left(\frac{(f+g)^{-p}f^p - \int f^p \left(\int (f+g)^p d\nu\right)^{-1}}{\int f^p d\nu \left(\int (f+g)^p d\nu\right)^{-1}}\right)^2 \frac{(f+g)^p}{\int (f+g)^p d\nu} d\nu\right]. \end{aligned} \tag{4.7}$$

Interchanging the roles of f and g yields

$$\begin{aligned} &\frac{\int g(f+g)^{p-1}d\nu}{\int (f+g)^p d\nu} \\ &\geqslant \left(\frac{\int g^p d\nu}{\int (f+g)^p d\nu}\right)^{\frac{1}{p}} \\ &\times \left[1+\left(\frac{1}{p}-1\right)\int \left(\left(\frac{(f+g)^{-p}g^p - \int g^p \left(\int (f+g)^p d\nu\right)^{-1}}{\int g^p d\nu \left(\int (f+g)^p d\nu\right)^{-1}}\right)^2 \frac{(f+g)^p}{\int (f+g)^p d\nu}\right) d\nu\right]. \end{aligned} \tag{4.8}$$

Adding the last two inequalities gives (4.5). Similarly together with Minkowski's inequality for $0<p<1$ we get (4.6) for $\frac{1}{2}\leqslant p<1$. □

5. Jensen and Slater-Pečarić type inequalities for Steffensen's coefficients

In Section 2 we dealt with Jensen's type and Slater-Pečarić type inequalities when the coefficients $\alpha_i \geqslant 0,\ i=1,\dots,n$.

We prove now a Jensen-Steffensen type inequality and a Slater-Pečarić type inequality for N-quasiconvex functions, when N is an integer, and the coefficients are not necessarily non-negative.

An extension of Jensen Steffensen inequality is proved in [2] for a non-negative superquadratic function which is therefore also increasing and convex:

THEOREM 11. *Let $\psi:[0,\infty)\to\mathbb{R}$ be differentiable superquadratic and nonnegative. Let $\mathbf{x}$ be a nonnegative monotonic n-tuple in $\mathbb{R}^n$, and $\boldsymbol{\rho}$ a real n-tuple satisfying Steffensen's coefficients. Let $\overline{x}$ be defined by $\overline{x}=\frac{1}{P_n}\sum_{i=1}^n \rho_i x_i$. Then*

$$\begin{aligned} \sum_{i=1}^n \rho_i\psi(x_i) - P_n\psi(\overline{x}) \geqslant &\sum_{j=1}^{k-1} P_j\psi\left(\left|x_j-x_{j+1}\right|\right) + P_k\psi\left(\left|x_k-\overline{x}\right|\right) \\ &+\overline{P}_{k+1}\psi\left(\left|x_{k+1}-\overline{x}\right|\right) + \sum_{j=k+2}^n \overline{P}_j\psi\left(\left|x_j-x_{j-1}\right|\right) \end{aligned}$$

$$\geqslant \left(\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{n} \overline{P_i}\right) \psi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|x_i - \overline{x}\right|\right)}{\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{n} \overline{P_i}}\right)$$
$$\geqslant \left(\left(n-1\right) P_n\right) \psi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|x_i - \overline{x}\right|\right)}{\left(n-1\right) P_n}\right)$$

holds where $k \in \{1,\dots,n-1\}$ *satisfies* $x_k \leqslant \overline{x} \leqslant x_{k+1}$, *unless one of the following two cases occurs:*

(1) either $\overline{x} = x_1$ *or* $\overline{x} = x_n$,

(2) there exists $k \in \{3,\dots,n-2\}$ *such that* $\overline{x} = x_k$ *and* $P_j\left(x_j - x_{j+1}\right) = 0,\ j = 1,\dots,k-1,\ \overline{P}_j\left(x_j - x_{j-1}\right) = 0,\ j = k+1,\dots,n.$

In these two cases $\sum_{i=1}^{n} \rho_i \psi\left(x_i\right) - P_n \psi\left(\overline{x}\right) = 0.$

An extension of Slater-Pečarić inequality is proved in [2], for a non-negative superquadratic function which is therefore also increasing and convex:

THEOREM 12. [2] *Let* $\psi : [0,\infty) \to \mathbb{R}$ *be a differentiable nonnegative superquadratic function. Let* $\boldsymbol{\rho} = \left(\rho_1,\dots,\rho_n\right)$ *be Jensen-Steffensen coefficients and* $\boldsymbol{x} = \left(x_1,\dots,x_n\right)$ *be a non-negative increasing* n*-tuple. If* $\sum_{i=1}^{n} \rho_i \psi'\left(x_i\right) \neq 0$ *we define* $M = \frac{\sum_{i=1}^{n} \rho_i x_i \psi'\left(x_i\right)}{\sum_{i=1}^{n} \rho_i \psi'\left(x_i\right)}$. *Then:*

Case A: for s *satisfying* $x_s \leqslant M \leqslant x_{s+1},\ s+1 \leqslant n$,

$$\sum_{i=1}^{n} \rho_i \psi\left(x_i\right)$$
$$\leqslant P_n \psi(M) - \left(\sum_{j=1}^{s-1} P_j \psi(x_{j+1} - x_j) + P_s \psi(M - x_s) + \overline{P}_{s+1} \psi(x_{s+1} - M) + \sum_{j=s+2}^{n} \overline{P}_j \psi(x_j - x_{j-1})\right)$$
$$\leqslant P_n \psi\left(M\right) - \left(\sum_{j=1}^{s} P_j + \sum_{j=s+1}^{n} \overline{P}_j\right) \psi\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - M\right|}{\sum_{j=1}^{s} P_j + \sum_{j=s+1}^{n} \overline{P}_j}\right)$$
$$\leqslant P_n \psi\left(M\right) - \left(\left(n-1\right) P_n\right) \psi\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - M\right|}{\left(n-1\right) P_n}\right)$$

holds, unless one of the following two cases occurs:

(1) either $\overline{x} = x_1$ *or* $\overline{x} = x_n$,

(2) there exists $k \in \{3,\dots,n-2\}$ *such that* $\overline{x} = x_k$ *and* $P_j\left(x_j - x_{j+1}\right) = 0,\ j = 1,\dots,s-1\ ,\overline{P}_j\left(x_j - x_{j-1}\right) = 0,\ j = s+1,\dots,n.$ *In these two cases* $\sum_{i=1}^{n} \rho_i \psi\left(x_i\right) - P_n \psi\left(M_{\psi_N}\right) = 0.$

Case B: for $M > x_n$: $\sum_{i=1}^{n} \rho_i \psi\left(x_i\right) \leqslant P_n \psi\left(M\right) - \left(nP_n\right) \psi\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - M\right|}{nP_n}\right)$.

For 1-quasiconvex function ψ we present a Jensen's type inequality obtained in [7, Theorem 3]:

THEOREM 13. *Let $\rho_1,\dots,\rho_n$ be Jensen-Steffensen coefficients, that is, $0\leqslant P_k=\sum_{i=1}^{k}\rho_i\leqslant P_n$, $\overline{P}_k=\sum_{i=k}^{n}\rho_i\geqslant 0$, $P_n>0$, $k=1,\dots,n$, and let $\mathbf{x}=(x_1,\dots,x_n)>0$ satisfy $0<x_1\leqslant\dots\leqslant x_n$. Let φ be non-negative, increasing differentiable convex function defined on $x\geqslant 0$, and let $\psi(x)=x\varphi(x)$. Let $\overline{x}=\sum_{i=1}^{n}\frac{\rho_i x_i}{P_n}$. Let s be the integer that satisfies $0<x_s\leqslant\overline{x}\leqslant x_{s+1}\leqslant x_n$. Then we get*

$$\begin{aligned}
&\sum_{i=1}^{n}\rho_i\psi(x_i)-P_n\psi(\overline{x})\\
&\geqslant \varphi'(x_1)\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}\overline{P}_j\right)\left(\frac{\sum_{i=1}^{n}\rho_i|x_i-\overline{x}|}{\sum_{j=1}^{s}P_i+\sum_{j=s+1}^{n}\overline{P}_j}\right)^2\\
&\geqslant \varphi'(x_1)P_n\max\{s,n-s\}\left(\frac{\sum_{i=1}^{n}\rho_i|x_i-\overline{x}|}{P_n\max\{s,n-s\}}\right)^2\\
&\geqslant \varphi'(x_1)(n-1)P_n\left(\frac{\sum_{i=1}^{n}\rho_i|x_i-\overline{x}|}{(n-1)P_n}\right)^2\geqslant 0.
\end{aligned}$$

We state now a Jensen-Steffensen type inequality and Slater Pečarić type inequality for N-quasiconvex functions, when N is an integer. The proof of this theorem uses (1.3) and some of the techniques used in [2]. This is done using identity (1.6) for the convex function ψ_N.

THEOREM 14. *Let $\rho_1,\dots,\rho_n$ be Jensen-Steffensen coefficients, and let $\mathbf{x}=(x_1,\dots,x_n)$ satisfy $0<x_1\leqslant\dots\leqslant x_n$. Let φ be non-negative, increasing differentiable convex function defined on $x\geqslant 0$, and let $\psi_N(x)=x^N\varphi(x)$ where N is an integer. Let $\overline{x}=\sum_{i=1}^{n}\frac{\rho_i x_i}{P_n}$. Let s be the integer that satisfies $0<x_s\leqslant\overline{x}\leqslant x_{s+1}\leqslant x_n$. Then*

$$\begin{aligned}
&\sum_{i=1}^{n}\rho_i\psi_N(x_i)-P_n\psi_N(\overline{x}) \qquad (5.1)\\
&\geqslant \sum_{k=1}^{N}x_1^{k-1}\psi'_{N-k}(x_1)\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}\overline{P}_j\right)\left(\frac{\sum_{j=1}^{n}\rho_j|x_j-\overline{x}|}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)^2\\
&=\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}\overline{P}_j\right)\left(\frac{\sum_{j=1}^{n}\rho_j|x_j-\overline{x}|}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)^2\frac{\partial}{\partial x}\left(\frac{x^N-x_1^N}{x-x_1}\varphi(x)\right)/_{x=x_1}\\
&\geqslant (P_n\max\{s,n-s\})^{-1}\left(\sum_{i=1}^{n}\rho_i|x_i-\overline{x}|\right)^2\frac{\partial}{\partial x}\left(\frac{x^N-x_1^N}{x-x_1}\varphi(x)\right)/_{x=x_1}\\
&\geqslant ((n-1)P_n)^{-1}\left(\sum_{i=1}^{n}\rho_i|x_i-\overline{x}|\right)^2\frac{\partial}{\partial x}\left(\frac{x^N-x_1^N}{x-x_1}\varphi(x)\right)/_{x=x_1}\geqslant 0
\end{aligned}$$

holds, unless one of the following two cases occurs:

(1) either $\overline{x}=x_1$ or $\overline{x}=x_n$,

(2) there exists $k \in \{3,...,n-2\}$ *such that* $\overline{x} = x_k$ *and* $P_j\left(x_j - x_{j+1}\right) = 0,\ j = 1,...,k-1,\ \overline{P}_j\left(x_j - x_{j-1}\right) = 0,\ j = k+1,...,n.$

In these two cases $\sum_{i=1}^{n} \rho_i \psi(x_i) - P_n \psi(\overline{x}) = 0.$

A refinement of Slater-Pečarić inequality in case that ψ_N is N-quasiconvex functions uses the same techniques as in Theorem 12 and in the proof of Theorem 13 is as follows:

THEOREM 15. *Under the same conditions as in Theorem 14 on* $(\rho_1,...,\rho_n)$, *on* $(x_1,...,x_n)$ *and on* $\psi_k(x) = x^k \varphi(x),\ k = 0,1,...,N,$ *if* $\sum_{i=1}^{n} \rho_i \psi_N'(x_i) \neq 0$, *we define* $M_{\psi_N} = \frac{\sum_{i=1}^{n} \rho_i x_i \psi_N'(x_i)}{\sum_{i=1}^{n} \rho_i \psi_N'(x_i)}$. *Then,*

Case A: for s *satisfying* $x_s \leqslant M_{\psi_N} \leqslant x_{s+1},\ s+1 \leqslant n$,

$$\begin{aligned}
&\sum_{i=1}^{n} \rho_i \psi_N(x_i) - P_n \psi_N\left(M_{\psi_N}\right) \qquad (5.2)\\
&\leqslant -\sum_{k=1}^{N} x_1^{k-1} \psi_{N-k}'(x_1) \left(\sum_{j=1}^{s} P_j + \sum_{j=s+1} \overline{P}_j\right) \left(\frac{\sum_{j=1}^{n} \rho_j \left|x_j - \overline{x}\right|}{\sum_{j=1}^{s} P_j + \sum_{j=s+1}^{n} \overline{P}_j}\right)^2\\
&= -\left(\sum_{j=1}^{s} P_j + \sum_{j=s+1} \overline{P}_j\right)^{-1} \left(\sum_{j=1}^{n} \rho_j \left|x_j - \overline{x}\right|\right)^2 \frac{\partial}{\partial x}\left(\frac{x^N - x_1^N}{x - x_1} \varphi(x)\right) /_{x=x_1}\\
&\leqslant -\left(P_n \max\{s, n-s\}\right)^{-1} \left(\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|\right)^2 \frac{\partial}{\partial x}\left(\frac{x^N - x_1^N}{x - x_1} \varphi(x)\right) /_{x=x_1}\\
&\leqslant -\left((n-1) P_n\right)^{-1} \left(\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|\right)^2 \frac{\partial}{\partial x}\left(\frac{x^N - x_1^N}{x - x_1} \varphi(x)\right) /_{x=x_1} \leqslant 0
\end{aligned}$$

holds, unless one of the following two cases occurs:

(1) either $\overline{x} = x_1$ *or* $\overline{x} = x_n$,

(2) there exists $k \in \{3,...,n-2\}$ *such that* $\overline{x} = x_k$ *and* $P_j\left(x_j - x_{j+1}\right) = 0,\ j = 1,...,s-1,\ \overline{P}_j\left(x_j - x_{j-1}\right) = 0,\ j = s+1,...,n.$

In these two cases $\sum_{i=1}^{n} \rho_i \psi(x_i) - P_n \psi\left(M_{\psi_N}\right) = 0.$

Case B: for $M_{\psi_N} > x_n$,

$$\begin{aligned}
&\sum_{i=1}^{n} \rho_i \psi_N(x_i) - P_n \psi_N\left(M_{\psi_N}\right)\\
&\leqslant -(n P_n)^{-1} \left(\sum_{i=1}^{n} \rho_i \left|x_i - M_{\psi_N}\right|\right)^2 \frac{\partial}{\partial x}\left(\frac{x^N - x_1^N}{x - x_1} \varphi(x)\right) /x = x_1.
\end{aligned}$$

Proof (of Theorem 14). The proof follows step by step the proof of [7, Theorem 3]. Here we only replace $\varphi'(x_1)$ with $\frac{\partial}{\partial x}\left(\frac{x^N-x_1^N}{x-x_1}\varphi(x)\right)/_{x=x_1}$ which is non-negative when φ is non-negative increasing and convex. Therefore the detailed proof is omitted. □

Proof (of Theorem 15). It was proved in [1] that when $\boldsymbol{\rho}$ is satisfying (1.5), $\boldsymbol{x}$ is increasing, and ψ_N is non-negative increasing and convex and that $\sum_{i=1}^n \rho_i \psi_N'(x_i) > 0$, $\sum_{i=1}^n \rho_i x_i \psi_N'(x_i) \geqslant 0$, we get that $x_1 \leqslant \frac{\sum_{i=1}^n \rho_i x_i \psi_N'(x_i)}{\sum_{i=1}^n \rho_i \psi_N'(x_i)} = M_{\psi_N}$ holds.

Case A: For $x_1 \leqslant x_s \leqslant M_{\psi_N} \leqslant x_{s+1} \leqslant x_n$, we use identity (1.6) for $s \in \{1,\dots,n-1\}$, and as $P_j \geqslant 0$, $\overline{P}_j \geqslant 0$, $j = 1,\dots,n$, and φ is non-negative increasing and convex function, we get that the N-quasiconvex function ψ_N satisfies

$$
\begin{aligned}
& P_n \psi_N\left(M_{\psi_N}\right) - \sum_{i=1}^n \rho_i \psi_N(x_i) \qquad (5.3)\\
&= \sum_{j=1}^{s-1} P_j\left(\psi_N(x_{j+1}) - \psi_N(x_j)\right) + P_s\left(\psi_N\left(M_{\psi_N}\right) - \psi_N(x_s)\right)\\
&\quad + \overline{P}_{s+1}\left(\psi_N\left(M_{\psi_N}\right) - \psi_N(x_{s+1})\right) + \sum_{j=s+2}^n \overline{P}_j\left(\psi_N(x_{j-1}) - \psi_N(x_j)\right)\\
&\geqslant \Bigg[\sum_{j=1}^{s-1} P_j \psi_N'(x_j)(x_{j+1}-x_j) + P_s \psi_N'(x_s)\left(M_{\psi_N}-x_s\right)\\
&\quad + \overline{P}_{s+1}\psi_N'(x_{s+1})\left(M_{\psi_N}-x_{s+1}\right) + \sum_{j=s+2}^n \overline{P}_j \psi_N'(x_j)(x_{j-1}-x_j)\Bigg]\\
&\quad + \Bigg[\sum_{j=1}^{s-1} P_j (x_{j+1}-x_j)^2 \frac{\partial}{\partial x_j}\left(\frac{x_j^N - x_{j+1}^N}{x_j - x_{j+1}}\varphi(x_j)\right)\\
&\quad + P_s\left(M_{\psi_N}-x_s\right)^2 \frac{\partial}{\partial x_s}\left(\frac{x_s^N - M_{\psi_N}^N}{x_s - M_{\psi_N}}\varphi(x_s)\right)\\
&\quad + \overline{P}_{s+1}\left(x_{s+1}-M_{\psi_N}\right)^2 \frac{\partial}{\partial x_{s+1}}\left(\frac{x_{s+1}^N - M_{\psi_N}^N}{x_{s+1} - M_{\psi_N}}\varphi(x_{s+1})\right)\\
&\quad + \sum_{j=s+2}^n \overline{P}_j (x_j-x_{j-1})^2 \frac{\partial}{\partial x_j}\left(\frac{x_j^N - x_{j-1}^N}{x_j - x_{j-1}}\varphi(x_j)\right)\Bigg].
\end{aligned}
$$

It is shown in [2] that under our conditions on ρ, the first parenthesis in the right handside of (5.3) for the convex functions ψ_N is non-negative. Then from the N-quasiconvexity of ψ_N, the convexity of $f(x) = x^2$ we get from (5.3) that

$$
\begin{aligned}
& P_n \psi_N\left(M_{\psi_N}\right) - \sum_{i=1}^n \rho_i \psi_N(x_i)\\
&\geqslant [0] + \sum_{j=1}^{s-1} P_j (x_{j+1}-x_j)^2 \frac{\partial}{\partial x_j}\left(\frac{x_j^N - x_{j+1}^N}{x_j - x_{j+1}}\varphi(x_j)\right)
\end{aligned}
$$

$$+P_s\left(M_{\psi_N}-x_s\right)^2\frac{\partial}{\partial x_s}\left(\frac{x_s^N-M_{\psi_N}^N}{x_s-M_{\psi_N}}\varphi\left(x_s\right)\right)$$
$$+\overline{P}_{s+1}\left(x_{s+1}-M_{\psi_N}\right)^2\frac{\partial}{\partial x_{s+1}}\left(\frac{x_{s+1}^N-M_{\psi_N}^N}{x_{s+1}-M_{\psi_N}}\varphi\left(x_{s+1}\right)\right)$$
$$+\sum_{j=s+2}^{n}\overline{P}_j\left(x_j-x_{j-1}\right)^2\frac{\partial}{\partial x_j}\left(\frac{x_j^N-x_{j-1}^N}{x_j-x_{j-1}}\varphi\left(x_j\right)\right)$$
$$\geqslant\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\left(\frac{\sum_{j=1}^{s-1}P_j\left(x_{j+1}-x_j\right)+P_s\left(M_{\psi_N}-x_s\right)}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right.$$
$$\left.+\frac{\overline{P}_{s+1}\left(x_{s+1}-M_{\psi_N}\right)+\sum_{j=s+2}^{n}\overline{P}_j\left(x_j-x_{j-1}\right)}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)^2\frac{\partial}{\partial x}\left(\frac{x_1^N-x^N}{x_1-x}\varphi\left(x\right)\right)/x=x_1$$
$$=\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)^{-1}\left(\sum_{i=1}^{n}\rho_i\left|x_i-M_{\psi_N}\right|\right)^2\frac{\partial}{\partial x}\left(\frac{x_1^N-x^N}{x_1-x}\varphi\left(x\right)\right)/x=x_1$$

In the same way as in the proof of the Theorem 13, we conclude that

$$\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)^{-1}\left(\sum_{j=1}^{n}\rho_i\left(\left|x_i-M\right|\right)\right)^2$$
$$\geqslant\left((n-1)P_n\right)^{-1}\left(\sum_{i=1}^{n}\rho_i\left(\left|x_i-M_{\psi_N}\right|\right)\right)^2$$

holds and hence we get (5.2).

The proof of the special cases (1) and (2) in the theorem are the same as the proofs of the equality cases in Theorem 2 and as proved in [1] and in [2].

Case B for $M>x_n$ is proved similarly to Case A.

Hence the proof of Theorem 15 is complete. □

Theorem 15, besides being a refinement of Slater-Pečarić inequality is also an analog of Theorem 12 which deals with non-negative superquadratic functions.

6. Bounds for differences of "Jensen's gap" for N-quasiconvex functions

In this section we state one of many results that can be derived from the previous theorems. First we quote a result from [3] about the difference between two "Jensen's gaps" $\sum_{i=1}^{n}p_i\psi(x_i)-\psi(\overline{x}_p)$ and $\sum_{i=1}^{n}q_i\psi(x_i)-\psi(\overline{x}_q)$. Then we present a new theorem with results when ψ is a N-quasiconvex function. In particular for a 1-quasiconvex function ψ the result is interesting.

The proofs in this section like the proofs in [3] employ some of the techniques used in [8].

In [3, Theorem 2] the following is proved:

THEOREM 16. *Suppose that $\psi : I \to \mathbb{R}$, where I is $[0,a]$ or $[o,\infty)$ is superquadratic. Let $x_i \in I$, $i = 1,\ldots,n$, $\overline{x}_p = \sum_{i=1}^{n} p_i x_i$, $p_i \geqslant 0$, $i = 1,\ldots,n$, $\sum_{i=1}^{n} p_i = 1$ and $\overline{x}_q = \sum_{i=1}^{n} q_i x_i$, $q_i \geqslant 0$, $i = 1,\ldots,n$, $\sum_{i=1}^{n} q_i = 1$. Then, for $m = \min_{1\leqslant i\leqslant n}\left(\frac{p_i}{q_i}\right)$*

$$\left(\sum_{i=1}^{n} p_i\psi(x_i) - \psi(\overline{x}_p)\right) - m\left(\sum_{i=1}^{n} q_i\psi(x_i) - \psi(\overline{x}_q)\right) \tag{6.1}$$
$$\geqslant m\psi\left(\left|\sum_{i=1}^{n}(q_i - p_i)x_i\right|\right) + \sum_{i=1}^{n}(p_i - mq_i)\,\psi\left(\left|x_i - \sum_{j=1}^{n} p_j x_j\right|\right)$$

and for $M = \max_{1\leqslant i\leqslant n}\left(\frac{p_i}{q_i}\right)$

$$\left(\sum_{i=1}^{n} p_i\psi(x_i) - \psi(\overline{x}_p)\right) - M\left(\sum_{i=1}^{n} q_i\psi(x_i) - \psi(\overline{x}_q)\right) \tag{6.2}$$
$$\leqslant -\sum_{i=1}^{n}(Mq_i - p_i)\,\psi\left(\left|x_i - \sum_{j=1}^{n} q_j x_j\right|\right) - \psi\left(\left|\sum_{i=1}^{n}(p_i - q_i)x_i\right|\right).$$

If the superquadratic function is also nonnegative and therefore is also convex, then (6.1) and (6.2) refine the following theorem by Dragomir in [8]:

THEOREM 17. *Under the same conditions on $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, $\overline{x}_p$, $\overline{x}_q$, m and M, as in Theorem 16, if ψ is convex then*

$$M\left(\sum_{i=1}^{n} q_i\psi(x_i) - \psi(\overline{x}_q)\right) \geqslant \sum_{i=1}^{n} p_i\psi(x_i) - \psi(\overline{x}_p) \tag{6.3}$$
$$\geqslant m\left(\sum_{i=1}^{n} q_i\psi(x_i) - \psi(\overline{x}_q)\right).$$

Now we show another refinement of Theorem 17 this time for N-quasiconvex function ψ_N.

THEOREM 18. *Suppose that $\psi_N : I \to \mathbb{R}$ where I is $[a,b)$, $0 \leqslant a$, $b \leqslant \infty$, is N-quasiconvex function, that is $\psi_N = x^N\varphi(x)$, $N = 1,2,\ldots$ where φ is convex on $[a,b)$. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, $\overline{x}_p$, $\overline{x}_q$, m and M be as in Theorem 16, then*

$$\left(\sum_{i=1}^{n} p_i\psi_N(x_i) - \psi_N(\overline{x}_p)\right) - m\left(\sum_{i=1}^{n} q_i\psi_N(x_i) - \psi_N(\overline{x}_q)\right) \tag{6.4}$$
$$\geqslant \sum_{i=1}^{n}(p_i - mq_i)(x_i - \overline{x}_p)^2\frac{\partial}{\partial\overline{x}_p}\left(\frac{x_i^N - \overline{x}_p^N}{x_i - \overline{x}_p}\varphi(\overline{x}_p)\right)$$
$$+m(\overline{x}_q - \overline{x}_p)^2\frac{\partial}{\partial\overline{x}_p}\left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p}\varphi(\overline{x}_p)\right),$$

and

$$\left(\sum_{i=1}^{n} p_i \psi_N(x_i) - \psi_N(\overline{x}_p)\right) - M\left(\sum_{i=1}^{n} q_i \psi_N(x_i) - \psi_N(\overline{x}_q)\right) \tag{6.5}$$

$$\leqslant \sum_{i=1}^{n} (p_i - Mq_i)(x_i - \overline{x}_q)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{x_i^N - \overline{x}_q^N}{x_i - \overline{x}_q}\varphi(\overline{x}_q)\right)$$

$$-M(\overline{x}_q - \overline{x}_p)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p}\varphi(\overline{x}_q)\right).$$

For $N = 1$ we get that

$$\left(\sum_{i=1}^{n} p_i \psi_1(x_i) - \psi_1(\overline{x}_p)\right) - m\left(\sum_{i=1}^{n} q_i \psi_1(x_i) - \psi_1(\overline{x}_q)\right) \tag{6.6}$$

$$\geqslant \varphi'(\overline{x}_p)\left(\left(\sum_{i=1}^{n} p_i x_i^2 - (\overline{x}_p)^2\right) - m\left(\sum_{i=1}^{n} q_i x_i^2 - (\overline{x}_q)^2\right)\right),$$

and

$$\left(\sum_{i=1}^{n} p_i \psi_1(x_i) - \psi_1(\overline{x}_p)\right) - M\left(\sum_{i=1}^{n} q_i \psi_1(x_i) - \psi_1(\overline{x}_q)\right) \tag{6.7}$$

$$\leqslant \varphi'(\overline{x}_q)\left(\left(\sum_{i=1}^{n} p_i x_i^2 - (\overline{x}_p)^2\right) - M\left(\sum_{i=1}^{n} q_i x_i^2 - (\overline{x}_q)^2\right)\right).$$

In particular if φ is also non-negative increasing then (6.4)–(6.7) are refinements of (6.3).

Proof. To prove (6.4) we define $\mathbf{y}$ and $\mathbf{d}$ as

$$y_i = \begin{cases} x_i, & i = 1, \ldots, n \\ \overline{x}_q, & i = n+1 \end{cases}, \qquad d_i = \begin{cases} p_i - mq_i, & i = 1, \ldots, n \\ m, & i = n+1 \end{cases}. \tag{6.8}$$

From (6.8) we get that $\overline{y} = \sum_{i=1}^{n+1} d_i y_i = \sum_{i=1}^{n} p_i x_i = \overline{x}_p$ Then (2.3) for $\mathbf{y}$ and $\mathbf{d}$ is

$$\left(\sum_{i=1}^{n} p_i \psi_N(x_i) - \psi_N(\overline{x}_p)\right) - m\left(\sum_{i=1}^{n} q_i \psi_N(x_i) - \psi_N(\overline{x}_q)\right)$$

$$= \sum_{i=1}^{n} (p_i - mq_i)\psi_N(x_i) + m\psi_N(\overline{x}_q) - \psi_N(\overline{x}_p)$$

$$= \sum_{i=1}^{n+1} d_i \psi_N(y_i) - \psi_N(\overline{x}_p) \geqslant \sum_{i=1}^{n+1} d_i (y_i - \overline{y})^2 \frac{\partial}{\partial \overline{y}}\left(\frac{\overline{y}^N - y_i^N}{\overline{y} - y_i}\varphi(\overline{y})\right).$$

which after using again (6.8) is (6.4).

To get (6.5), we choose $\mathbf{z}$ and $\mathbf{r}$ as

$$z_i = \begin{cases} x_i, & i = 1, \dots, n \\ \overline{x}_p, & i = n+1 \end{cases}, \qquad r_i = \begin{cases} q_i - \frac{p_i}{M}, & i = 1, \dots, n \\ \frac{1}{M}, & i = n+1 \end{cases}.$$

Then, as $\sum_{i=1}^{n+1} r_i = 1,\ r_i \geqslant 0,\ i = 1, \dots, n+1$ and $\sum_{i=1}^{n+1} r_i z_i = \sum_{i=1}^{n} q_i x_i = \overline{x}_q$, we get that

$$\begin{aligned}
&\left(\sum_{i=1}^{n} q_i \psi_N(x_i) - \psi_N(\overline{x}_q)\right) - \frac{1}{M}\left(\sum_{i=1}^{n} p_i \psi_N(x_i) - \psi_N(\overline{x}_p)\right) \\
&= \sum_{i=1}^{n}\left(q_i - \frac{p_i}{M}\right)\psi_N(x_i) + \frac{1}{M}\psi_N(\overline{x}_p) - \psi_N(\overline{x}_q) \\
&= \sum_{i=1}^{n+1} r_i \psi_N(z_i) - \psi_N\left(\sum_{i=1}^{n+1} r_i z_i\right) \\
&\geqslant \sum_{i=1}^{n}\left(q_i - \frac{p_i}{M}\right)(x_i - \overline{x}_q)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{\overline{x}_q^N - x_i^N}{\overline{x}_q - x_i}\varphi(\overline{x}_q)\right) \\
&\quad + \frac{1}{M}(\overline{x}_p - \overline{x}_q)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p}\varphi(\overline{x}_q)\right),
\end{aligned}$$

which is equivalent to (6.5). □

Acknowledgements. The author wish to thank L.-E. Persson whose many advices are as usual very useful and to the referee for the illuminating remarks.

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(Received February 1, 2016)

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